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FORMULÆ AND TABLES FOR THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE.

By Edward B. Rosa and Louis Cohen.

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INTRODUCTION.

A great many formulæ have been given for calculating the mutual and self-inductance of the various cases of electrical circuits occurring in practice. Some of these formulæ have subsequently been shown to be wrong, and of those which are correct and applicable to any given case there is usually a choice, because of the greater accuracy or greater convenience of one as compared with the others. For the convenience of those having such calculations to make we have brought together in this paper all the formulæ with which we are acquainted which are of value in the calculation of mutual and self-inductance, particularly in nonmagnetic circuits where the frequency of the current is low enough to assure sensibly uniform distribution of current. A considerable number of formulæ which have been shown to be unreliable or which have been replaced by others that are less complicated or more accurate have been omitted, although in most cases we have given references to such omitted formulæ. Where several formulæ are applicable to the same case we have pointed out the especial advantage of each and indicated which one is best adapted to precision work.

In the second part of the paper we give a large number of examples to illustrate and test the formulæ. Some of these examples are taken from previous papers by the present authors, but many are new. We have given the work in many cases in full to serve as a guide in such calculations in order to make the formulæ as useful as possible to students and others not familiar with such calculations, and also to facilitate the work of checking up the results by anyone going over the subject. We have been impressed with the advantage of this in reading the work of others.

In the appendix to the paper are a number of tables that will be found useful in numerical calculations of inductance.

In most cases we have given the name of the author of a formula in connection with the formula. This is not merely for the sake of historical interest, or to give proper credit to the authors, but also because we have found it helpful to distinguish in this way the various formulæ instead of denoting each merely by a number. The formulæ of sections 8 and 9, which are taken largely from a paper by one of the present authors,[1] are, however, not so designated, although the authorship of those that are not new is indicated where known.

---

[1] Rosa, this Bulletin, 4, p. 301; 1907.

I. FORMULÆ.

1. MUTUAL INDUCTANCE OF TWO COAXIAL CIRCLES.

MAXWELL'S FORMULÆ IN ELLIPTIC INTEGRALS.

The first and most important of the formulæ for the mutual inductance of coaxial circles is the formula in elliptic integrals given by Maxwell:[1]

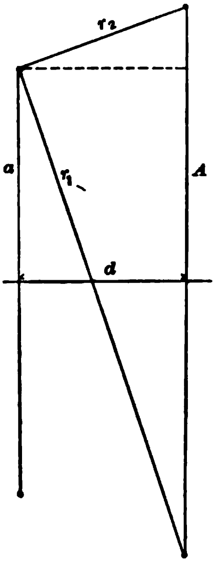

Fig. 1.

$$M=4\pi\sqrt{Aa}\left\{\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right\} \qquad [1]$$

in which A and a are the radii of the two circles, d is the distance between their centers, and

$$k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}}=\sin\gamma$$

F and E are the complete elliptic integrals of the first and second kind, respectively, to modulus k. Their values may be obtained from the tables of Legendre, or the values of $M\div 4\pi\sqrt{Aa}$ may be obtained from Table I in the appendix of this paper, the values of γ being the argument.

The notation of Maxwell is slightly altered in the above expressions in order to bring it into comformity with the formulæ to follow.

Formula (1) is an absolute one, giving the mutual inductance of two coaxial circles of any size at any distance apart. If the two circles have equal or nearly equal radii, and are very near each other, the quantity k will be very nearly equal to unity and γ will be near to 90°. Under these circumstances it may be difficult to obtain a sufficiently exact value of F and E from the tables, as the quantities are varying rapidly and the tabular differences are very large. Under such circumstances the following formula, also given by Maxwell[1] (derived by means of Landen's transformation), is more suitable:

$$M=8\pi\frac{\sqrt{Aa}}{\sqrt{k_1}}\left\{F_1-E_1\right\} \qquad [2]$$

---

[1] Electricity and Magnetism, Vol. II, § 701.

in which F_1 and E_1 are complete elliptic integrals to modulus k_1, and

$$k_1 = \frac{r_1 - r_2}{r_1 + r_2} = \sin \gamma_1$$

r_1 and r_2 are the greatest and least distances of one circle from the other (Fig. 1); that is,

$$r_1 = \sqrt{(A+a)^2 + d^2}$$
$$r_2 = \sqrt{(A-a)^2 + d^2}$$

The new modulus k_1 differs from unity more than k, hence γ_1 is not so near to 90° as γ and the values of the elliptic integrals can be taken more easily from the tables than when using formula (1) and the modulus k.

Another way of avoiding the difficulty when k is nearly unity is to calculate the integrals F and E directly, and thus not use the tables of elliptic integrals, expanding F and E in terms of the complementary modulus k', where $k' = \sqrt{1-k^2}$. The expressions for F and E are very convergent when k' is small.

$$\left.\begin{aligned}
F = \log\frac{4}{k'} &+ \frac{1^2}{2^2}k'^2\left(\log\frac{4}{k'} - \frac{2}{1.2}\right) \\
&+ \frac{1^2 3^2}{2^2 4^2}k'^4\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4}\right) \\
&+ \frac{1^2 3^2 5^2}{2^2 4^2 6^2}k'^6\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6}\right) \\
&+ \frac{1^2 3^2 5^2 7^2}{2^2 4^2 6^2 8^2}k'^8\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} - \frac{2}{7.8}\right) \\
&+ \dots\dots\dots \\
E = 1 &+ \frac{1}{2}k'^2\left(\log\frac{4}{k'} - \frac{1}{1.2}\right) \\
&+ \frac{1^2 3}{2^2 4}k'^4\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{1}{3.4}\right) \\
&+ \frac{1^2 3^2 5}{2^2 4^2 6}k'^6\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{1}{5.6}\right) \\
&+ \frac{1^2 3^2 5^2 7}{2^2 4^2 6^2 8}k'^8\left(\log\frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} - \frac{1}{7.8}\right) \\
&+ \dots\dots\dots
\end{aligned}\right\} \quad [3]$$

WEINSTEIN'S FORMULA.

Weinstein[3] gives an expression for the mutual inductance of two coaxial circles, in terms of the complementary modulus k' used in the preceding series (3). Substituting in equation (1) the values of F and E given above we have Weinstein's equation, which is as follows:

$$M=4\pi\sqrt{Aa}\left\{\left(1+\frac{3}{4}k'^2+\frac{33}{64}k'^4+\frac{107}{256}k'^6+\frac{5913}{16384}k'^8+\ .\right)\left(\log\frac{4}{k'}-1\right)\right.$$
$$\left.-\left(1+\frac{15}{128}k'^4+\frac{185}{1536}k'^6+\frac{7465}{65536}k'^8+\ .\ .\ .\right)\right\} \qquad [4]$$

This expression is rapidly convergent when k' is small, and hence will give an accurate value of M when the circles are near each other. Otherwise formula (1) may be more suitable.

NAGAOKA'S FORMULÆ.

Nagaoka[4] has given formulæ for the calculation of the mutual inductance of coaxial circles, without the use of tables of elliptic integrals. These formulæ make use of Jacobi's q-series, which is very rapidly convergent. The first is to be used when the circles are not near each other, the second when they are near each other. Either may be employed for a considerable range of distances between the extremes, although the first is more convenient. The *first* formula is as follows:

$$\begin{aligned}M&=16\pi^2\sqrt{Aa}.q^{\frac{3}{2}}(1+\epsilon)\\&=4\pi\sqrt{Aa}\{4\pi q^{\frac{3}{2}}(1+\epsilon)\}\end{aligned} \qquad [5]$$

where A and a are the radii of the two circles. The correction term ϵ can be neglected when the circles are quite far apart.

$$\epsilon=3q^4-4q^6+9q^8-12q^{10}+\ .\ .\ .\ .$$
$$q=\frac{l}{2}+\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+\ .\ .\ .\ .$$
$$l=\frac{1-\sqrt{k'}}{1+\sqrt{k'}} \qquad k'=\frac{r_2}{r_1}=\frac{\sqrt{(A-a)^2+d^2}}{\sqrt{(A+a)^2+d^2}}$$

---

[3] Wied. Ann. **21**, p. 344; 1884.

[4] Phil. Mag., **6**, p. 19; 1903.

d being the distance between the centers of the circles, and k' the complementary modulus occurring in equations (3) and (4).

Nagaoka's *second* formula is as follows:

$$M=4\pi\sqrt{Aa}.\frac{1}{2(1-2q_1)^2}\left\{\log\frac{1}{q_1}[1+8q_1(1-q_1+4q_1^2)]-4\right\} \qquad [6]$$

$$q_1=\frac{l_1}{2}+2\left(\frac{l_1}{2}\right)^5+15\left(\frac{l_1}{2}\right)^9+\ \cdot\ \cdot\ \cdot$$

$$l_1=\frac{1-\sqrt{k}}{1+\sqrt{k}} \qquad k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+d^2}}$$

k is the modulus of equation (1), but is employed here to obtain the value of the *q*-series instead of the values of the elliptic integrals employed in (1). This formula is ordinarily simpler in use than it appears, because some of the terms in the expressions above are usually negligible.

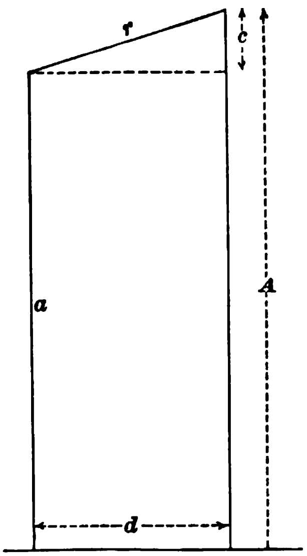

Fig. 2.

MAXWELL'S SERIES FORMULA.

Maxwell[5] obtained an expression for the mutual inductance between two coaxial circles in the form of a converging series which is often more convenient to use than the elliptical integral formula, and when the circles are nearly of the same radii and relatively near each other the value given is generally sufficiently exact. In the following formula *a* is the smaller of the two radii, *c* is their difference, $A-a$, *d* is the distance apart of the circles as before. and $r=\sqrt{c^2+d^2}$. The mutual inductance is then

$$M=4\pi a\left\{\log\frac{8a}{r}\left(1+\frac{c}{2a}+\frac{c^2+3d^2}{16a^2}-\frac{c^3+3cd^2}{32a^3}+\ \cdot\ \cdot\ \cdot\right)\right.$$
$$\left.-\left(2+\frac{c}{2a}-\frac{3c^2-d^2}{16a^2}+\frac{c^3-6cd^2}{48a^3}-\ \cdot\ \cdot\ \cdot\right)\right\} \qquad [7]$$

---

[5] Electricity and Magnetism, Vol. II, § 705.

When c and d are small compared with a, we have for an approximate value of the mutual inductance the following simple expression:[6]

$$M=4\pi a\left\{\log\frac{8a}{r}-2\right\} \qquad [8]$$

When the two radii are equal, as is often the case in practice, the equation (7) is somewhat simplified, as follows:

$$M=4\pi a\left\{\log\frac{8a}{d}\left(1+\frac{3d^2}{16a^2}\right)-\left(2+\frac{d^2}{16a^2}\right)\right\} \qquad [9]$$

The above formulæ (7) and (9) are sufficiently exact for very many cases, the terms omitted in the series being unimportant when $\frac{c}{a}$ and $\frac{d}{a}$ are small. For example, if $\frac{d}{a}$ is 0.1, the largest term neglected in (9) is less than two parts in a million. If, however $d=a$, this term will be more than one per cent, and the formula will be quite inexact.

Coffin[7] has extended Maxwell's formula (9) for two equal circles by computing three additional terms for each part of the expression. This enables the mutual inductance to be computed with considerable exactness up to $d=a$. Formula (1) is exact, as stated above, for all distances, and either it or (5) should be used in preference to (10) when d is large. Coffin's formula is as follows:

$$\begin{aligned}M=4\pi a\Bigg\{\log\frac{8a}{d}&\left(1+\frac{3d^2}{16a^2}-\frac{15d^4}{8\times128a^4}+\frac{35d^6}{128^2a^6}-\frac{1575d}{2\times128^3a^8}+\ldots\right)\\&-\left(2+\frac{d^2}{16a^2}-\frac{31d^4}{16\times128a^4}+\frac{247d^6}{6\times128^2a^6}-\frac{7795d^8}{8\times128^3d^8}+\ldots\right)\Bigg\}\end{aligned} \qquad [10]$$

We have extended Maxwell's formula (7) for unequal circles as follows:[8]

---

[6] This is equivalent to the approximate formula given by Wiedemann, $M=4\pi a\left\{\log\frac{2l}{c}-2.45\right\}$, where l is the circumference of the smaller circle and c is the same as r above.

[7] J. G. Coffin, this Bulletin, **2**, p. 113; 1906.

[8] This Bulletin, **2**, p. 364; 1907.

$$M=4\pi a\left\{\log\frac{8a}{r}\left(1+\frac{c}{2a}+\frac{c^2+3d^2}{16a^2}-\frac{c^3+3cd^2}{32a^3}+\frac{17c^4+42c^2d^2-15d^4}{1024a^4}\right.\right.$$
$$\left.-\frac{19c^5+30c^3d^2-45cd^4}{2048a^5}+\ldots\right)-\left(2+\frac{c}{2a}-\frac{3c^2-d^2}{16a^2}+\frac{c^3-6cd^2}{48a^3}\right.$$
$$\left.\left.+\frac{19c^4+534c^2d^2-93d^4}{6144a^4}-\frac{379c^5+3030c^3d^2-1845cd^4}{61440a^5}\right)\right\}\quad[11]$$

When $c=0$, this gives the first part of series (10). When $d=0$, the case of two circles in the same plane, with radii a and $a+c$, we have

$$M=4\pi a\left\{\log\frac{8a}{c}\left(1+\frac{c}{2a}+\frac{c^2}{16a^2}-\frac{c^3}{32a^3}+\frac{17c^4}{1024a^4}-\frac{19c^5}{2048a^5}+\ldots\right)\right.$$
$$\left.-\left(2+\frac{c}{2a}-\frac{3c^2}{16a^2}+\frac{c^3}{48a^3}+\frac{19c^4}{6144a^4}-\frac{379c^5}{61440a^5}+\ldots\right)\right\}\quad[12]$$

These formulæ (11) and (12) give the mutual inductance with great precision when the coils are not too far apart. The degree of convergence, of course, indicates approximately in any case the accuracy of the result.

The necessity for accurate formulæ for the mutual inductance of coaxial circles, which arises in connection with the development and testing of other formulæ as well as in the determination of the mutual inductance of coils by the methods of the next section, is fully met by the preceding formulæ. It is only necessary to use a sufficient number of decimal places to get any required accuracy when using absolute formulæ like (1) and (2), and some of the series formulæ give very high accuracy in many cases. The considerable number of formulæ available in most cases makes it possible to check important calculations by independent formulæ, and in general to choose for any particular case the formula that is on the whole best adapted.

For illustrations and tests of the above formulæ see examples 1–11, page 65.

2. MUTUAL INDUCTANCE OF TWO COAXIAL COILS.

ROWLAND'S FORMULA.

Let there be two coaxial coils of mean radii A and a, axial breadth of coils b_1 and b_2, radial depth c_1 and c_2, and distance apart of their mean planes d. Suppose them uniformly wound with n_1 and n_2 turns of wire. The mutual inductance M_0 of the two central turns of the coils (Fig. 3), will be given by formula (1) or (4), and the mutual inductance M of the two coils of n_1 and n_2 turns will then be, to a *first approximation*,

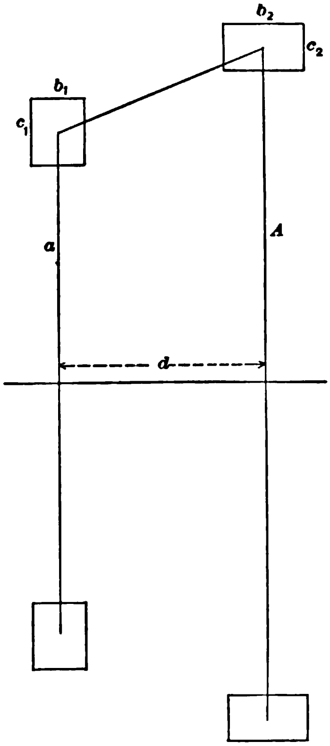

Fig. 3.

$$M=n_1n_2M_0$$

The following second approximation was obtained by Rowland by means of Taylor's theorem, following Maxwell, § 700:

$$M=M_0+\frac{1}{24}\Big\{(b_1^2+b_2^2)\frac{d\,M_0}{dx^2}+c_1^2\frac{d^2M_0}{da^2}+c_2^2\frac{d^2M_0}{dA^2}\Big\}$$

If the two coils are of equal radii but unequal section,

$$M=M_0+\frac{1}{24}\Big\{(b_1^2+b_2^2)\frac{d^2M_0}{dx^2}+(c_1^2+c_2^2)\frac{d^2M_0}{da^2}\Big\} \qquad [13]$$

If the two coils are of equal radii and equal section, this becomes

$$M=M_0+\frac{1}{12}\Big\{b^2\frac{d^2M}{dx^2}+c^2\frac{d^2M}{da^2}\Big\} \qquad [14]$$

The value of M_0 is preferably calculated by formula (1), but any one of the foregoing formulæ for the mutual inductance of coaxial circles adapted to the particular case may be used. The correction terms will be calculated by means of the following:

$$\left.\begin{aligned} \frac{d^2M_0}{da^2} &= \pi\frac{k}{a}\left\{(2-k^2)\,F-\left(2-k^2\frac{1-2k^2}{1-k^2}\right)E\right\} \\ \frac{d^2M_0}{dx^2} &= \pi\frac{k^2}{a}\left\{F-\frac{1-2k^2}{1-k^2}E\right\} \end{aligned}\right\} \quad [15]$$

The equation (14) is equivalent to Rowland's equation, where 2ξ and 2η are the breadth and depth of the section of the coil, instead of b and c, except that there is an error in the formula as printed in Rowland's[9] paper, ξ and η being interchanged. The equations (15) are equivalent to those given by Rowland, being somewhat simpler.[10]

Formula (14) gives a very exact value for the mutual inductance of two coils, provided the cross sections are relatively small and the distance apart d is not too small. But when b or c is large or d is small the fourth differential coefficients which have been neglected become appreciable and the expression may not be sufficiently exact.

RAYLEIGH'S FORMULA.

Maxwell[11] gives a formula, suggested by Rayleigh, for the mutual inductance of two coils, which has a very different form from Rowland's, but is nearly equivalent to it when the coils are not near each other. It has been used by Rayleigh in calculating the mutual inductance of a Lorenz apparatus and by Glazebrook (Phil. Trans., 1883) in calculating the mutual inductance of parallel coils of rectangular section employed in a determination of the ohm. It may also be employed in calculating the attraction between two coils.[12] It is sometimes called the formula of quadratures, and is as follows:[13]

$$M=\frac{1}{6}\left(M_1+M_2+M_3+M_4+M_5+M_6+M_7+M_8-2M_0\right) \quad [16]$$

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[9] Collected Papers, p. 162. Am. Jour. Sci. [3], XV, 1878.

[10] Gray, Absolute Measurements, Vol. II, Part II, p. 322.

[11] Electricity and Magnetism, Vol. II, Appendix II, Chapter XIV.

[12] Gray, Absolute Measurements, Vol. II, Part II, p. 403.

[13] This Bulletin, 2, p. 370–372; 1906.

where M_1 is the mutual inductance of the circle O_2 and a circle through the point 1 of radius $A-\frac{c_1}{2}$, and similarly for the others, Fig. 4.

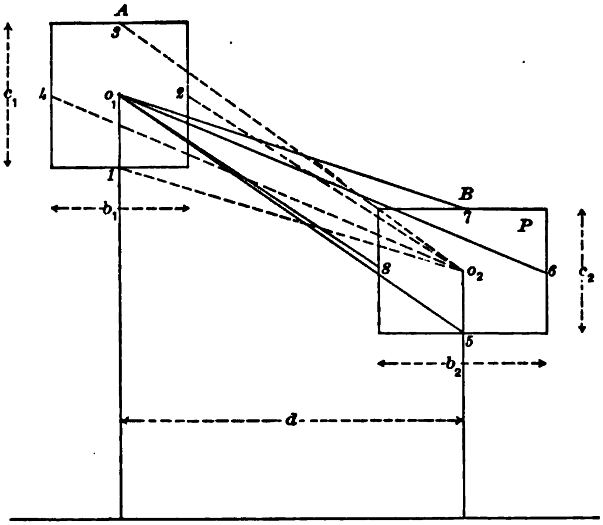

Fig. 4.

For two coils of equal radii and equal section this becomes

$$M=\frac{1}{3}\left(M_1+M_2+M_3+M_4-M_0\right) \qquad [17]$$

Equation (16) is Rayleigh's formula, or the formula of quadratures. Instead of computing the correction to M_0 by means of the differential coefficients (13), eight additional values are computed, corresponding to the mutual inductances of the single turns at the eight numbered points indicated in Fig. 4, each with reference to the central turn of the other coil. These *M's* may be computed by formulæ (7) and (9) or (10) and (11), and the values of the constants for the case of two coils of *equal radii* are given in the following table, the radius being *a* in every case.

	Axial distance.	**Radial distance.**	**r**
Using (7)	d	$-\frac{c_1}{2}$	$\sqrt{d^2+\frac{c_1^2}{4}}$
"	d	$+\frac{c_1}{2}$	$\sqrt{d^2+\frac{c_1^2}{4}}$
"	d	$-\frac{c_2}{2}$	$\sqrt{d^2+\frac{c_2^2}{4}}$
"	d	$+\frac{c_2}{2}$	$\sqrt{d^2+\frac{c_2^2}{4}}$
Using (9)	$d-b_1/2$	0	
"	$d+b_1/2$	0	
" .	$d+b_2/2$	0	
"	$d-b_2/2$	0	

MAGNITUDE OF THE ERRORS IN ROWLAND'S AND RAYLEIGH'S FORMULÆ.

The error ϵ_1 in equation (17), for two coils of equal radii a, distance between centers being d, and section $b \times c$, depends on the dimensions of the coil in a manner shown by the following expression:[14]

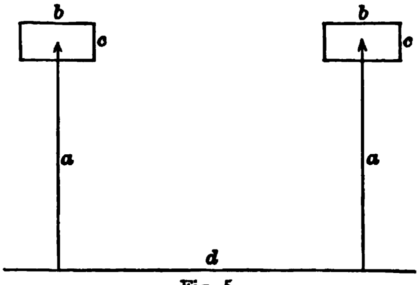

Fig. 5.

$$\epsilon_1 \propto 4\pi a \left\{\frac{3b^4+3c^4-20b^2c^2}{480d^4}\right\} \quad [18]$$

For a square coil the correction is a negative quantity, showing that M by equation (17) is too large, and the error is proportional to the fourth power of $\frac{1}{d}$, the reciprocal of the distance between the mean planes of the coils. For a rectangular coil in which b is greater than c the correction is negative so long as b is not more than 2.5 times c. When b is still larger with respect to c the correction becomes plus, the value of M by (17) being too small.

Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (18) as we vary the shape of cross section, keeping $bc=4$.

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[14] This Bulletin, 2, p. 373; 1906.

Dimensions of coil.		Error proportional to
$b=2$	$c=2$	$-\ 224$
$b=2.5$	$c=1.6$	$-\ 183$
$b=3$	$c=1.33$	$-\ 67.5$
$b=4$	$c=1$	$+\ 451$
$b=8$	$c=0.5$	$+11{,}988$

Thus we see that the value of M as given by the formula of quadratures may be too large or too small according to the shape of the section, and that the error is proportional directly to the fourth power of the dimensions of the section and inversely to the fourth power of the distance between the mean planes of the coils. When the section is small and d large the error will become negligible.

The error by Rowland's formula is[14]

$$\epsilon_2 \propto 4\pi a \frac{6}{d^4}\left\{\frac{b^4+c^4}{360}-\frac{b^2c^2}{144}\right\} \propto 4\pi a\left\{\frac{8b^4+8c^4-20b^2c^2}{480d^4}\right\} \qquad [19]$$

This is negative for a square coil, but smaller than ϵ_1. For a coil of section such that $b=c\sqrt{2}$, the error is zero, and for sections such that $\frac{b}{c}>\sqrt{2}$, the error is positive. Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (19) which is proportional to the error by Rowland's formula.

Dimensions of coil.		Error proportional to—
$b=2$	$c=2$	$-\ 64$
$b=2.5$	$c=1.6$	$+\ 45$
$b=3$	$c=1.33$	$+\ 353$
$b=4$	$c=1$	$+\ 1{,}736$
$b=8$	$c=0.5$	$+32{,}448$

Thus the error is smaller by Rowland's formula for coils having square or nearly square section, but larger for coils having rectangular sections not nearly square.

LYLE'S FORMULA.

Professor Lyle[15] has recently proposed a very convenient method for calculating the mutual inductance of coaxial coils, which gives very accurate results for coils at some distance from each other.

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[14] This Bulletin, **2**, p. 373; 1906.

[15] Phil. Mag., **8**, p. 310; 1902. Also this Bulletin, **2**, p. 374–378; 1906.

The mutual inductance is calculated from formula (1) or any other formula for two coaxial circles, using, however, a modified radius r instead of the mean radius a, r being given by the following equation when the section is square, b being the side of the square section:

$$r=a\left(1+\frac{b^2}{24a^2}\right) \qquad [20]$$

If the coil has a rectangular section not square, it can be replaced by two filaments, the distance apart of the filaments being called the *equivalent breadth* or the *equivalent depth* of the coil.

$$\left.\begin{array}{l} \beta^2=\dfrac{b^2-c^2}{12}, \text{ 2 } \beta \text{ is the equivalent breadth of A} \\ \\ \delta^2=\dfrac{c^2-b^2}{12}, \text{ 2 } \delta \text{ is the equivalent depth of B} \end{array}\right\} \qquad [21]$$

The equivalent radius of A is given by the same expression which holds for a square coil, viz:

$$r=a\left(1+\frac{c^2}{24a^2}\right)$$

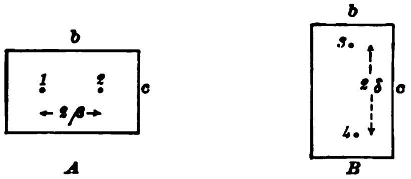

Fig. 6.

In the coil B the equivalent filaments have radii $r+\delta$ and $r-\delta$, respectively, where

$$r=a\left(1+\frac{b^2}{24a^2}\right)$$

The mutual inductance of two coils may now be readily calculated. If each has a square section, it is necessary only to calculate the mutual inductance of the two equivalent filaments. For coils of rectangular sections, as A, B, the mutual inductance will be the sum of the mutual inductances of the two filaments of A on the two filaments of B, counting $n/2$ turns in each. Or, it is n_1n_2 times the mean of the four inductances M_{13}, M_{14}, M_{23}, M_{24}, where M_{13} is the mutual inductance of filament 1 on filament 3, etc.

Lyle's method is of special value in computing mutual inductances because it applies to coils of unequal as well as of equal radii.

ROSA'S FORMULÆ.[16]

Writing the mutual inductance of two coaxial coils of equal radii and equal section as $M = M_0 + \varDelta M$, where M_0 is the mutual inductance of the central circles of the two equal coils of sections $b \times c$, Fig. 5, and $\varDelta M$ is the correction for the section of the coil, the value of $\varDelta M$ is as follows:

$$\begin{aligned}\varDelta M = 4\pi a n^2 \Bigg\{ & \frac{3b^2+c^2}{96a^2} \cdot \log\frac{8a}{d} - \frac{11b^2-3c^2}{192a^2} + \frac{b^2-c^2}{12d^2} + \frac{2b^4+2c^4-5b^2c^2}{120d^4} \\ & + \frac{6b^4+6c^4+5b^2c^2}{5760a^2d^2} + \frac{3b^6-3c^6+14b^2c^4-14b^4c^2}{504d^6} + \frac{7c^2d^2}{1024a^4}\left(\log\frac{8a}{d} - \frac{163}{84}\right) \\ & - \frac{15b^2d^2}{1024a^4}\left(\log\frac{8a}{d} - \frac{97}{60}\right)\Bigg\} \end{aligned} \qquad [22]$$

For a square section, when $b = c$, this becomes

$$\varDelta M = \frac{\pi b^2 n^2}{6a}\left\{\log\frac{8a}{d} - 1 - \frac{a^2b^2}{5d^4} - \frac{3d^2}{16a^2}\left(\log\frac{8a}{d} - \frac{4}{3}\right) + \frac{17b^2}{240d^2}\right\} \qquad [23]$$

The last two terms of equation (23) are relatively small, so that we may write, *approximately:*

$$\varDelta M = \frac{\pi b^2 n^2}{6a}\left\{\log\frac{8a}{d} - 1 - \frac{a^2b^2}{5d^4}\right\} \qquad [24]$$

These expressions for $\varDelta M$ are very exact where the coils are near together or even where they are separated by a considerable distance, but become less exact as d is greater. They are therefore most reliable where formulæ (14), (17), and (20) are least reliable. As formula (24) is exact enough for most purposes, it affords a very easy method of getting the correction for equal coils of square section.

Stefan's formula for the mutual inductance of two equal coaxial coils (originally published[17] without demonstration) is incorrect and is not given here. It resembles equation (22), but is seriously in error for coils at considerable distances.

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[16] This Bulletin, **4**, p. 348, (38) and (39).

[17] Wied. Annalen, **22**, p. 107; 1884.

THE ROSA-WEINSTEIN FORMULA.

Weinstein's formula[18] for the mutual inductance of equal coaxial coils has been revised and corrected by Rosa, and the value of ΔM, the correction for section, expressed separately. The expression for ΔM is as follows:[19]

$$\Delta M = 4\pi a n_1 n_2 \sin\gamma\left\{(F-E)\left(A+\frac{c^2}{24a^2}\right)+EB\right\} \qquad [25]$$

where F and E are the complete elliptic integrals to modulus $\sin\gamma$, Fig. 7 (as in equation 1) and

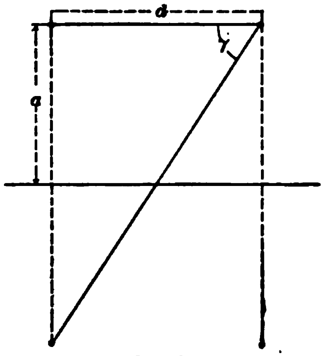

Fig. 7.

$$A=\frac{\cos^2\gamma}{12d^2}\left(a_1-a_2-a_3+(2a_2-3a_3)\cos^2\gamma+8a_3\cos^4\gamma\right)$$

$$B=\frac{\sin^2\gamma}{12d^2}\left(a_1+\frac{a_2}{2}+2a_3+(2a_2+3a_3)\cos^2\gamma+8a_3\cos^4\gamma\right)$$

The values of a_1, a_2, and a_3 are as follows:

$$a_1=b^2-c^2+\frac{c^4}{30a^2} \qquad \text{For square section: } a_1=\frac{b^4}{30a^2}$$

$$a_2=\frac{5b^2c^2-4c^4}{60a^2} \qquad \text{`` `` ``} \quad a_2=\frac{b^4}{60a^2}$$

$$a_3=\frac{2b^4+2c^4-5b^2c^2}{20d^2} \qquad \text{`` `` ``} \quad a_3=-\frac{b^4}{20d^2}$$

Formula (25) is a very exact formula for all positions of the two coils, except when they are very close together.

Weinstein's original formula,[18] which is much less accurate than (25) for coils relatively near together, is not here given.

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[18] Wied. Annalen, **21**, p. 350; 1884.

[19] This Bulletin, **4**, p. 342, equation (20).

USE OF FORMULÆ FOR SELF-INDUCTANCE IN CALCULATING MUTUAL INDUCTANCE.

One can sometimes obtain the mutual inductance of adjacent coils, or of coils at a distance from one another, by means of a formula for the self-inductance of coils. Thus, suppose we have a coil of rectangular section, which we subdivide into three equal parts, 1, 2, 3, Fig. 8. Let L be the self-inductance of the whole coil, L_1 be the self-inductance of any one of the three equal smaller coils, and L_2 be the self-inductance of two adjacent coils taken together. Also let M_{12} be the mutual inductance of coil 1 on coil 2, or of coil 2 on coil 3, and M_{13} be the mutual inductance of coil 1 on coil 3. Then,

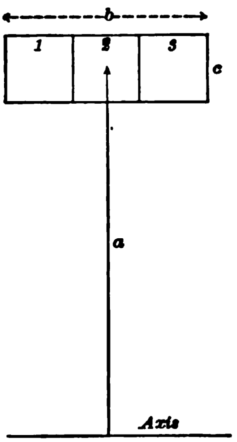

Fig. 8.

$$L = 3L_1 + 4M_{12} + 2M_{13}$$

$$\text{Also, } L_2 = 2L_1 + 2M_{12}$$

$$\left.\begin{aligned} \therefore\ M_{12} &= \frac{L_2 - 2L_1}{2} \\ \text{and } M_{13} &= \frac{L + L_1 - 2L_2}{2} \end{aligned}\right\} \qquad [26]$$

Formula (26) will thus enable us to find the mutual inductance of two coils of equal radii adjacent or near each other by the calculation of self-inductances from such formulæ as those of Weinstein (67) and Stefan (69). These latter formulæ are not, however, exact enough when the section is large to permit us to apply them to coils at any considerable distance from one another.

GEOMETRIC MEAN DISTANCE FORMULA.

The mutual inductance of two coaxial coils adjacent or very near can sometimes be obtained by means of the geometric mean distances. This method is accurate only when the sections are very small relatively to the radius. It can often be used to advantage in testing other formulæ, but not often in determining the mutual inductance of actual coils.

Formula (7) gives the mutual inductance of two very near coaxial coils in terms of the geometric mean distance, if r be replaced by R, the geometric mean distance of the two sections. Formula (7) gives M_0 if r be used, where r is the distance between centers. Thus,

$$\Delta M = 4\pi a n^2 \left(1 + \frac{c}{2a}\right) \log \frac{r}{R} \qquad [27]$$

For coils A and C, $R<r$ and ΔM is positive; $R=0.99770\ r$
" " A " B, $R>r$ and ΔM is negative; $R=1.00655\ r$

The same formula may also be used for squares not adjacent, but only when quite near.[30]

For illustrations and tests of the above formulæ see examples 12–21, pages 71–77.

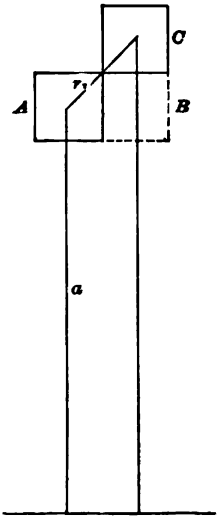

Fig. 9.

The preceding formulæ can be used with entire satisfaction to calculate the mutual inductance of coaxial coils, especially those of coils of equal radii. Formulæ (16), (17), (20), and (21) apply also to coils of unequal radii, but unfortunately they are not as accurate as some of the others, except when the coils are relatively distant or have very small cross sections. The difficulty can be overcome by subdividing each of the two coils into two, four, or more equal parts, and taking the sum of the mutual inductances of all of the parts of one on all the parts of the other. This is a laborious operation, but in important cases it should be done. As the subdivision is carried further the results will approach a final value, and hence the results themselves show when the subdivision has been carried far enough.

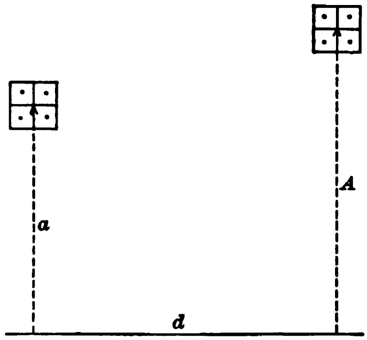

Fig. 9a.

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[30] For other values of the geometric mean distances of squares in a plane see this Bulletin, 8, p. 1; 1907.

Thus, suppose two coils A, B of square section are subdivided into four equal parts and by the method of Lyle, formula (20), the mutual inductance of the whole of B is computed on each of the four parts of A. If the sum differs appreciably from the result obtained by taking A and B as wholes in one calculation, then the four parts of B may be taken separately with respect to the separate parts of A. If one is doubtful whether this is sufficiently accurate, one of the sections of A may be subdivided further and calculated with respect to one section of B, to see whether there is any appreciable difference due to this further subdivision. For coils of equal radii very accurate results for near coils can be obtained much more easily by using some of the other formulæ.

3. MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

There are several formulæ for the calculation of the mutual inductance of coaxial solenoids. Although few of these formulæ are exact, the approximate formulæ often permit inductances to be calculated with very great accuracy by using a sufficient number of terms of the series by which they are expressed.

MAXWELL'S FORMULA.[21]

CONCENTRIC, COAXIAL SOLENOIDS OF EQUAL LENGTH.

The mutual inductance M of two coaxial solenoids of equal length is given by the following expression, due to Maxwell, where A and a are the radii of the outer and inner solenoids, respectively, l is the common length, and n_1 and n_2 the number of turns of wire per cm on the single layer winding of the outer and inner solenoids, respectively:

$$\left.\begin{aligned} M&=4\pi^2a^2n_1n_2[l-2A\alpha] \\ \text{where}\quad \alpha&=\frac{l-r+A}{2A}-\frac{a^2}{16A^2}\left(1-\frac{A^3}{r^3}\right)-\frac{a^4}{64A^4}\left(\frac{1}{2}+2\frac{A^5}{r^5}-\frac{5}{2}\frac{A^7}{r^7}\right) \\ &\quad-\frac{35a^6}{2048A^6}\left(\frac{1}{7}-\frac{8A^7}{7r^7}+\frac{4A^9}{r^9}-\frac{3A^{11}}{r^{11}}\right)+\ldots \end{aligned}\right\} \quad [28]$$

Putting

$$M=M_0-\varDelta M$$

---

[21] Electricity and Magnetism, Vol. II, § 678.

$M_0=4\pi^2a^2n_1n_2l$ is the mutual inductance of an infinite outer solenoid and the finite inner solenoid, while $\varDelta M$ is the correction due to the ends.

Equation (28) is Maxwell's expression, except that we have carried it out further than Maxwell did. This expression for M is rapidly convergent when a is considerably smaller than A, Fig. 10. Equation (28) shows that the mutual inductance is proportional to $l-2A\alpha$; or the length l must be reduced by $A\alpha$ on each end. When a is small and l is large α is 1/2, approximately. That is, the length l is reduced by A, the radius of the outer solenoid.

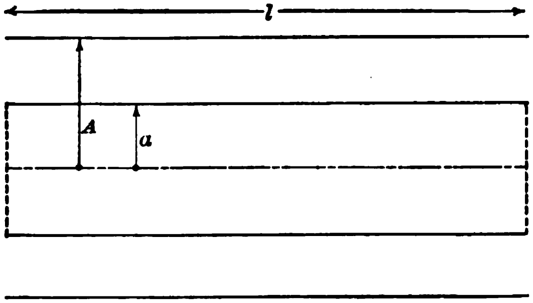

Fig. 10.

For the case of two coils each of more than one layer the above formula may be used, A and a being the mean radii, and n_1 and n_2 the total number of turns per cm in all the layers. The result will be only approximate, but usually less in error than if one uses the formula of Maxwell § 679 quoted by Mascart and Joubert.[22]

When the solenoids are very long in comparison with the radii, formula (28) may be simplified by omitting the terms in A/l, A^3/r^3, A^5/r^5, etc. The expression for α then becomes

$$\alpha=\frac{1}{2}-\frac{a^2}{16A^2}-\frac{a^4}{128A^4}-\frac{5a^6}{2048A^6}-\dots \qquad [29]$$

Heaviside[23] gives an extension of formula (29), but as it neglects $\frac{A}{l}$, $\frac{A^3}{r^3}$, etc., the additional terms are of no importance, being smaller than the terms already neglected in (29).

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[22] Electricity and Magnetism, Vol. I, p. 533.

[23] There are some misprints in Heaviside, 2, p. 277. The radius of the *inner* solenoid should be c_2, of the *outer* c_1, and ρ is c_2^2/c_1^2.

RÒITI'S FORMULA.

For a pair of concentric, coaxial solenoids of which the inner solenoid is considerably shorter than the outer, we have the following formula:[24]

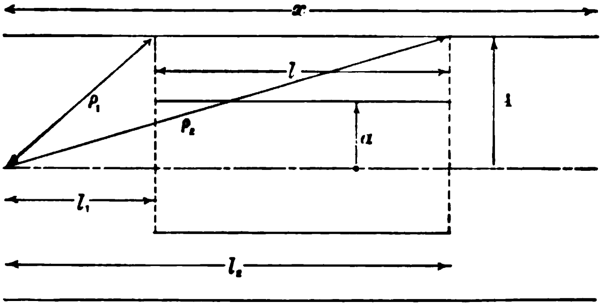

Fig. 11.

$$M=4\pi^2a^2n_1n_2\left[\rho_2-\rho_1+\frac{a^2A^2}{8}\left(\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right)-\frac{a^4A^2}{16}\left(\frac{1}{\rho_1^5}-\frac{1}{\rho_2^5}\right)\right.$$
$$+\frac{5a^4A^4}{64}\left(\frac{1}{\rho_1^7}-\frac{1}{\rho_2^7}\right)+\frac{5a^6A^2}{128}\left(\frac{1}{\rho_1^7}-\frac{1}{\rho_2^7}\right)$$
$$\left.-\frac{35a^6A^4}{256}\left(\frac{1}{\rho_1^9}-\frac{1}{\rho_2^9}\right)+\frac{105a^6A^6}{1024}\left(\frac{1}{\rho_1^{11}}-\frac{1}{\rho_2^{11}}\right)+\,..\right] \quad [30]$$

in which

$\rho_1=\sqrt{l_1^2+A^2}$ where $l_1=\frac{x-l}{2}$

$\rho_2=\sqrt{l_2^2+A^2}$ " $l_2=\frac{x+l}{2}$

$l=l_2-l_1=$ length of inner solenoid.

$x=$ length of outer solenoid and A and a the radii.

This is for many cases a very convenient and very accurate formula.

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[24] For the derivation and extension of this formula see this Bulletin, **3**, pp. 309–310; 1907. This formula was originally given (without proof and without the last three terms) in this Bulletin, **2**, p. 130; 1906.

GRAY'S FORMULA.

Gray[25] gives a general expression for the mutual kinetic energy of two solenoidal coils which may or may not be concentric, and their axes may be at any angle ϕ. The most important case in practice is when the two coils are concentric and coaxial. In that case the zonal harmonic factors in each term reduce to unity, and half the terms become zero. Putting the current in each equal to unity, the mutual kinetic energy becomes the mutual inductance M.

Let $2x$ = the length of outer solenoid
$2l$ = " " " inner "
A = radius of outer "
a = " " inner "
n_1 = number of turns per cm on outer solenoid
n_2 = " " " " " " inner "

Gray's expression with these changes becomes

$$M=\pi^2a^2A^2n_1n_2[K_1k_1+K_3k_3+K_5k_5+\ \ldots\ ] \qquad [31]$$

where K_1, K_3, etc., are functions of x and A, and k_1, k_3, etc., are functions of l and a.[26] When the ratio of the length of the winding of the outer coil to the radius is $\sqrt{3}$ to 1, $K_5=0$, and if the same condition holds for the inner coil, $k_3=0$. If in addition a is considerably smaller than A, the terms of higher order become negligible and (31) reduces to

$$M=\frac{2\pi^2a^2N_1N_2}{d} \qquad [32]$$

where d is half the diagonal of the outer coil, $=\sqrt{x^2+A^2}$. When the dimensions depart slightly from these theoretical ratios the small correction terms to (32) can be calculated.[26]

SEARLE AND AIREY'S FORMULA.

The following expression for the mutual inductance of two concentric, coaxial solenoidal coils has been given by Searle and Airy.[27]

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[25] Absolute Measurements, **2**, Part I, p. 274, equation 53.

[26] Rosa, this Bulletin, **3**, p. 221.

[27] The Electrician (London), **56**, p. 318; 1905.

$$M=g_1G_1+g_3G_3+g_5G_5+g_7G_7+\ \ldots\ldots$$

$$=\frac{2\pi^2a^2N_1N_2}{d}\left[1-\frac{A^2}{2d^4}\cdot\frac{4l^2-3a^2}{4}-\frac{A^4(4x^2-3A^2)}{8d^8}\cdot\frac{8l^4-20l^2a^2+5a^4}{8}\right.$$

$$\left.-\frac{A^6(8x^4-20x^2A^2+5A^4)}{16d^{12}}\cdot\frac{(64l^6-336l^4a^2+280l^2a^4-35a^6)}{64}-\ \ldots\right]$$

[33]

The notation of (33) differs slightly from that used by Searle and Airey.

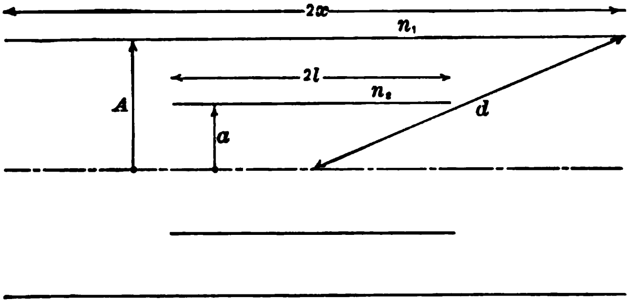

Fig. 12.

Equation (33) has been extended and put for greater convenience in calculation into the following form:[28]

$$M=\frac{2\pi^2a^2N_1N_2}{d}\left[1+\frac{A^2a^2}{8d^4}L_2+\frac{A^4a^4}{32d^8}X_2L_4\right.$$

$$\left.+\frac{A^6a^6}{32d^{12}}X_4L_6+\frac{A^8a^8}{32d^{16}}X_6L_8+\ \ldots\right]\quad[34]$$

where

$$X_2=3-4\frac{x^2}{A^2}\qquad L_2=3-4\frac{l^2}{a^2}$$

$$X_4=\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4}\qquad L_4=\frac{5}{2}-10\frac{l^2}{a^2}+4\frac{l^4}{a^4}$$

$$X_6=\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6}\qquad L_6=\frac{35}{16}-\frac{35}{2}\frac{l^2}{a^2}+21\frac{l^4}{a^4}-4\frac{l^6}{a^6}$$

$$L_8=\frac{63}{32}-\frac{105}{4}\frac{l^2}{a^2}+63\frac{l^4}{a^4}-36\frac{l^6}{a^6}+4\frac{l^8}{a^8}$$

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[28] Rosa, this Bulletin, **3**, p. 224.

This reduces to (32) when the terms after the first are negligible, as they are when the conditions assumed for (32) are fulfilled. The above expressions for L_2, X_2 show what these conditions are in order to make the second and third terms zero. If l^2/a^2 is slightly more or less than ¾, (34) gives the value of the second term which is neglected in (32), etc.

COHEN'S FORMULA.[29]

This is an absolute formula for two coaxial, concentric solenoids of lengths $2l.$ and $2l_2$, Fig. 13.

$$\left.\begin{aligned} M&=4\pi n_1 n_2(V-V_1)\\ V&=-(A^2-a^2)c[F\{F(k',\theta)-E(k',\theta)\}-EF(k',\theta)]\\ &\quad+\frac{c^4-(A^2-6Aa+a^2)c^2-2(A^2-a^2)^2}{3\sqrt{(A+a)^2+c^2}}.F\\ &\quad+\frac{2(A^2+a^2)-c^2}{3}\sqrt{(A+a)^2+c^2}.\,E-c(A^2-a^2)\frac{\pi}{2} \end{aligned}\right\}\quad[35]$$

V_1 is obtained from V by replacing c by c_1,

$$c=l_1+l_2, \qquad c_1=l_1-l_2,$$

F and E are the complete elliptic integrals of the first and second kind to modulus k, where $k^2=\dfrac{4Aa}{(A+a)^2+c^2}$

$F(k', \theta)$ and $E(k', \theta)$ are the incomplete elliptic integrals of modulus k' and amplitude θ,

$$k'^2=1-k^2=1-\frac{4Aa}{(A+a)^2+c^2}$$

$$=\frac{(A-a)^2+c^2}{(A+a)^2+c^2}$$

$$\sin^2\theta=\frac{(A^2-a^2)^2+c^2(A-a)^2}{(A^2-a^2)^2+c^2(A+a)^2}$$

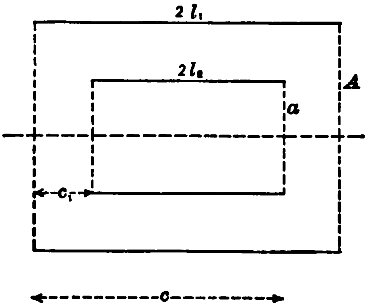

Fig. 13.

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[29] This Bulletin, 3, p. 301; 1907.

RUSSELL'S FORMULÆ.[30]

Russell's formula for coaxial solenoids in the notation of this paper is

$$M=4\pi^2a^2n_1n_2\Bigg[R_1\Bigg\{1-\frac{1}{2}q_2k_1^2-\frac{1}{2}\cdot\frac{1}{4}q_3k_1^4-\frac{1\cdot1\cdot3}{2\cdot4\cdot6}q_4k_1^6$$

$$-\frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}q_5k_1^8-\frac{1\cdot1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot10}q_6k_1^{10}-\;.\;.\;.\;.\Bigg\}$$

$$-R_2\Bigg\{1-\frac{1}{2}q_2k_2^2-\frac{1}{2}\cdot\frac{1}{4}q_3k_2^4-\text{terms with above coefs.}\Bigg\}\Bigg] \qquad [36]$$

where

$$R_1^2=(A+a)^2+(l_1+l_2)^2 \qquad k_1^2=\frac{4\,Aa}{R_1^2}$$

$$R_2^2=(A+a)^2+(l_1-l_2)^2 \qquad k_2^2=\frac{4\,Aa}{R_2^2}$$

$$q_n=\frac{(A+a)^2}{4\,Aa}q_{n-1}-\frac{1}{n}\cdot\frac{1}{2}\cdot\frac{3\cdot5\;.\;.\;.\;.\;2n-3}{4\,.\,6\;.\;.\;.\;.\;2n-2}\cdot\frac{A}{a}$$

$$q_2=\frac{(A+a)^2}{4\,Aa}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{A}{a}$$

$$q_3=\frac{(A+a)^2}{4\,Aa}q_2-\frac{1}{3}\cdot\frac{1\,.\,3}{2\,.\,4}\cdot\frac{A}{a}$$

etc.

A and *a* are the radii of the outer and inner cylinders respectively, $2l_1$ and $2l_2$ their lengths, Fig. 13, and n_1, n_2 the number of turns of wire per cm in the two windings. This formula applies only when the inner coil is shorter than the outer. For two coils of equal length the second part of the above formula is not convergent, and hence it must be replaced by an expression in elliptic integrals. The formula thus becomes (equation 42 in Russell's paper)

$$M=4\pi^2a^2n_1n_2\Bigg[R_1\Bigg\{1-\frac{1}{2}\,q_2k_1^2-\frac{1}{8}q_3k_1^4=.\;.\;.\text{ as above}\Bigg\}\Bigg]$$

$$+\frac{8\pi Aa}{3(A+a)}n_1n_2[(A^2+a^2)(F-E)-2\,AaF] \qquad [37]$$

---

[30] Alexander Russell, Phil. Mag., Apr. 1907, p. 420.

This formula gives an accurate result for equal solenoids of considerable length, but Maxwell's formula (28) is just as accurate and much more convenient.

For short coils neither (36) nor (37) will apply, but for that case as well as other cases Russell's general formula may be used. As the latter is equivalent to (35) it is not here given.

ROSA'S FORMULA FOR SINGLE LAYER COILS OF EQUAL RADII AND EQUAL BREADTH.

The mutual inductance of two coaxial single layer coils of equal radii and equal breadth is given by the following expression:

$$M = M_0 + \Delta M$$

where M_0 is the mutual inductance of the two parallel circles at the centers of the coils and ΔM is given by the following expression:[31]

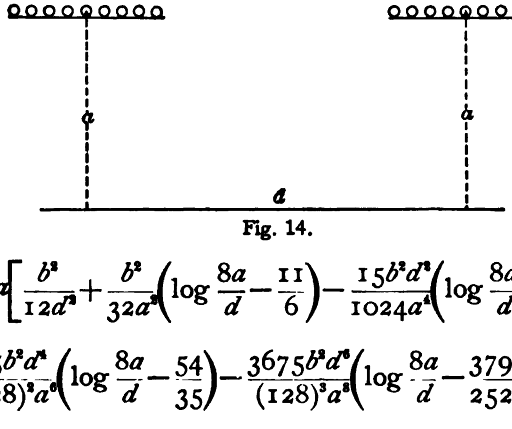

Fig. 14.

$$\Delta M = 4\pi n^2 a\left[\frac{b^2}{12d^2} + \frac{b^2}{32a^2}\left(\log\frac{8a}{d} - \frac{11}{6}\right) - \frac{15b^2d^2}{1024a^4}\left(\log\frac{8a}{d} - \frac{97}{60}\right)\right.$$
$$+ \frac{175b^2d^4}{2(128)^2a^6}\left(\log\frac{8a}{d} - \frac{54}{35}\right) - \frac{3675b^2d^6}{(128)^3a^8}\left(\log\frac{8a}{d} - \frac{3793}{2520}\right) \qquad [38]$$
$$\left. + \frac{b^4}{960a^2d^2} + \frac{b^4}{60d^4} - \frac{b^4}{1024a^4}\left(\log\frac{8a}{d} - \frac{187}{60}\right) + \frac{b^6}{168d^6} + \frac{b^8}{360d^8}\right]$$

This expression will give a very accurate value of ΔM for two coils not nearer together than their breadth if a is considerably greater than b, the breadth of the coil.

OTHER FORMULÆ.

Himstedt has given several formulæ for different cases of coaxial solenoids. The first[32] is for the case of a short secondary on the outside of a long primary. The formula is very complicated and the calculation tedious. By putting the shorter coil inside, the formula of Rôiti or of Searle and Airey may be used to much better advantage.

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[31] Rosa, this Bulletin, **2**, p. 351.

[32] Wied. Annalen, **26**, p. 551; 1885.

Himstedt's second expression is for the case of two coaxial solenoids not concentric, the distance between their mean planes having any value; the radius of one is supposed to be considerably smaller than the other. This also is a very complicated formula, involving second and fourth derivatives of expressions containing the elliptic integrals F and E. Gray's general equation is much simpler to calculate. This is not, however, an important case in practice, and we do not therefore give Himstedt's equation. Himstedt's third equation is general and applies to two coaxial solenoids of nearly equal or very different radii, which may or may not be concentric. This expression of Himstedt's consists of four terms, each of which is a somewhat complicated expression involving both complete and incomplete elliptic integrals. A less inconvenient general expression for M in elliptic integrals is given above (35).

For illustrations and tests of the above formulæ see examples 22–24, page 77.

4. THE MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SINGLE LAYER COIL.

LORENZ'S FORMULA.

The problem of finding the mutual inductance of a circle and a coaxial single layer winding was first solved by Lorenz.[33] Assuming that the current was uniformly distributed over the surface of the cylinder, forming a current sheet, he integrated over the length of the cylinder the expression for the mutual inductance of a circular element and the given circle. This expression is an elliptic integral. Lorenz reduced the integrated form to a series and gave the following formula for the mutual inductance of the disk and solenoid of what is now called the Lorenz apparatus. He called it, however, the constant of the apparatus instead of mutual inductance, and denoted it by C. It is of course the whole number of lines of magnetic force passing through the disk due to unit current in the surrounding solenoid.

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[33] Wied. Annalen, **25**, p. 1; 1885.
Ouvres Scientifiques, **2**, p. 162.

$$\left.\begin{aligned} M &= \frac{\pi q r^2}{d}\Big[Q(a_1)+Q(a_2)\Big] \\ Q(a) &= 2\pi q\sqrt{\frac{a-1}{a}}\Big[1+\frac{3}{8}\frac{q^2}{a^2}+\frac{5}{16}\frac{q^4}{a^4}\Big(\frac{7}{4}-a\Big) \\ &\qquad +\frac{35}{128}\frac{q^6}{a^6}\Big(\frac{33}{8}-\frac{9}{2}a+a^2\Big)+\ldots\Big] \end{aligned}\right\} \quad [39]$$

ρ = radius of the disk, Fig. 1.
r = radius of the solenoid.
$2x$ = length of winding of solenoid.
$q=\rho/r$ = ratio of the two radii.
$d=\frac{2x}{n}$ = distance between centers of successive turns of wire.

$$a=\frac{x^2+r^2}{r^2}$$

If the disk be not exactly in the mean plane of the solenoid, and x_1 be the distance from the plane of the disk to one end of the solenoid and x_2 to the other,

$$a_1=\frac{x_1^2+r^2}{r^2} \qquad a_2=\frac{x_2^2+r^2}{r^2}$$

Then $Q(a_1)$ is found by substituting the values of a_1 in equation (39) above, and $Q(a_2)$ by using the value of a_2 for a in the same equation. The sum of these two quantities multiplied by $\frac{\pi q r^2}{d}$ gives the constant of the instrument; that is, the mutual inductance sought.

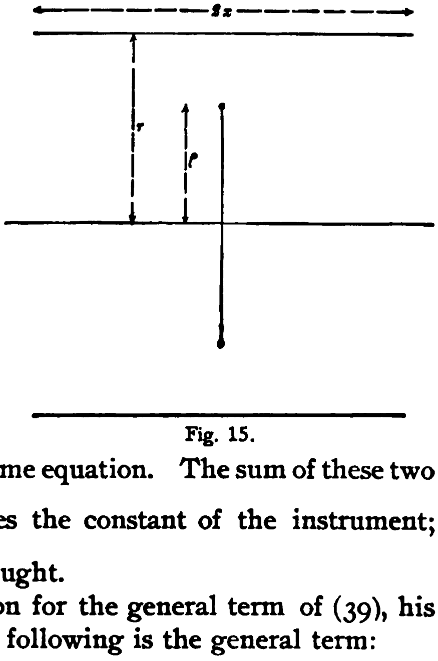

Fig. 15.

As Lorenz gave the expression for the general term of (39), his equation can be extended. The following is the general term:

$$Q(a)=2\pi\sum_{m=0}^{m=\infty} q^{2m+1}\frac{1.3\ldots.2m-1}{2.4\ldots..2m}\cdot\frac{1}{1.2\ldots.(m+1)}\cdot\frac{d^m}{da^m}\Big(\frac{a-1}{a}\Big)^{m+\frac{1}{2}}$$

JONES'S FORMULÆ.

Two solutions of the above problem were given by Jones,[34] both in terms of elliptic integrals. The current was considered to flow not in a current sheet, but along a spiral winding or helix. The first solution was in the form of a series, convergent only when O_1A, Fig. 16, is less than the difference in the radii of inner and outer coils; that is, when O_1A is less than $A-a$. As this is a serious limitation, and the formula is a laborious one to use, it is not here given. The second solution is exact and general, and is in terms of elliptic integrals of all three kinds. The second formula is as follows:

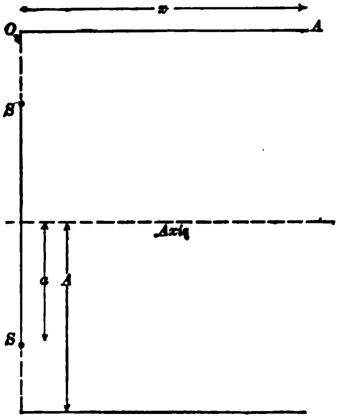

Fig. 16.

$$M_\theta = \Theta(A+a)ck\left\{\frac{F-E}{k^2}+\frac{c'^2}{c^2}(F-\Pi)\right\} \quad [40]$$

M_θ = mutual inductance of helix O_1A, Fig. 16, with respect to the disk S in the plane of one end.

$\Theta = 2\pi n$, $1/n$ = pitch of winding, Θ = whole angle of winding.

F, E, and Π are the complete elliptic integrals to modulus k, where

$$k^2=\frac{4Aa}{(A+a)^2+x^2}=\sin^2\gamma, \quad c^2=\frac{4Aa}{(A+a)^2}, \quad c'^2=1-c^2.$$

Π, the complete elliptic integral of the third kind, can be expressed in terms of incomplete integrals of the first and second kinds, and the value of M_θ can then be calculated by the help of Legendre's tables; see example 27. The calculation is, however, extremely tedious, especially when the value is to be determined with high precision.

Campbell has given Jones's formula (40) a slightly different form,[35] somewhat more convenient in calculation, as follows:

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[34] J. V. Jones, Proc. Roy. Soc., **68**, p. 198; 1898. Also, Trans. Roy. Soc., **182**, A; 1891. Jones's first formula was given in Phil. Mag., **27**, p. 61; 1889.

[35] A. Campbell, Proc. Roy. Soc., A, **79**, p. 428; 1907. There is a misprint in the formula as given in Campbell's paper. It was, however, used correctly in the numerical calculations given in the paper.

$$M=2\pi n_1 n_2 (A+a)\left\{\frac{c}{k}(F-E)+\frac{A-a}{b}\psi\right\} \qquad [41]$$

where n_1 is the same as n above, the number of turns per cm on the solenoid, n_2 is the number of turns in the secondary coil (in the above case it was taken as one), A is the greater and a the less of the two radii (in the above case A was the radius of the solenoid and a of the circle within), and

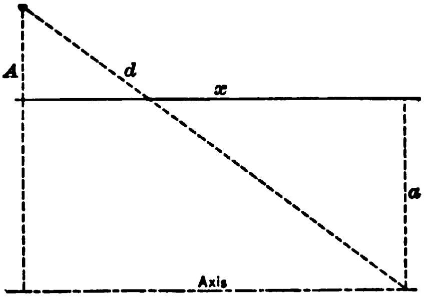

Fig. 17.

$$\psi=F(k)E(k',\beta)-$$

$$[F(k)-E(k)]F(k',\beta)-\frac{\pi}{2}$$

where $F(k)$ and $E(k)$ are the complete elliptic integrals to modulus k, and $F(k',\beta)$ and $E(k',\beta)$ are the incomplete elliptic integrals to modulus k' and amplitude β; $k'=\cos\gamma$, $\beta=c'/k'$; k, c, and c' are given above. If a secondary circle or coil has a radius greater than that of the solenoid, the same formula can be used if A is taken for the radius of the larger secondary and a is the radius of the solenoid.

ROSA'S FORMULA.[36]

The following formula gives the mutual inductance of a single layer coil of length x and a coaxial circle of radius a in the plane of one end of the coil, as shown in Fig. 16. It is the same quantity represented by M of equations (39) and (41) and M_0 of (40).

$$M_{0A}=\frac{2\pi^2a^2N}{d}\left[1+\frac{3}{8}\frac{a^2A^2}{d^4}+\frac{5}{64}\frac{a^4A^4}{d^8}X_2+\frac{35}{512}\frac{a^6A^6}{d^{12}}X_4+\frac{63}{1024}\frac{a^8A^8}{d^{16}}X_6\right.$$

$$\left.+\frac{231}{4096}\frac{a^{10}A^{10}}{d^{20}}X_8+\frac{429}{16384}\frac{a^{12}A^{12}}{d^{24}}X_{10}+\ \ldots\right] \qquad [42]$$

$$X_2=3-4\frac{x^2}{A^2}$$

$$X_4=\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4}$$

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[36] This Bulletin, 3, p. 209; 1907.

$$X_6=\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6}$$

$$X_8=\frac{63}{32}-\frac{105}{4}\frac{x^2}{A^2}+63\frac{x^4}{A^4}-36\frac{x^6}{A^6}+4\frac{x^8}{A^8}$$

$$X_{10}=\frac{231}{128}-\frac{1155}{32}\frac{x^2}{A^2}+\frac{1155}{8}\frac{x^4}{A^4}-165\frac{x^6}{A^6}+55\frac{x^8}{A^8}-4\frac{x^{10}}{A^{10}}$$

a = radius of disk or circle S, Fig. 2.

A = radius of the solenoid.

x = length O_1A of one end of the solenoid.

$d=\sqrt{x^2+A^2}$ = half the diagonal of the solenoid.

N is the whole number of turns of wire in the length x.

This formula is very easy to use in numerical calculation, notwithstanding it looks somewhat elaborate. The logarithm of $\frac{a^2A^2}{d^4}$, multiplied by 2, 3, 4, etc., gives the logarithm of the corresponding factor in each of the other terms. Similarly, the various terms X_2, X_4 etc., contain only powers of $\frac{x^2}{A^2}$ besides the numerical coefficients. It is hence a far simpler matter to compute M with high precision by this formula than by Jones's formula, the latter containing as it does elliptic integrals of all three kinds and involving the tedius interpolations for incomplete elliptic integrals.

If the secondary circle has a larger radius than the solenoid, A will be the radius of the circle and a the radius of solenoid. In every case A is the greater and a the less of the two radii, and d is $\sqrt{A^2+x^2}$.

Equation (42) may be written

$$M=\frac{2\pi^2a^2n_1x}{d}S$$

where n_1 is the number of turns of wire per cm, x is the length of the coil, Fig. 16, and S is the value of the quantity in brackets in (42), which is always somewhat greater than unity. This may also be put as follows:

$$M=a^2n_1\left(\frac{2\pi^2x}{d}\right)S=a^2n_1RS$$

or, [43]

$$M=a^2n_1K$$

The quantity R depends on x/d; that is, only upon the shape of the solenoid. S depends upon x/A, a/A, and A/d; that is, upon the relative sizes of the inner circle and the solenoid and the shape of the solenoid. If we have the value of RS, or K of equation (43) for a given solenoid and circle, we can get M by multiplying by a^2n_1, and for any other system of similar shape but different size by multiplying the same value of K by a^2n_1. The values of the constant K for various values of a/A and x/A are given in Table III, page 113.

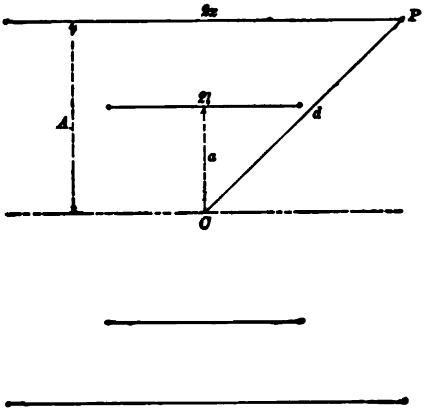

Fig. 18.

If the disk or circle be in the center of a solenoid of length $2x$ (Fig. 18), the value of M is of course double that given by using x. If it be not quite in the center, the value of M must be calculated for each end separately.

For illustrations and tests of the above formulæ see examples 25, 26, and 27, page 83.

5. THE SELF-INDUCTANCE OF A CIRCULAR RING OF CIRCULAR SECTION.

KIRCHHOFF'S FORMULA.

The formula for the self-inductance of a circle was first given by Kirchhoff[37] in the following form:

$$L = 2l\left\{\log\frac{l}{\rho} - 1.508\right\} \qquad [44]$$

where l is the circumference of the circular conductor and ρ is the radius of its cross section. This is equivalent to the following:

$$L = 4\pi a\left\{\log\frac{8a}{\rho} - 1.75\right\} \qquad [45]$$

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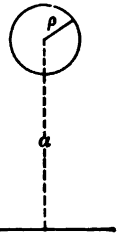
[37] Pogg. Annalen, **121**, p. 551; 1864.

a being the radius of the circle, Fig. 19. These formulæ are approximate, being more nearly correct as the ratio ρ/a is smaller.

MAXWELL'S FORMULA.

A more accurate expression, obtained by means of Maxwell's principle of the geometrical mean distance, is the following:



$$L=4\pi a\left\{\left(1+\frac{3}{16}\frac{R^2}{a^2}\right)\log\frac{8a}{R}-\left(2+\frac{R^2}{16a^2}\right)\right\} \qquad [46]$$

Substituting in this equation the value of the geometrical mean distance for a circular area, $R=\rho e^{-\frac{1}{4}}=.7788\rho$, we obtain[38]

$$L=4\pi a\left\{\left(1+0.1137\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.0095\frac{\rho^2}{a^2}-1.75\right\} \qquad [47]$$

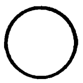


Fig. 19.

This is a very accurate formula for circles in which the radius of section ρ is very small in comparison with the radius a of the circle. The geometrical mean distance R has, however, been computed on the supposition of a linear conductor, and can only be applied to circles of relatively small value of ρ/a, and the square of the geometrical mean distance is used for the arithmetical mean square distance in the second order terms. We must therefore expect an appreciable error in formula (47) when the ratio ρ/a is not very small. Formulæ (44), (45), and (47) have been deduced on the supposition of a uniform distribution of the current over the cross section of the ring.

If the ring is a hollow circular thin tube, or if the current in the ring is alternating and of extremely high frequency, so that it can be regarded as flowing on the surface of the ring, the geometrical mean distance for the section would be the radius ρ, and we should have instead of (47) the following by substituting $R=\rho$,

$$L=4\pi a\left\{\left(1+\frac{3}{16}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-\frac{\rho^2}{16a^2}-2\right\} \qquad [48]$$

---

[38] Wied. Annalen, **58**, p. 928; 1894.

In the case of solid rings carrying alternating currents of moderate frequency the value of L would be somewhere between the values given by (47) and (48).

WIEN'S FORMULÆ.

Max Wien[38] has given the most accurate formula for the self-inductance of a circle, as follows:

$$L=4\pi a\left\{\left(1+\frac{1}{8}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.0083\frac{\rho^2}{a^2}-1.75\right\} \qquad [49]$$

It will be noticed that the formula differs very slightly from (47). Neglecting the terms in ρ^2/a^2 we obtain from either (47) or (49) Kirchhoff's approximate formula.

If the current be not distributed uniformly over the section of the wire, but the current density at any point is proportional to the distance from the axis of the ring, Wien's formula for the self-inductance is

$$L=4\pi a\left\{\left(1+\frac{3}{8}\frac{\rho^2}{a^2}\right)\log\frac{8a}{\rho}-.092\frac{\rho^2}{a^2}-1.75\right\} \qquad [50]$$

which differs very slightly from (49).

This would apply to the case of a ring revolving about a diameter in a uniform magnetic field.

As would be expected, (50) gives a greater value than (49).

RAYLEIGH AND NIVEN'S FORMULA.

Rayleigh and Niven gave[39] the following formula for a circular coil of n turns and of circular section:[40]

$$L=4\pi n^2 a\left\{\left(1+\frac{\rho^2}{8a^2}\right)\log\frac{8a}{\rho}+\frac{\rho^2}{24a^2}-1.75\right\} \qquad [51]$$

When $n=1$, this will be the self-inductance of a single circular ring. It agrees with Wien's, except as to one term, which is

$$+\frac{\rho^2}{24a^2}\text{ instead of }-0.0083\frac{\rho^2}{a^2}.$$

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[38] Wied. Annalen, **58**, p. 928; 1894.

[39] Rayleigh's Collected Papers, Vol. II, p. 15.

[40] Neglecting the correction for effect of insulation and shape of section of the separate wires.

If used for a coil of more than one turn, the expression for L (whether obtained from (51) or from one of the preceding more accurate expressions) must be corrected for the space occupied by the insulation between the wires and for the shape of the section.[41]

J. J. THOMSON'S FORMULA FOR RING OF ELLIPTICAL SECTION.

If the circular ring has an elliptical section the approximate formula for its self-inductance (corresponding to (45) for a circular section) is[42]

$$L = 4\pi a \left\{ \log \frac{16a}{\alpha+\beta} - 1.75 \right\} \qquad [52]$$

where α and β are the semi-axes of the ellipse, and a is the mean radius of the circular ring.

The formulæ of Minchin,[43] Hicks,[44] and Blathy[45] we have elsewhere[46] shown to be incorrect, and hence they are not here given.

6. THE SELF-INDUCTANCE OF A SINGLE LAYER COIL OR SOLENOID.

The following approximate formula for the self-inductance of a long solenoid is often given:

$$L = 4\pi^2 a^2 n_1^2 l \qquad [53]$$

where a is the mean radius, n_1 is the number of turns of wire per cm, and l is the length, supposed great in comparison with a. There is a considerable error in this formula, due to the end effect, but the variations in L due to changes in l are almost exactly proportional to the changes in l, and hence this formula may be used for calculating the corresponding variations in L.

RAYLEIGH AND NIVEN'S FORMULÆ.

The following formula[47] for the self-inductance of a single layer winding on a solenoid is very accurate when the length b is small compared with the radius a, Fig. 20:

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[41] See Rosa, this Bulletin, **3**, p. 1; 1907.

[42] J. J. Thomson, Phil. Mag., **23**, p. 384; 1886.

[43] Phil. Mag., **37**, p. 300; 1894.

[44] Phil. Mag., **38**, p. 456; 1894.

[45] London Electrician, **24**, p. 630; April 25, 1890.

[46] This Bulletin, **4**, p. 149; 1907.

[47] Proc. Roy. Soc., **32**, pp. 104–141; 1881. Rayleigh's Collected Papers, **2**, p. 15.

$$L_s = 4\pi a n^2\left\{\log\frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2}\left(\log\frac{8a}{b} + \frac{1}{4}\right)\right\} \quad [54]$$

n is the whole number of turns of wire on the coil, and the radius is measured to the center of the wire. The length b is the *mean over-all length including the insulation on the first and last wires* if the coil is wound closely with insulated wire. See also page 41.

The self-inductance L_s is, however, not the actual self-inductance of the coil, but the current sheet value; that is, it is the value of the self-inductance if the winding were of infinitely thin tape, so that the current would cover the entire length b. To get the actual self-inductance L for any given case one must correct L_s by formula (59) below. The same remark applies to all the formulæ in this section for L_s. The approximate formula (53) is too rough to make it worth while to apply such a correction.

For a coil in which the axial dimension b is zero and the radial depth is c, the following current sheet formula of Rayleigh and Niven gives the self-inductance:

$$L_s = 4\pi a n^2\left\{\log\frac{8a}{c} - \frac{1}{2} + \frac{c^2}{96a^2}\left(\log\frac{8a}{c} + \frac{43}{12}\right)\right\} \quad [55]$$

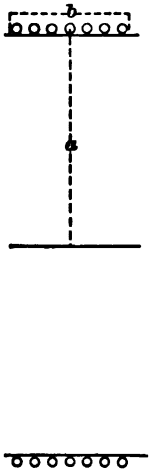

Fig. 20.

This is not an important case in practice.

Formulæ (54) and (55) may be obtained from (67) by making first $c=0$ and then $b=0$.

COFFIN'S FORMULA.

Coffin[48] has extended formula (54) so that it is very accurate for coils of length as great as the radius, and sufficiently accurate for most purposes for coils considerably longer than this.

$$L_s = 4\pi a n^2\left\{\log\frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2}\left(\log\frac{8a}{b} + \frac{1}{4}\right) - \frac{1}{1024}\frac{b^4}{a^4}\left(\log\frac{8a}{b} - \frac{2}{3}\right)\right.$$
$$\left. + \frac{10}{131072}\frac{b^6}{a^6}\left(\log\frac{8a}{b} - \frac{109}{120}\right) - \frac{35}{4194304}\frac{b^8}{a^8}\left(\log\frac{8a}{b} - \frac{431}{420}\right)\right\} \quad [56]$$

---

[48] This Bulletin, **2**, p. 113; 1906.

LORENZ'S FORMULA.

Lorenz first gave[49] an exact formula for the self-inductance of a single layer solenoid. It is, like the others, a current sheet formula, and requires correction by (59) for a winding of wire, but applies to a solenoid of any length. Changing the notation slightly Lorenz's formula as originally given is as follows:

$$L_s = \frac{32}{3}\frac{\pi n^2 a^3}{b^2}\left\{\frac{2k^2-1}{k^3}E + \frac{1-k^2}{k^3}F - 1\right\} \qquad [57]$$

where $k = \frac{4a^2}{4a^2+b^2}$ and F and E are complete elliptic integrals of the first and second kind to modulus k, and a, b, and n are the radius, length, and whole number of turns of wire, respectively. By simple substitutions the formula may be put into the following form, where d is the diagonal of the solenoid $= \sqrt{4a^2+b^2}$;

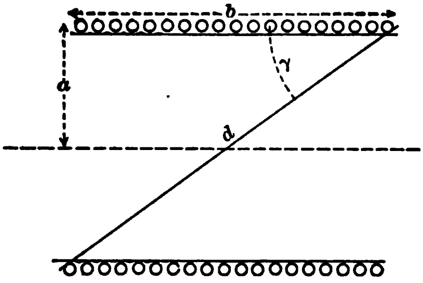

Fig. 21.

$$L_s = \frac{4\pi n^2}{3b^2}\left\{d\,(4a^2-b^2)E + db^2F - 8a^3\right\} \qquad [58]$$

Coffin derived[50] an expression for L in elliptic integrals which is equivalent to (58), and also obtained (58) from an expression[51] attributed to Kirchhoff.

Formula (58) may be written

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[49] Wied. Annal., 7, p. 161; 1879. Oeuvres Scientifiques de L. Lorenz, Tome 2, 1, p. 196.

[50] This Bulletin, 2, p. 123, equation (31).

[51] This Bulletin, 2, p. 127, equation (36). The notation is slightly different.

$$L_s = an^2\left[\frac{8\pi}{3}\left\{\sqrt{1+\frac{b^2}{4a^2}}\left(\frac{4a^2}{b^2}-1\right)E+\sqrt{1+\frac{b^2}{4a^2}}\,F-\frac{4a^2}{b^2}\right\}\right]$$

or $L_s = an^2Q$ [58]a

where a is the radius of the solenoid, n is the whole number of turns on the coil, and Q is the function of $\frac{2a}{b}$ ($=\tan\gamma$) contained in the square brackets. We have calculated Q for various values of $\tan\gamma$ from 0.2 to 4.0 and given them in Table IV, p. 114. This table will be found useful in calculating L_s for solenoids when $\tan\gamma$ has one of the values given in the table, as all calculation of elliptic integrals is avoided. In problems where the length and diameter can be chosen at will, as in the designing of apparatus, this method of calculating L will be most frequently useful. The values of the constant Q given in the table have been computed with great care, so that they give very accurate values of L_s, for long as well as short solenoids.

In calculating the value of L_s by means of formula (54), (56), (58), or (58a) one should use for the length b the *over-all length including the insulation* (A B, Fig. 22, and not a b) for a close winding of insulated wire, or *n times the pitch* for a uniform winding of bare or covered wire, which is, of course, the same as the length from center to center of $n+1$ turns. The radius a is the mean radius to the center of the wire. The same method of taking the breadth and depth b and c applies in the formulæ of section 7. See also remarks under example 24.

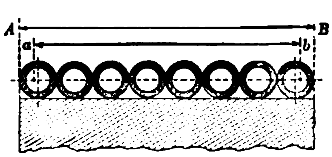

Fig. 22.

ROSA'S CORRECTION FORMULA.

Rosa has shown[52] that the above formulæ (54 to 58) apply accurately only to a winding of infinitely thin strip which completely covers the solenoid (the successive turns being supposed to meet at the edges without making electrical contact) and so realizing the uniform distribution of current over the cylindrical surface which has been assumed in the derivation of all the formulæ. A winding of insulated wire or of bare wire in a screw thread may have a greater or less self-inductance than that given by the current sheet formulæ above according to the ratio of the diameter of the wire to the pitch of the winding. Putting L for the actual self-inductance of a winding and L_s for the current sheet value given by one of the above formulæ,

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[52] This Bulletin, **2**, pp. 161–187; 1906.

$$L = L_s - \Delta L$$

The correction ΔL is given by the following expression:

$$\Delta L = 4\pi a n \ [A + B] \qquad [59]$$

where as above a is the radius, n the whole number of turns of wire and A and B are constants given in Tables VII and VIII, pp. 116 and 117.

The correction term A depends on the size of the (bare) wire (of diameter d) as compared with the pitch D of the winding; that is, on the value of the ratio d/D. For values of d/D less than 0.58, A is negative, and in such cases when the numerical values of A are greater than the value of B, which is always positive, the correction ΔL will be negative, and hence L will be *greater* than L_s. See examples 32 and 33.

THE SUMMATION FORMULA FOR L.[53]

If we have a single layer winding on a cylinder the self-inductance is equal to the sum of the self-inductances of the separate turns plus the sum of the mutual inductances of each wire on all the others. Thus if there are n turns

$$L = nL_1 + 2(n-1)M_{12} + 2(n-2)M_{13} + 2(n-3)M_{14} + \ . \ . \ . \ . \ 2M_{1n} \qquad [60]$$

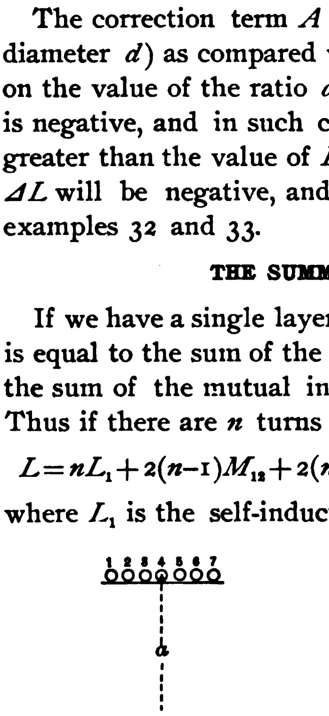

Fig. 23.

where L_1 is the self-inductance of a single turn, M_{12} is the mutual inductance of the first and second turns or any two adjacent turns, M_{13} is the mutual inductance of the first and third or of any two turns separated by one, etc., and M_{1n} is the mutual inductance of the first and last turns. For a coil of four turns this becomes

$$L = 4L_1 + 6M_{12} + 4M_{13} + 2M_{14}$$

L_1 should be calculated by formula (49) or any formula for circles, and M_{12}, etc., by (9) or (10). When the number of turns on the coil is small formula (60) is very convenient, and gives very accurate results.

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[53] Kirchhoff, Gesammelte Abhandlungen, p. 177.

STRASSER'S FORMULA.

Strasser[54] has derived a formula for the self-inductance of a single layer coil of few turns from (60) by substituting for L_1 its value as given by formula (45) and for the various M's their values as given by (9). Strasser's formula with slight correction and some changes in notation is as follows:[55]

$$L=4\pi a\left[n\left(\log\frac{8a}{\rho}-1.75\right)+n(n-1)\left(\log\frac{8a}{d}-2\right)-A\right.$$
$$\left.+\frac{d^2}{8a^2}\left\{\left(3\log\frac{8a}{d}-1\right)\left(\frac{n^2(n^2-1)}{12}\right)-B\right\}\right] \qquad [61]$$

where n is the whole number of turns, d is the pitch, or distance between the centers of two adjacent turns, a is the mean radius of the coil, ρ is the radius of the section of the wire, and A and B are constants given by Table V, page 115, for values of n up to 30. For coils of a larger number of turns (or indeed any number of turns) the value of L can be accurately calculated by (69) and (72) or by (58) and (59).

SELF-INDUCTANCE OF TOROIDAL COIL OF RECTANGULAR SECTION.

The first approximation to the self-inductance of a toroidal coil (that is, a circular solenoid) of rectangular section, wound with a single layer of n turns of wire is

$$L_s=2n^2h\log\frac{r_2}{r_1} \qquad [62]$$

where h is the axial depth of the coil, and r_1 and r_2 are the inner and outer radii of the ring, Fig. 24. Formula (62) is exact for a toroidal core enveloped by a current sheet, or for a winding of n turns of infinitely thin tape covering the core completely, the core within the current sheet being h cm in axial height and (r_2-r_1) cm in radial breadth.

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[54] Wied. Annal., **17**, p. 763; 1905.

[55] Strasser uses the formula for L as: $L=4\pi a\left(\log\frac{a}{\rho}+0.333\right)$. This is not quite correct. It should be

$$L_1=4\pi a\left(\log\frac{8a}{\rho}-1.75\right)=4\pi a\log\left(\frac{a}{\rho}-1.75+\log_e 8\right)=4\pi a\left(\log\frac{a}{\rho}+0.32944\right).$$

When the core is wound with round insulated wire, the self-inductance is affected by those lines of force within the cross section of the wire itself, and by those linked with each separate turn of wire in addition to those running through the core. Rosa has shown[56] that the total self-inductance may be more or less than the current sheet value given by (62) according to the size of the wire and the pitch of the winding. In every case, however, the correct value of the self-inductance is derived from the current sheet value L_s by subtracting a correction term ΔL, which is equal to twice the length of the wire multiplied by the sum of two quantities A and B. Thus

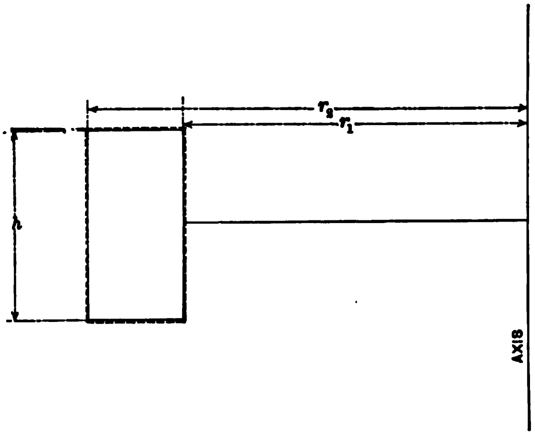

Fig. 24.

$$L = L_s - 2nl(A+B) \qquad [63]$$

where n is the whole number of turns in the winding, l is the length of one turn, A is a quantity, depending on the diameter of the wire and the pitch of the winding, given in Table VII, and B is 0.332. When A is negative and greater than B, L is greater than L_s. This occurs when the pitch of the winding is more than 2.5 times the diameter of the (uncovered) wire.

Fröhlich's formula[57] based on the assumption that a winding of round wires is equivalent to a thick current sheet has been shown to be incorrect.[58]

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[56] This Bulletin, **4**, p. 141; 1907.

[57] Wied. Annal., **68**, p. 142; 1897.

[58] This Bulletin, **4**, p. 141; 1907.

7. THE SELF-INDUCTANCE OF A CIRCULAR COIL OF RECTANGULAR SECTION.

MAXWELL'S APPROXIMATE FORMULA.

Maxwell first gave[59] an approximate formula for the important case of a circular coil or conductor of rectangular section, Fig. 25, as follows:

$$L=4\pi a n^2\left(\log\frac{8a}{R}-2\right) \qquad [64]$$

where R is the geometrical mean distance of the cross section of the coil or conductor. The current is supposed uniformly distributed over this section.

The value of R for any given shape of rectangular section is given by (103). Its value for several particular cases is given in the table of page 60. It is very nearly proportional to the perimeter of the rectangle and approximately equal to 0.2235 $(\alpha+\beta)$ where α and β are the length and breadth of the rectangle.

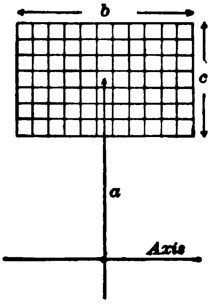

Fig. 25.

Formula (64) is derived from (8) by putting R, the geometrical mean distance of the area of the section of the coil from itself, in place of r, the distance between two circles. If we use (9) instead of (8) for this purpose, we shall have a closer approximation to the value of L. Thus,

$$L=4\pi a n^2\left\{\log\frac{8a}{R}\cdot\left(1+\frac{3R^2}{16a^2}\right)-\left(2+\frac{R^2}{16a^2}\right)\right\} \qquad [65]$$

We have placed R^2 in place of d^2 in the second order terms, which is of course not strictly correct, as we should use an arithmetical mean square distance instead of a geometrical mean square distance. (See p. 63.) Nevertheless, (65) is a much closer approximation than (64).

PERRY'S APPROXIMATE FORMULA.

Professor Perry has given[60] the following empirical expression for the self-inductance of a short circular coil of rectangular section:

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[59] Elect. and Mag., § 706.

[60] John Perry, Phil. Mag., **30**, p. 223; 1890.

$$L=\frac{4\pi n^2a^2}{0.2317a+0.44b+0.39c} \qquad [66]$$

in which n is the whole number of turns of wire, a the mean radius, b the axial breadth, c the radial depth. As in all the formulæ of this paper, the dimensions are in centimeters and the value of L is in centimeters. This formula gives a good approximation to L as long as b and c are small compared with a.

WEINSTEIN'S FORMULA.

Maxwell's more accurate expression for the self-inductance of a circular coil of rectangular section[61] was not quite correct. The investigation was repeated by Weinstein,[62] who gave the following formula:

$$L_u=4\pi an^2\,(\lambda+\mu)$$

where

$$\left.\begin{aligned}\lambda=&\log\frac{8a}{c}+\frac{1}{12}-\frac{\pi x}{3}-\frac{1}{2}\log\,(1+x^2)+\frac{1}{12x^2}\log\,(1+x^2)\\&+\frac{1}{12}x^2\log(1+\frac{1}{x^2})+\frac{2}{3}(x-\frac{1}{x})\tan^{-1}x,\\\mu=&\frac{c^2}{96a^2}\left[\left(\log\frac{8a}{c}-\frac{1}{2}\log\,(1+x^2)\right)(1+3x^2)+3.45x^2+\frac{221}{60}\right.\\&\left.-1.6\pi x^3+3.2x^3\tan^{-1}x-\frac{1}{10}\frac{1}{x^2}\log(1+x^2)+\frac{1}{2}x^4\log(1+\frac{1}{x^2})\right]\end{aligned}\right\} \quad [67]$$

b and c are the breadth and depth of the coil and $x=\frac{b}{c}$.

Weinstein's formula for the case of a square section, where $b=c$, reduces to the following simpler expression:

$$L_u=4\pi an^2\left\{\left(1+\frac{c^2}{24a^2}\right)\log\frac{8a}{c}+.03657\frac{c^2}{a^2}-1.194914\right\} \qquad [68]$$

This is a very accurate formula as long as c/a is a small quantity. The current is supposed distributed uniformly over the section of the coil, and hence for a winding of round insulated wire correction must be made by formula (72).

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[61] Phil. Trans., 1865, and collected works.

[62] Wied. Annal., **21**, p. 329; 1884.

STEFAN'S FORMULA.

Stefan[63] simplified Weinstein's expression (67) by collecting together terms depending on the ratio of b to c and computing two short tables of constants y_1 and y_2. His formula is as follows:

$$L=4\pi a n^2\left\{\left(1+\frac{3b^2+c^2}{96a^2}\right)\log\frac{8a}{\sqrt{b^2+c^2}}-y_1+\frac{b^2}{16a^2}y_2\right\} \qquad [69]$$

The values of y_1 and y_2 are given in Table VI, page 115, as functions of $x=b/c$ or c/b; that is, x is the ratio of the breadth to the depth of the section, or vice versa, being always less than unity.

For the method of taking the dimensions b and c of the cross section see p. 99, section 6. Also example 40, p. 99.

LONG COIL OF RECTANGULAR SECTION; I. E., SOLENOID OF MORE THAN ONE LAYER.

ROSA'S METHOD.

When the coil is so long that the formula of Stefan is no longer accurate, the self-inductance may be accurately calculated by a method given by Rosa.[64]

In Figs. 26, 27, and 28 are shown three coils, having the same length and mean radius. The first is a single winding of thin tape and the self-inductance, calculated by a current sheet formula, is L_s. The second is a single layer of wire of square section (length b, depth c, and b/c turns) and its self-inductance is L_u, the current being supposed uniformly distributed over the area of the square conductors. The third is a winding of round insulated wire of length b, depth c, and any number of layers, and its self-inductance is L. These different self-inductances are related as follows:

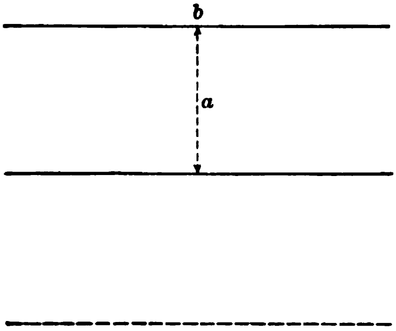

Fig. 26.

$$L_s-\Delta_1 L=L_u$$
$$L_u+\Delta_2 L=L$$
$$\therefore\ L=L_s-\Delta_1 L+\Delta_2 L$$

L_s is calculated by any current sheet formula as (54), (56), (57), or

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[63] Wied. Annal., **22**, p. 113; 1884.

[64] This Bulletin, **4**, p. 369; 1907.

(58). The correction $\Delta_1 L$ for the depth of the coil is given by the following formula:

$$\Delta_1 L = 4\pi a n' \; [A_s + B_s] \qquad [70]$$

This formula has the same form as (59), but some of the quantities have a different meaning; a is the mean radius as before, n' is b/c, the number of square conductors in the length b, Fig. 26, and A_s and B_s are given in Tables IX and X.

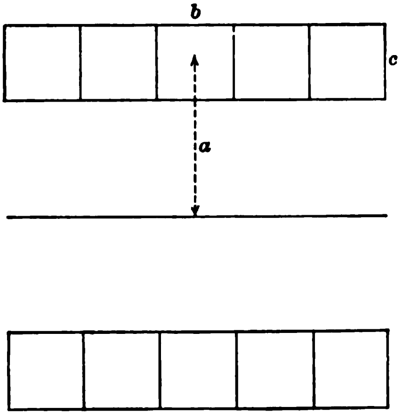

Fig. 27.

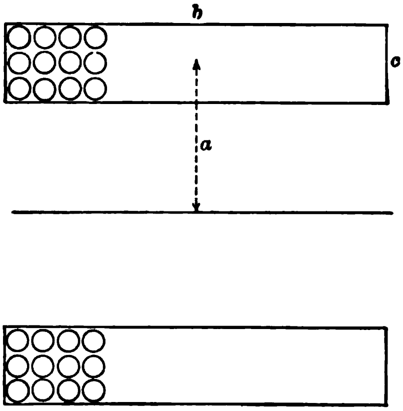

Fig. 28.

The correction $\Delta_2 L$ is calculated in precisely the same way as for a short coil, as described below, formula (72). The above formula for $\Delta_1 L$ gives a very accurate value of the correction to be applied to L_s to obtain L_u, and permits a test to be made for the error of Stefan's formula when applied to longer coils than the latter is intended for. Such a calculation shows that for a coil as long as its diameter Stefan's formula (and Weinstein's also, of course) is 1 per cent in error, giving too large a value.

COHEN'S APPROXIMATE FORMULA.

Cohen has given the following approximate formula[65] for the self-inductance of a long coil or solenoid of several layers:

$$L = 4\pi^2 n^2 m \left\{ \frac{2a_0^4 + a_0^2 l^2}{\sqrt{4a_0^2 + l^2}} - \frac{8a_0^3}{3\pi} \right\} + 8\pi^2 n^2 \Big\{ [(m-1)a_1^2 + (m-2)a_2^2 + \ldots] \left(\sqrt{a_1^2 + l^2} - \frac{7}{8}a_1 \right) + \frac{1}{2}[m(m-1)a_1^2 + \ldots] \left(\frac{a_1 \delta a}{\sqrt{a_1^2 + l^2}} - \delta a \right) \Big\} \qquad [71]$$

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[65] This Bulletin, 4, p. 389; 1907.

where a_0 is the mean radius of the solenoid, a_1, a_2, . . . a_m are the mean radii of the various layers, m is the number of layers and δa is the distance between centers for any two consecutive layers.

For long solenoids, where the length is, say, four times the diameter, we can neglect the last term in equation (71).

This formula is sufficiently accurate for most purposes; it will give results accurate to within one half of one per cent even for short solenoids, where the length is only twice the diameter.

MAXWELL'S CORRECTION FORMULA.[66]

GIVING THE VALUE OF $\Delta_2 L$.

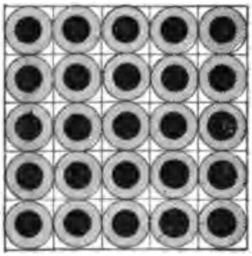

Fig. 29.

Maxwell has shown that when a coil of rectangular section (Fig. 28) is wound with round insulated wire and the self-inductance is calculated by a formula in which the current is assumed to be distributed uniformly over the section, as in Weinstein's and Stefan's, the calculated value L_u is subject to three corrections, each of which tends to increase the calculated value of the self-inductance. Thus:

$$L = L_u + \Delta_2 L$$

and

$$\Delta_2 L = 4\pi a n \left\{ \log_e \frac{D}{d} + 0.13806 + E \right\} \quad [72]$$

Maxwell showed that the first term takes account of the effect of the insulation, d and D being the diameters of the bare and covered wire respectively. The second correction term (0.13806) reduces from a square section to a circular section for the conductor. The third correction term E takes account of the differences in the mutual inductances of the separate turns of wire on one another when the wire has a round section from what the mutual inductances would be if the wire were of square section and no space was occupied by insulation. This term was stated by Maxwell to be equal to $-.01971$; it was subsequently stated by Stefan to be equal to $+.01688$. Rosa has shown[67] that its value is variable, depending on the number of turns of wire in the coil and the shape of the cross section of the latter, and has given the values of E for a number of particular cases.

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[66] Elect. and Mag., vol. 2, § 693.

[67] This Bulletin, 8, p. 37.

From the following table one can interpolate for E for any particular case not included in the table.

Summary of the values of E found for the various cases considered:

2	turns		$E=$	.006528
3	"	(one layer)	$E=$	.009045
4	"	(two layers)	$E=$	.01691
4	"	(one layer)	$E=$	.01035
8	"	(two layers)	$E=$	.01335
10	"	(one layer)	$E=$	.01276
20	"	(one layer)	$E=$	.01357
16	"	(four layers)	$E=$	.01512
100	"	(ten layers)	$E=$	.01713
400	"	(20 × 20)	$E=$	.01764
1,000	"	(50 × 20)	$E=$	.01778
Infinite number of turns			$E=$	.01806

8. SELF AND MUTUAL INDUCTANCE OF LINEAR CONDUCTORS.[68]

SELF-INDUCTANCE OF A STRAIGHT CYLINDRICAL WIRE.

The self-inductance of a length l of straight cylindrical wire of radius ρ is

$$L=2\left[l\log\frac{l+\sqrt{l^2+\rho^2}}{\rho}-\sqrt{l^2+\rho^2}+\frac{l}{4}+\rho\right] \qquad [73]$$

$$=2l\left[\log\frac{2l}{\rho}-\frac{3}{4}\right] \text{ approximately.} \qquad [74]$$

Where the permeability of the wire is μ, and that of the medium outside is unity,

$$L=2l\left[\log\frac{2l}{\rho}-1+\frac{\mu}{4}\right] \qquad [75]$$

This formula was originally given by Neumann.

For a straight cylindrical tube of infinitesimal thickness, or for alternating currents of great frequency, when there is no magnetic field within the wire, the self-inductance is

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[68] See paper by E. B. Rosa, this Bulletin, 4, p. 301; 1907.

$$L = 2l\left[\log\frac{2l}{\rho} - 1\right] \qquad [76]$$

This is obtained by subtracting from (74) $l/2$ or from (75) $\mu\, l/2$, the magnetic flux within the conductor due to unit current.

THE MUTUAL INDUCTANCE OF TWO PARALLEL WIRES.

The mutual inductance of two parallel wires of length l, radius ρ, and distance apart d is the number of lines of force due to unit current in one which cut the other when the current disappears. This is

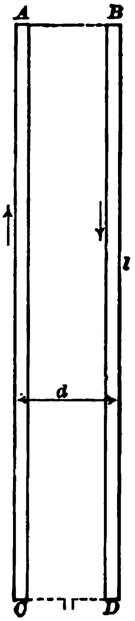

Fig. 30.

$$M = 2\left[l\log\frac{l+\sqrt{l^2+d^2}}{d} - \sqrt{l^2+d^2} + d\right] \qquad [77]$$

$$\therefore M = 2l\left[\log\frac{2l}{d} - 1 + \frac{d}{l}\right] \text{approximately} \qquad [78]$$

when the length l is great in comparison with d.

Equation (77), which is an exact expression when the wires have no appreciable cross section, is not an exact expression for the mutual inductance of two parallel cylindrical wires, but is not appreciably in error even when the section is large and d is small if l is great compared with d.

THE SELF-INDUCTANCE OF A RETURN CIRCUIT.

If we have a return circuit of two parallel wires each of length l (the current then flowing in opposite direction in the two wires) the self-inductance of the circuit, neglecting the effect of the end connections shown by dotted lines, Fig. 30, will be very approximately

$$L = 4l\left[\log\frac{d}{\rho} + \frac{\mu}{4} - \frac{d}{l}\right] \qquad [79]$$

In the usual case of $\mu = 1$ this will be, when d/l is small

$$L = 4l\left[\log\frac{d}{\rho} + \frac{1}{4}\right] \qquad [80]$$

If the end effect is large, as when the wires are relatively far apart, use the expression for the self-inductance of a rectangle below (86); or, better, add to the value of (79) the self-inductance of AB+CD, using equation (71) in which $l = 2$AB.

[This is equivalent to the following formula in which the logarithms are common:

$$L = 0.7411 \log_{10} \frac{d}{\rho} + .0805 \text{ in millihenrys per mile of conductor,}$$

$$= 0.4605 \log_{10} \frac{d}{\rho} + .050 \text{ in millihenrys per kilometer of conductor,}$$

d and ρ being expressed in centimeters, inches, or any other unit.]

MUTUAL INDUCTANCE OF TWO LINEAR CONDUCTORS IN THE SAME STRAIGHT LINE.

The mutual inductance of two adjacent linear conductors of lengths l and m in the same straight line is

$$M_{lm} = l \log \frac{l+m}{l} + m \log \frac{l+m}{m}, \text{ approximately.} \qquad [81]$$

This approximation is very close indeed if the radius of the conductor (which has been assumed zero) is very small.

THE SELF-INDUCTANCE OF A STRAIGHT RECTANGULAR BAR.

The self-inductance of a straight bar of rectangular section is, to within the accuracy of the approximate formula (75), the same as the mutual inductance of two parallel straight filaments of the same length separated by a distance equal to the geometrical mean distance of the cross section of the bar. Thus,

$$L = 2l\left[\log \frac{2l}{R} - 1 + \frac{R}{l}\right] \qquad [82]$$

where R is the geometrical mean distance of the cross section of the rod or bar. If the section is a square, $R = .447\ a$, a being the side of the square. If the section is a rectangle, the value of R is given by Maxwell's formula (103).

This is equivalent to the following:

$$L = 2l\left[\log \frac{2l}{\alpha+\beta} + \frac{1}{2} + \frac{0.2235(\alpha+\beta)}{l}\right] \qquad [83]$$

In the above formula L is the self-inductance of a straight bar or wire of length l and having a rectangular section of length α and breadth β.

TWO PARALLEL BARS. SELF AND MUTUAL INDUCTANCE.

The mutual inductance of two parallel straight, square, or rectangular bars is equal to the mutual inductance of two parallel wires or filaments of the same length and at a distance apart equal to the geometrical mean distance of the two areas from one another. This is very nearly equal in the case of square sections to the distance between their centers for all distances, the g. m. d. being a very little greater for parallel squares, and a very little less for diagonal squares[69] (Fig. 31). We should, therefore, use equation (78) with d equal to the g. m. d. of the sections from one another; that is, substantially, to the distances between the centers.

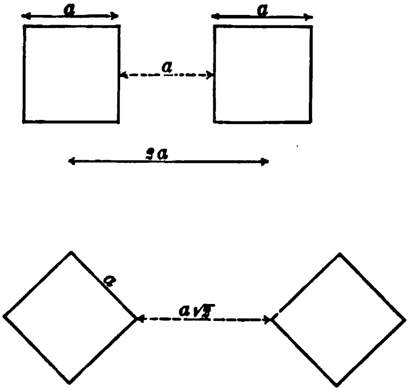

Fig. 31.

The self-inductance of a return circuit of two such parallel bars is equal to twice the self-inductance of one minus twice their mutual inductance. That is,

$$L = 2[L_1 - M]$$

in which L_1 is calculated by (83) and M by (78).

SELF-INDUCTANCE OF A SQUARE.

The self-inductance of a square may be derived from the expressions for the self and mutual inductance of finite straight wires from the consideration that the self-inductance of the square is the sum of the self-inductances of the four sides minus the mutual inductances. That is,

$$L = 4L_1 - 4M$$

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[69] Rosa, this Bulletin, 8, p. 1.

the mutual inductance of two mutually perpendicular sides being zero. Substituting a for l and d in formulæ (73) and (77) we have, neglecting ρ^2/a^2, $L = 8a\left(\log\frac{a}{\rho}+\frac{\rho}{a}-.524\right)$ [84]

where a is the length of one side of the square and ρ is the radius of the wire. If we put $l=4a=$ whole length of wire in the square,

$$L=2l\left(\log\frac{l}{\rho}+\frac{4\rho}{l}-1.910\right)$$

$$\text{or, } L=2l\left(\log\frac{l}{\rho}-1.910\right)\text{, approximately.} \qquad [85]$$

Formulæ (84) and (85) were first given by Kirchhoff[70] in 1864.

SELF-INDUCTANCE OF A RECTANGLE.

(*a*) *The conductor having a circular section.*

The self-inductance of the rectangle of length a and breadth b is

$$L=2(L_a+L_b-M_a-M_b)$$

where L_a and L_b are the self-inductances of the two sides of length a and b taken alone, M_a and M_b are the mutual inductances of the two opposite pairs of length a and b, respectively.

From (73) and (77) we therefore have, neglecting ρ^2/a^2, and putting d for the diagonal of the square $=\sqrt{a^2+b^2}$

$$L=4\left[(a+b)\log\frac{2ab}{\rho}-a\log(a+d)-b\log(b+d)\right.$$
$$\left.-\frac{7}{4}(a+b)+2(d+\rho)\right] \qquad [86]$$

(*b*) *The conductor having a rectangular section.*

For a rectangle made up of a conductor of rectangular section $\alpha\times\beta$,

$$L=4\left[(a+b)\log\frac{2ab}{\alpha+\beta}-a\log(a+d)-b\log(b+d)\right.$$
$$\left.-\frac{a+b}{2}+2d+0.447(\alpha+\beta)\right] \qquad [87]$$

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[70] Gesammelte Abhandlungen, p. 176. Pogg. Annal., **121**, 1864.

where as before d is the diagonal of the square. This is equivalent to Sumec's exact formula[71] (6a).

For $a=b$, a square,

$$L=8a\left[\log\frac{a}{\alpha+\beta}+0.2235\frac{\alpha+\beta}{a}+0.726\right] \qquad [88]$$

If $\alpha=\beta$, that is, the section of the conductor is a square,

$$L=8a\left[\log\frac{a}{\alpha}+.447\frac{\alpha}{a}+.033\right] \qquad [89]$$

MUTUAL INDUCTANCE OF TWO EQUAL PARALLEL RECTANGLES.

For two equal parallel rectangles of sides a and b and distance apart d the mutual inductance, which is the sum of the several mutual inductances of parallel sides, is,

$$\begin{aligned}M=4\Bigg[a\log\left(\frac{a+\sqrt{a^2+d^2}}{a+\sqrt{a^2+b^2+d^2}}\cdot\frac{\sqrt{b^2+d^2}}{d}\right)\\ +b\log\left(\frac{b+\sqrt{b^2+d^2}}{b+\sqrt{a^2+b^2+d^2}}\cdot\frac{\sqrt{a^2+d^2}}{d}\right)\Bigg]\\ +8\left[\sqrt{a^2+b^2+d^2}-\sqrt{a^2+d^2}-\sqrt{b^2+d^2}+d\right] \qquad [90]\end{aligned}$$

For a square, where $a=b$, we have

$$\begin{aligned}M=8\left[a\log\left(\frac{a+\sqrt{a^2+d^2}}{a+\sqrt{2a^2+d^2}}\cdot\frac{\sqrt{a^2+d^2}}{d}\right)\right]\\ +8\left[\sqrt{2a^2+d^2}-2\sqrt{a^2+d^2}+d\right] \qquad [91]\end{aligned}$$

Formula (90) was first given by F. E. Neumann[72] in 1845.

SELF AND MUTUAL INDUCTANCE OF THIN TAPES.

The self-inductance of a straight thin tape of length l and breadth b (and of negligible thickness) is equal to the mutual inductance of two parallel lines of distance apart R_1 equal to the geometrical mean distance of the section, which is 0.22313b, or $\log R=\log b-\frac{3}{2}$.

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[71] Elektrotech. Zs., p. 1175; 1906.

[72] Allgemeine Gesetze der Inducirten Ströme, Abh. Berlin Akad.

Thus we have approximately,

$$L=2l\left[\log\frac{2l}{R_1}-1\right]$$

$$=2l\left[\log\frac{2l}{b}+\frac{1}{2}\right] \qquad [92]$$

If the thickness of the tape is not negligible this formula becomes, when a is the thickness of the tape,

$$L=2l\left[\log\frac{2l}{b}-\frac{a}{b}+\frac{1}{2}\right] \qquad [93]$$

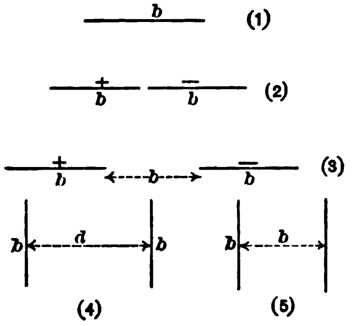

Fig. 32.

A closer approximation to L is given by (83) in which a is the thickness and β is the breadth of the tape. For two such tapes in the same plane, coming together at their edges without making electrical contact, the mutual inductance is

$$\left.\begin{aligned}M&=2l\left[\log\frac{2l}{R_2}-1\right]\\&=2l\left[\log\frac{2l}{b}-0.8863\right]\end{aligned}\right\} \qquad [94]$$

where R_2 is the geometrical mean distance of one tape from the other, which in this case is $0.89252b$. For a return circuit made up of these two tapes the self-inductance is

$$L=2L_1-2M$$

$$=4l\left(\log\frac{R_2}{R_1}\right)=4l\log_e 4 \qquad [95]$$

$$=5.545\times \textit{length of one tape.}$$

Thus the self-inductance of such a circuit is independent of the width of the tapes. If the tapes are separated by the distance b equal to the width of the tapes, $R_2=1.95653$ and $L=8.685\,l$.

If the two tapes are not in the same plane but parallel,

$$L=2L_1-2M=4l\log\frac{R_2}{R_1} \qquad [96]$$

and when the distance apart is equal to the breadth of the tapes we have

$$\log \frac{R_2}{R_1} = \frac{\pi}{2}$$

and

$$L = 4l\frac{\pi}{2} = 2\pi\, l \tag{97}$$

In this case also the self-inductance [2π cm per unit of length] of the pair of thin strips is independent of their width so long as the distance apart is equal to their width. Formula (96) may be employed to calculate the self-inductance of a non-inductive shunt made up of a sheet of thin metal doubled on itself.

CONCENTRIC CONDUCTORS.

The self-inductance of a thin, straight tube of length l and radius a_2, when a_2/l is very small, is given by (76),

$$L_2 = 2l\left[\log \frac{2l}{a_2} - 1\right]$$

The mutual inductance of such a tube on a conductor within it is equal to its self-inductance, since all the lines of force due to the outer tube cut through the inner when they collapse on the cessation of current. The self-inductance of the inner conductor, suppose a solid cylinder, is

$$L_1 = 2l\left[\log \frac{2l}{a_1} - \frac{3}{4}\right]$$

If the current goes through the latter and returns through the outer tube, the self-inductance of the circuit is

$$L = L_1 + L_2 - 2M = L_1 - L_2$$

since M equals L_2

$$\therefore\ L = 2l\left[\log \frac{a_2}{a_1} + \frac{1}{4}\right] \tag{98}$$

This result can also be obtained by integrating the expression for the force outside a_1 between the limits a_1 and a_2, and adding the term for the field within a_1, there being no magnetic field outside a_2.

If the outer tube has a thickness $a_3 - a_2$ and the current is distributed uniformly over its cross section the self-inductance will be a little greater, the geometrical mean distance from a_1 to the tube, which is more than a_2 and less than a_3, being given by the expression

$$\log a_g = \frac{a_3^2 \log a_3 - a_2^2 \log a_2}{a_3^2 - a_2^2} - \frac{1}{2}$$

Putting this value of log a in (56) in place of log a_2, we should have the self-inductance of the return circuit.

If the current is alternating and of very high frequency, the current would flow on the outer surface of a_1 and on the inner surface of the tube, and L for the circuit would be

$$L = 2l \log \frac{a_2}{a_1} \qquad [99]$$

MULTIPLE CONDUCTORS.

If a current be divided equally between two wires of length l, radius ρ and distance d apart, the self-inductance of the divided conductor is the sum of their separate self-inductances plus twice their mutual inductance.

Thus, when d/l is small,

$$L = 2l\left[\log\frac{2l}{(\rho d)^{\frac{1}{2}}} - \frac{7}{8}\right] = 2l\left[\log\frac{2l}{(r_g d)^{\frac{1}{2}}} - 1\right] \qquad [100]$$

where r_g, the g. m. d. of the section of the wire is 0.7788ρ for a round section.

If there are three straight conductors in parallel and distance d apart, the self-inductance is similarly

$$L = 2l\left[\log\frac{2l}{(r_g d^2)^{\frac{1}{3}}} - 1\right] \qquad [101]$$

The expression $(r_g d^2)^{\frac{1}{3}}$ is the g. m. d. of the multiple conductor.

9. FORMULÆ FOR GEOMETRICAL AND ARITHMETICAL MEAN DISTANCES.

GEOMETRICAL MEAN DISTANCES.

Maxwell showed how to calculate mutual and self-inductances in several important cases by means of what he called the geometrical mean distances, either of one conductor from another or of a conductor from itself. On account of the importance of this method we give below some of the most useful of these formulæ. The geometrical mean distance of a point from a line is the *n^{th} root of the product of the n distances from the point P to the various points in the line, n being increased to infinity in determining the value of R.* Or, the logarithm of R is the *mean value of log d for all the infinite values of the distance d.* Similarly, the geometrical mean distance of a line from itself is the *n^{th} root of the product of the n distances between all the various pairs of points in the line, n being infinity.*[78]

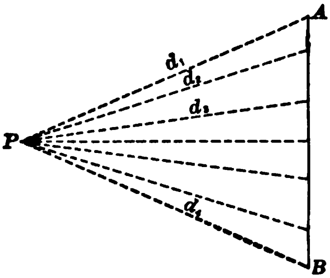

Fig. 33.

Similar definitions apply to the g. m. d. of one area from another, or of an area from itself.

The geometrical mean distance R of a *line* of length a from itself is given by

$$\left.\begin{aligned} \log R &= \log a - \frac{3}{2} \\ R &= ae^{-\frac{3}{2}} \\ \text{or } R &= 0.22313a \end{aligned}\right\} \quad [102]$$

The g. m. d. of a *rectangular area* of sides a and b from itself is given by

$$\log R = \log \sqrt{a^2+b^2} - \frac{1}{6}\frac{a^2}{b^2}\log\sqrt{1+\frac{b^2}{a^2}} - \frac{1}{6}\frac{b^2}{a^2}\log\sqrt{1+\frac{a^2}{b^2}}$$
$$+\frac{2}{3}\frac{a}{b}\tan^{-1}\frac{b}{a} + \frac{2}{3}\frac{b}{a}\tan^{-1}\frac{a}{b} - \frac{25}{12} \quad [103]$$

When the area is a *square*, and hence $a = b$,

$$\left.\begin{aligned} \log R &= \log a + \frac{1}{3}\log 2 + \frac{\pi}{3} - \frac{25}{12} \\ \therefore R &= 0.44705\, a \end{aligned}\right\} \quad [104]$$

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[78] Rosa, this Bulletin, 4, p. 326.

For a *circular area* of radius a,

$$\left.\begin{aligned} \log R &= \log a - \frac{1}{4} \\ R &= a e^{-\frac{1}{4}} \\ R &= 0.7788\, a \end{aligned}\right\} \quad [105]$$

For an *ellipse* of semi-axes a and b,

$$\log R = \log \frac{a+b}{2} - \frac{1}{4} \qquad [106]$$

An approximate expression for the g. m. d. of a *rectangular area* of length a and breadth b is

$$R = 0.2235(a+b) \qquad [107]$$

which is nearly true for all values of a and b; that is, the geometrical mean distance of the rectangular area from itself is approximately proportional to the perimeter of the rectangle. The following table gives the ratio $R/(a+b)$ for a series of rectangles of different proportions, from a square to a ratio of 20 to 1 between length and breadth, and finally when the breadth is infinitesimal in comparison with the length. By interpolating for any other case between the values given in the table one can obtain a quite accurate value without the trouble of calculating it by formula (103).

Geometrical Mean Distances of Rectangles of Different Proportions.

[a and b are the Length and Breadth of the Rectangles. R is the Geometrical Mean Distance of its Area.]

Ratio	R	$\frac{R}{a+b}$
1 :1	0.44705a	0.22353
1.25:1	0.40235a	0.22353
1.5 :1	0.37258a	0.22355
2 :1	0.33540a	0.22360
4 :1	0.27961a	0.22369
10 :1	0.24596a	0.22360
20 :1	0.23463a	0.22346
1 :0	0.22315a	0.22315

The g. m. d. of an *annular ring* of radii a_1 and a_2 from itself is given by

$$\log R = \log a_1 - \frac{a_2^4}{(a_1^2 - a_2^2)^2} \log \frac{a_1}{a_2} + \frac{1}{4} \frac{3a_2^2 - a_1^2}{a_1^2 - a_2^2} \qquad [108]$$

The g. m. d. of a *line* of length a from a second line of the same length, distant in the same straight line na, center to center, is given by the following formula:

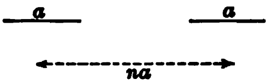

Fig. 34.

$$\log R_n = \frac{(n+1)^2}{2} \log (n+1)a - n^2 \log na + \frac{(n-1)^2}{2} \log (n-1)a - \frac{3}{2} \qquad [109]$$

This formula is equivalent to the following, which is more convenient for calculation for all values of n greater than one.[74]

$$\log R_n = \log n - \left[\frac{1}{12n^2} + \frac{1}{60n^4} + \frac{1}{168n^6} + \frac{1}{360n^8} + \frac{1}{660n^{10}} + \cdots\cdots\right] \qquad [110]$$

This formula is very convergent, and only two or three terms are generally required.

The following values of the geometrical mean distances (calling a unity) were calculated from the above formulæ, all after the second being obtained by (110):

$R_0 = 0.22313$	$R_5 = 4.98323$
$R_1 = 0.89252$	$R_6 = 5.98610$
$R_2 = 1.95653$	$R_7 = 6.98806$
$R_3 = 2.97171$	$R_8 = 7.98957$
$R_4 = 3.97890$	$R_9 = 8.99076$

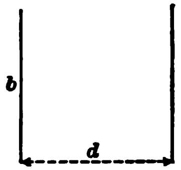

Fig. 34*a*.

If the strips are parallel and at distance d, the g. m. d. is given by

$$\log R = \frac{d^2}{b^2} \log d + \frac{1}{2}\left(1 - \frac{d^2}{b^2}\right) \log (b^2 + d^2) + 2\frac{d}{b} \tan^{-1}\frac{b}{d} - \frac{3}{2} \qquad [111]$$

If $d = b$,

$$\log R = \log b + \frac{\pi}{2} - \frac{3}{2} \qquad [112]$$

The g. m. d. from a point O_2 outside a circle to the circumference of the circle, or to the entire area of the circle *is the distance d from* O_2 *to the center of the circle.*

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[74] Rosa, this Bulletin, **2**, p. 168; 1906.

(1) The g. m. d. from the center O_1 to the circumference is of course the radius a. (2) The g. m. d. of any point (as O_3) within the circle from the circumference is also a. (3) The g. m. d. of any point on the circumference (as O_4) from all other points of the circumference is also a. (4) Therefore the g. m. d. of a circular line of radius a from itself is a; that is,

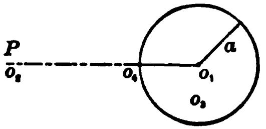

Fig. 35.

$$R = a \qquad [113]$$

for each of the four cases named above.

The g. m. d. of a point outside a circular ring from the ring *is the distance d to the center of the ring.* The g. m. d. of any point O_1, O_2, etc., within the ring is given by

$$\log R = \frac{a_1^4 \log a_1 - a_2^4 \log a_2}{a_1^2 - a_2^2} - \frac{1}{2} \qquad [114]$$

The same expression gives the g. m. d. of any figure, as S_1, within the ring from the ring. The g. m. d. of an external figure, as S_2, from the annular ring is equal to the g. m. d. of the center O_1 from the figure S_2.

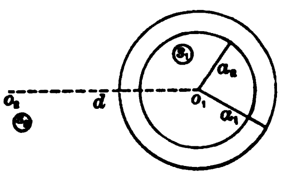

Fig. 36.

The g. m. d. from one circular area to another is the distance between their centers; that is,

$$R = d \qquad [115]$$

Fig. 37.

for the area S_1 with respect to S_2 as it is for the point O_1 with respect to S_2.

The g. m. d. of a line of length a from a second parallel line of length a' located symmetrically (Fig. 38) is given by Gray[75], equation (114). The g. m. d. of a line from a parallel and symmetrically

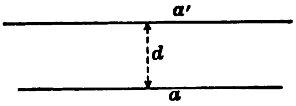

Fig. 38.

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[75] Absolute Measurements, Vol. II, Part I.

There are a number of misprints in equations 104, 109, 111, and 113 of Gray. The sign of the first term of equation 104 should be +. The signs before p^2 in the coefficients of the log in the first four terms of equation 109 should be all minus; thus $\frac{1}{4}(\beta^2-p^2), -\frac{1}{4}(\alpha^2-p^2), -\frac{1}{4}[(a-\beta)^2-p^2], +\frac{1}{4}[(a-\alpha)^2-p^2]$. Similarly in equation 111 the coefficients of the first two terms should be $\frac{1}{2}(\beta^2-p^2)$ and $-\frac{1}{2}(\alpha^2-p^2)$. In equation 113 the coefficient of β^4 in each of the first four terms should be $\frac{1}{6}$ instead of $\frac{1}{2}$ and the first term should have $\log[(p+b+b')^2+\beta^2]$ instead of $\log[(p+b+b')^2-\beta^2]$.

situated rectangle is given by Gray's equation (112). The g. m. d. of two unequal rectangles from one another is given by Gray's equation (113).[76]

The g. m. d. of two adjacent rectangles and of two obliquely situated rectangles are given by Rosa,[77] equations (8*a*) and (17). As these expressions are somewhat lengthy and not often required they are not repeated here. The values of the g. m. d. for two equal squares in various relative positions to one another have been accurately calculated[78] by these formulæ, and the results used in the determination[79] of the correction term E of formula (72).

ARITHMETICAL MEAN DISTANCES.

In the determination of self and mutual inductances by the method of geometrical mean distances it has been shown[80] that more accurate formulæ can be obtained by the use of certain arithmetical mean distances and arithmetical mean square distances taken in connection with geometrical mean distances.

The arithmetical mean distance of a point from a line is the arithmetical mean of the *n* distances of the point from the various points of the line, *n* being infinite. Similarly, the arithmetical mean distance of a line from itself is the *arithmetical mean of the distances of the n pairs of points in the line from one another, n being infinite.*

The a. m. d. of a line of length b from itself is[81]

$$S_1=\frac{b}{3} \qquad [116]$$

that is, while the g. m. d. of a line from itself is 0.22313 times its length, the a. m. d. is one-third the length.

The arithmetical mean square distance of a line from itself is of course larger than the square of the a. m. d. Putting S_2^2 for the arithmetical mean square distance (a. m. s. d.)

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[76] Also by Rosa, equation (8). This Bulletin, **8**, p. 6.

[77] This Bulletin, **8**, pp. 7 and 12.

[78] This Bulletin, **8**, pp. 9–19.

[79] This Bulletin, **8**, p. 37.

[80] Rosa, this Bulletin, **4**, p. 326–32.

[81] Rosa, this Bulletin **4**, p. 326.

$$S_2^2 = \frac{b^2}{6}, \text{ or } \sqrt{S_2^2} = \frac{b}{\sqrt{6}} \qquad [117]$$

The arithmetical mean distance of a point in the circumference of a circle from the circle is the same as the a. m. d. of the circle from itself; that is, for a circle of radius a,

$$S_1 = S_2 = \frac{4}{\pi}a \qquad [118]$$

The arithmetical mean square distance is

$$S_2^2 = 2a \text{ and } \sqrt{S_2^2} = a\sqrt{2} \qquad [119]$$

(The g. m. d. for this case is $R = a$, equation (113).)

The arithmetical mean distance of an external point P from the circumference of a circle is

$$S_1 = \sqrt{d^2 + a^2} \qquad [120]$$

which is the distance PA.

The arithmetical mean distance from P to the entire area of the circle is

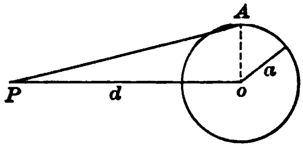

Fig. 39.

$$S_1 = \sqrt{d^2 + \frac{a^2}{2}} \qquad [121]$$

(The g. m. d. for each of these cases is $R = d$, equation (115).)

For the proof of these and other expressions for the arithmetical mean distances and applications of their use see the article referred to above.

II. EXAMPLES TO ILLUSTRATE AND TEST THE FORMULÆ.

1. COAXIAL CIRCLES.

EXAMPLE 1. MAXWELL'S FORMULA (1). FOR ANY TWO COAXIAL CIRCLES.

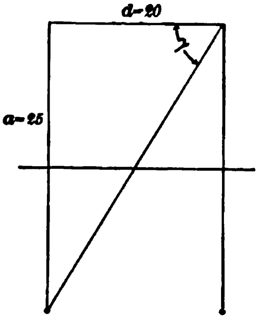

Fig. 40.

Let $a=A=25$ cm, Fig. 40,
$d=20$ cm.

$$k=\frac{50}{\sqrt{2500+400}}=0.9284766=\sin\gamma$$

$$\gamma=68^\circ\ 11'\ 54.''88=68^\circ.198578.$$

From Legendre's tables, we obtain

$$\begin{aligned}\log F&=0.3852191\\ \log E&=0.0547850\end{aligned}\quad \left(\frac{2}{k}-k\right)F-\frac{2}{k}E=0.5318500$$

$$4a=100 \qquad \therefore\ M=167.0856 \text{ cm.}$$

To facilitate calculations in such problems as this, we have prepared Table II, which gives F and log F, E and log E, as functions of tan γ. In the above case $\tan\gamma=\frac{50}{20}=2.5$, and from Table II we can take the values of log F and log E directly, avoiding the calculation of γ and the interpolation for log F and log E in Legendre's tables (or Table XIII). This is only applicable for circles of equal radii, and is especially advantageous when tan γ is one of the values given in the table, when interpolation is unnecessary.

The above problem may also be calculated by means of Table I, taken from Maxwell, as follows:

$$\log_{10}\frac{M}{4\pi a} \text{ for } 68^\circ.1 \qquad =\bar{1}.7230634$$

$$\text{for } 68^\circ.2 \qquad =\underline{\bar{1}.7258281}$$

$$\text{for } 68^\circ.198578=\bar{1}.7257888=\log\frac{M}{4\pi a}$$

$\therefore$ $M=167.0855$ cm, agreeing almost exactly with the above value.

The calculation of mutual inductance by the above methods is simplest for circles not near each other, as then the values of log F, log E, and log $\frac{M}{4\pi\sqrt{Aa}}$ are very exact when taken by simple interpolation. When γ is nearly 90°, however, second and third differences have to be used in interpolation.

EXAMPLE 2. MAXWELL'S SECOND EXPRESSION (2). FOR CIRCLES NEAR EACH OTHER.

$$\text{Let } a=A=25 \text{ cm}, \ d=1 \text{ cm}$$

$$\text{In this case } k=\sin\gamma=\frac{50}{\sqrt{2501}}=0.9998002 \qquad \gamma=88^\circ\ 51'\ 14''$$

This value of γ is so nearly 90° that it is difficult to obtain accurate values of F and E from tables of elliptic integrals, or of $\frac{M}{4\pi a}$ from Maxwell's table.

We may therefore use formula (2) instead of (1).

$$r_1=\sqrt{2501}=50.01 \text{ nearly}, \ r_2=1.0$$

$$\therefore\ k_1=\sin\gamma_1=\frac{r_1-r_2}{r_1+r_2}=\frac{49.01}{51.01}=0.960792$$

$$\gamma_1=73^\circ\ 54'\ 9.''7=73^\circ\!.9027$$

From Legendre's tables or Table XIII, for $\gamma_1=73^\circ\!.9027$,

$$F_1=2.7024553$$
$$E_1=1.0852167$$
$$F_1-E_1=1.6172386$$

$$\frac{8\pi\sqrt{Aa}}{\sqrt{k_1}}=\frac{200\pi}{\sqrt{.960792}} \quad \therefore \frac{8\pi\sqrt{Aa}}{\sqrt{k_1}}\left(F_1-E_1\right)=M=1036.667 \text{ cm.}$$

EXAMPLE 3. FORMULA (3). SERIES FOR F AND E, CIRCLES NEAR EACH OTHER.

Suppose that, in the last example, we calculate F and E by means of formula (3), instead of taking them from Table XIII.

$$A=a=25, \ d=1.$$

$$k'^2=1-k^2=1-\frac{2500}{2501}=\frac{1}{2501}$$

$$\therefore\ F=5.298947 \quad E=1.000960$$

If these values of F and E be substituted in formula (1), k being 0.9998002, we obtain the same value of M as by formula (2).

EXAMPLE 4. FORMULA (3). SECOND CASE, CIRCLES NOT NEAR.

$$A=25,\ a=20,\ d=10 \text{ cm.} \quad \text{(See Fig. 1.)}$$

$$k^2=\frac{4\times 20\times 25}{(45)^2+(10)^2}=\frac{16}{17} \quad \therefore k'^2=\frac{1}{17}$$

$$\log\frac{4}{k'}=\frac{1}{2}\log(16\times 17)=\frac{1}{2}\log_e 272 \qquad =2.8029010$$

$$\frac{k'^2}{4}\left(\log\frac{4}{k'}-1\right) \qquad = .0265132$$

$$\frac{9k'^4}{64}\left(\log\frac{4}{k'}-\frac{7}{6}\right) \qquad = .0007962$$

$$\frac{25k'^6}{256}\left(\log\frac{4}{k'}-\frac{111}{90}\right) \qquad = .0000312$$

$$\frac{1225k'^8}{16384}\left(\log\frac{4}{k'}-1.27\right) \qquad = \underline{.0000014}$$

$$\therefore F \quad =2.8302430$$

$$1+\frac{k'^2}{2}\left(\log\frac{4}{k'}-\frac{1}{2}\right) \qquad =1.0677324$$

$$\frac{3k'^4}{16}\left(\log\frac{4}{k'}-\frac{13}{12}\right) \qquad = .0011156$$

$$\frac{15k'^6}{128}\left(\log\frac{4}{k'}-1.20\right) \qquad = .0000381$$

$$\frac{175k'^8}{2048}\left(\log\frac{4}{k'}-1.25\right) \qquad = \underline{.0000017}$$

$$\therefore E \quad =1.0688878$$

To find the value of M we now use equation (1).

$$\left\{\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right\}=0.885388$$

Multiplying by $4\pi\sqrt{Aa}=4\pi\sqrt{500}$ gives

$$M=248.7875 \text{ cm.}$$

EXAMPLE 5. WEINSTEIN'S FORMULA (4). FOR ANY COAXIAL CIRCLES NOT TOO FAR APART.

Take the same circles as in example 4.

$$A=25,\ a=20,\ c=5,\ d=10.$$

$$k'^2=\frac{1}{17},\ \log\frac{4}{k}-1=1.802901$$

$$1+\frac{3}{4}k'^2=1.0441176 \qquad 1+\frac{15}{128}k'^4=\ 1.0004053$$

$$\frac{33}{64}k'^4=\ .0017842 \qquad \frac{185}{1536}k'^6=\ .0000245$$

$$\frac{107}{256}k'^6=\ .0000851 \qquad \frac{7465}{65536}k'^8=\ \underline{.0000012}$$

$$\frac{5913}{16384}k'^8=\ \underline{.0000042} \qquad 1.0004310=C$$

$$\text{Sum}\quad =1.0459911=B$$

$$B\log\left(\frac{4}{k'}-1\right)=1.8858184;\quad \left\{B\log\left(\frac{4}{k'}-1\right)-C\right\}=0.8853874$$

Multiplying by $4\pi\sqrt{500}$ gives $M=248.7873$ cm, agreeing almost exactly with the value previously found, example 4.

EXAMPLE 6. NAGAOKA'S Q-SERIES FORMULA (5). FOR CIRCLES NOT NEAR EACH OTHER.

$A=a=25,\quad d=20\quad$ See Fig. 40.

$$\sqrt{k'}=\sqrt{cos\gamma}=\left(\frac{20}{\sqrt{2900}}\right)^{\frac{1}{2}}=\ 0.6094183$$

$$\frac{l}{2}=\frac{1-\sqrt{k'}}{2(1+\sqrt{k'})}=\frac{1}{2}\,\frac{0.3905817}{1.6094183}=\ 0.1213425$$

$$2\left(\frac{l}{2}\right)^5=\ .0000526$$

$$15\left(\frac{l}{2}\right)^9=\ \underline{.0000000}$$

$$\therefore q=\ 0.1213951$$

$$3q^4=+\ .0006516$$

$$-4q^6=-\ .0000128$$

$$+9q^8=+\ \underline{.0000004}$$

$$\epsilon=\ .0006392$$

$$1+\epsilon=\ \underline{1.0006392}$$

$$\log(1+\epsilon)=\ 0.0002773$$

$$\log q^{\frac{3}{2}}=\ \bar{2}.6263018$$

$$\log 16\pi\sqrt{Aa}=\ \underline{3.0992099}$$

$$\log\frac{M}{\pi}=\ 1.7257890$$

$\therefore M=167.0855$ cm, as previously found by formula (1), example 1.

EXAMPLE 7. NAGAOKA'S SECOND FORMULA (6). FOR CIRCLES NEAR EACH OTHER.

$$A = a = 25, \ d = 4$$

$$k = \sin\gamma = \frac{50}{\sqrt{2516}}; \quad \sqrt{k} = 0.99840637$$

$$l_1 = \frac{1-\sqrt{k}}{1+\sqrt{k}} = \frac{0.00159363}{1.9984064}; \quad \frac{l_1}{2} = 0.00039872 = q_1$$

as $\left(\frac{l_1}{2}\right)$ and higher powers can be neglected.

$$\frac{1}{q_1} = 2508.04, \ \log_e\left(\frac{1}{q_1}\right) = \quad 7.827238$$

$$(1 - q_1 + 4q_1^2) = \quad 0.9996019$$

$$8q_1 = \quad 0.00318976$$

$$\left\{\log_e\left(\frac{1}{q_1}\right)[1 + 8q_1(1 - q_1 + 4q_1^2)] - 4\right\} = \quad 3.852195 = A$$

$$\frac{1}{2(1-2q_1)^2} = \quad 0.5007985 = B$$

$$4\sqrt{Aa} = \underline{100 \qquad\quad} = C$$

$$\text{Product } A \times B \times C = M = 606.0679 \text{ cm.}$$

There is a difficulty in using the above formula, owing to the fact that when k is nearly unity the numerator of the expression for l_1 is small, and unless the value of k is carried out to about eight decimal places the value of M may be appreciably in error. For approximate calculations a seven-place table of logarithms is sufficient, and it is not very troublesome to carry out this one number to the necessary number of places for precision calculations. Or, k can easily be computed to any degree of accuracy without logarithms. The same thing applies to formula (3), where k' must be computed with great precision when it is quite small.

Using Table II for the above problem, where $\tan\gamma = 12.5$, we have $\log F = 0.5932708$ and $\log E = 0.0047004$. Using these values in formula (1) we obtain for the mutual inductance

$$M = 606.0666 \text{ cm}$$

which differs from the value by Nagaoka's formula by 2 parts in a million.

EXAMPLE 8. MAXWELL'S SERIES FORMULA (7). FOR ANY TWO COAXIAL CIRCLES NEAR EACH OTHER.

$$A=26,\ a=25,\ d=1,\ c=1, \text{ and } r=\sqrt{2}$$

$$\text{Since } r=\sqrt{2},\ \log_e\frac{8a}{r}=\log_e\frac{200}{\sqrt{2}}=4.951744$$

$$\begin{array}{lcrlcr}
1+\dfrac{c}{2a} & = & 1.020000 & 2+\dfrac{c}{2a} & = & 2.020000 \\
\dfrac{c^2+3d}{16a^2} & = & .000400 & -\dfrac{3c^2-d^2}{16a^2} & =- & .000200 \\
-\dfrac{c^3+3cd^2}{32a^3} & =- & \underline{.000008} & -\dfrac{c^3-6cd^2}{48a^3} & =- & \underline{.000010} \\
 & & 1.020392=B & & & 2.019790=C
\end{array}$$

$$\begin{array}{lcl}
B\log\dfrac{8a}{r} & = & 5.052310 \\
C & = & \underline{2.019790} \\
\left\{B\log\dfrac{8a}{r}-C\right\} & = & 3.032520
\end{array}\quad \text{Multiply by } 4\pi a=100\pi \text{ and}$$

$$M=952.6943 \text{ cm.}$$

This formula would be less accurate for the circles of problem 4, but is accurate for circles close together, as this problem shows.

EXAMPLE 9. MAXWELL'S FORMULA (9). FOR CIRCLES OF EQUAL RADII NEAR EACH OTHER.

$$A=a=25,\quad d=1$$

$$\frac{8a}{d}=200,\ \log_e 200=5.298317$$

$$\log_e\frac{8a}{d}\left(1+\frac{3d^2}{16a^2}\right)=1.000300\times 5.298317=5.29990$$

$$\left(2+\frac{d^2}{16a^2}\right)=\frac{2.00010}{3.29980}$$

Multiply by $4\pi a=100\pi$

$$M=1036.663 \text{ cm.}$$

nearly agreeing with the more exact value found under problem 2.

This is a very simple and convenient formula for equal circles, and gives approximate results for circles still farther apart than in this problem.

EXAMPLE 10. COFFIN'S FORMULA (10). EXTENSION OF FORMULA (9) FOR CIRCLES OF EQUAL RADII.

$$A = a = 25, \quad d = 16$$

$$\frac{8a}{d} = 12.5, \quad \log_e 12.5 \quad = 2.5257286$$

$$\text{First series of terms} = B = 1.074478$$

$$\text{Second series of terms} = C = 2.023220$$

$$\therefore \left\{ B \log \frac{8a}{d} - C \right\} = 0.690620$$

$$4\pi a = 100\pi \quad \therefore M = 216.9647 \text{ cm.}$$

This agrees with the value given by formula (1) within 1 part in 200,000. As the distance apart of the circles increases the accuracy by this formula of course gradually decreases.

EXAMPLE 11. FORMULA (11). EXTENSION OF MAXWELL'S FORMULA (7) FOR CIRCLES OF UNEQUAL RADII.

$$A = 25, \quad a = 20, \quad c = 5, \quad d = 10.$$

$$r = \sqrt{c^2 + d^2} = 5\sqrt{5}, \quad \log_e \frac{8a}{r} = \log_e \frac{32}{\sqrt{5}} = \quad 2.6610169$$

$$\text{First series of terms} = B \log_e \frac{8a}{r} = 3.112060$$

$$\text{Second series of terms} = C \qquad = \underline{2.122114}$$

$$0.989946$$

$$\text{multiplying by } 4\pi a = 80\pi, \ M = 248.8006 \text{ cm.}$$

This result is correct to 1 part in 19,000 (see examples 4 and 5). Using only the first three terms for B and C (that is, formula 8), the result would be too large by 1 part in 1750.

2. MUTUAL INDUCTANCE OF COILS OF RECTANGULAR SECTIONS.

EXAMPLE 12. ROWLAND'S FORMULA (14). FOR COAXIAL COILS OF EQUAL RADII.

$$A = a = 25, \qquad b = c = 2 \text{ cm}, \qquad d = 10.$$

The mutual inductance of the two coils is $M = M_0 + \Delta M$.

We find M_0 by formula 1, 5, or 10, and ΔM by 14 and 15.

$$M_0 = 107.4885\pi$$

$$k = \sin\gamma = \frac{50}{\sqrt{2600}} = 0.9805807$$

$$k^2 = 0.9615383$$

$$\log_{10}F = 0.4821754$$

$$\log_{10}E = 0.0207625$$

By Table II, since $\tan\gamma = 5$, $\log F = 0.4821752$ and $\log E = 0.0207626$. These slight differences in the logarithms obtained in the two different ways amount to scarcely one part in two million of F and E, respectively, and may usually be neglected. If more accurate values are required they may be obtained by carrying the interpolations further in Legendre's table, provided the angle γ is obtained with sufficient accuracy.

Substituting these values in formula (15) we obtain

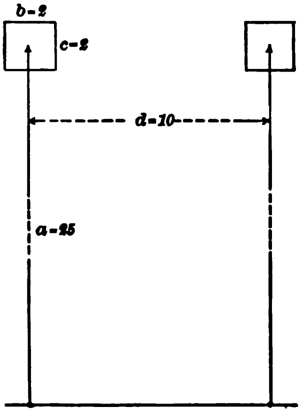

Fig. 41.

$$\frac{d^2M}{da^2} = -0.9081\,\pi$$

$$\frac{d^2M}{dx^2} = +1.0639\,\pi \qquad b^2 = c^2 = 4$$

Substituting these values in formula (14) we obtain

$$\Delta M = .05193\,\pi$$

$$\therefore\ M = M_0 + \Delta M = (107.4885 + 0.519)\pi$$

$$= 337.8481 \text{ cm.}$$

The correction ΔM thus amounts to about 1 part in 2,000 of M. At a distance $d = 20$ cm, the correction is over 1 part in 1,000. For a coil of section 4 × 4 cm at $d = 10$, ΔM would be four times as large as the value above, or about 1 part in 500, and at 20 cm 1 part in 250.

EXAMPLE 13. RAYLEIGH'S FORMULA (17). FOR COAXIAL COILS OF EQUAL RADII.

$$A = a = 25, \quad b = 4, \quad c = 1, \quad d = 10$$

We now find by formula (1) in accordance with formula (17) the mutual inductance of the following pairs of circles:

O, 1 when $a=25$, $A=25.5$, $d=10$; O, 4 when $a=25$, $A=24.5$, $d=10$; O, 2 when $a=A=25$ and $d=8$; O, 3 when $A=a=25$, $d=12$ and finally O, O′ when $A=a=25$, $d=10$. Thus:

$$\begin{aligned}
M_1 &= 109.3217\pi \\
M_4 &= 105.4287\pi \\
M_2 &= 127.3949\pi \\
M_3 &= \underline{\ \ 91.9206\pi} \\
& \quad 434.0659\pi \\
M_0 &= \underline{107.4885\pi} \\
& \quad 326.5774\pi \\
\therefore M &= 108.8591\pi \\
M_0 &= \underline{107.4885\pi} \\
\Delta M &= \quad 1.3706\pi \text{ cm.}
\end{aligned}$$

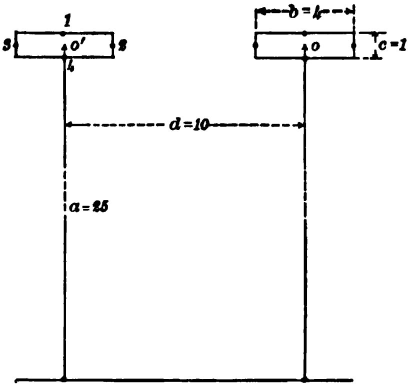

Fig. 42.

EXAMPLE 14. LYLE'S FORMULA (20). FOR COILS OF SQUARE SECTION.

$$A=a=25 \text{ cm}, \quad b=c=2 \text{ cm}, \quad d=10 \text{ cm}.$$

The equivalent radius $r=a\left(1+\frac{b^2}{24a^2}\right)$

$$r=25\left(1+\frac{4}{15000}\right)=25.00667 \text{ cm}.$$

M is now found by using formula 1, 5, or 10, employing r in place of a as the radius.

The result is $M=337.8475$, agreeing very closely with the result found under example 12.

$$M-M_0=\Delta M=.0517\pi$$

EXAMPLE 15. LYLE'S FORMULA (21). FOR COILS OF RECTANGULAR SECTION.

$$A=a=25, \quad b=4, \quad c=1, \quad d=10$$

$$r=25\left(1+\frac{1}{15000}\right)=25.00167$$

$\beta^2=\frac{b^2-c^2}{12}=\frac{15}{12}=1.25$, $2\beta=2.236$ cm, the distance apart of the two filaments which replace the coil. We now find by formula 1, 5, or 10 the mutual inductances of two circles 1, 2 on the two circles 3, 4, where $a=25.00167$ and d is 7.764, 10 and 12.236 cm, respectively. Thus:

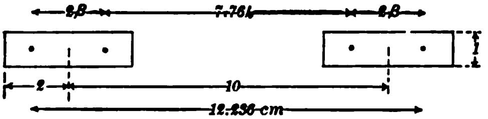

Fig. 43.

$$
\begin{aligned}
2\,M_{13} &= 215.002228\pi \\
M_{14} &= 90.313047\pi \\
M_{23} &= 130.140607\pi \\
\hline
4\,M &= 435.455922\pi \\
\therefore M &= 108.8640\ \pi \\
M_0 &= 107.4885\ \pi \\
\hline
\Delta M &= 1.3755\ \pi
\end{aligned}
$$

ΔM = the correction for section of the coils whose dimensions are given above. These values of M and ΔM agree nearly with the results obtained in example 13 above.

EXAMPLE 16. ROSA'S FORMULA (22). FOR COILS OF EQUAL RADII.

$A=a=25,\quad b=4,\quad c=1,\quad d=10$

(same coils as examples 13, 15).

$$\log_e\frac{8a}{d}=\log_e 20=2.9957$$

$$
\begin{aligned}
\frac{3b^2+c^2}{96a^2}\log_e\frac{8a}{d} &= \frac{49\times2.9957}{60{,}000} = .0024465 \\
\frac{b^2-c^2}{12d^2} &= \frac{15}{1200} = .0125000 \\
\frac{2b^4+2c^4-5b^2c^2}{120d^4} &= \frac{434}{1{,}200{,}000} = .0003617 \\
\frac{3b^6-3c^6+14b^2c^4-14b^4c^2}{504d^6} &= \frac{8925}{504\times10^6} = .0000177 \\
\frac{6b^4+6c^4+5b^2c^2}{5760a^2d^2} &= \frac{1622}{360\times10^6} = .0000045 \\
\frac{7c^2d^2}{1024a^4}\left(\log_e\frac{8a}{d}-\frac{163}{84}\right) &= \underline{.0000018} \qquad .0153322 \\
-\frac{11b^2-3c^2}{192a^2} &= -\frac{173}{120{,}000} = -.0014417 \\
-\frac{15b^2d^2}{1024a^4}\left(\log_e\frac{8a}{d}-\frac{97}{60}\right) &= -\underline{.0000827} \qquad -\underline{.0015244} \\
& \qquad\qquad\qquad\qquad\qquad .0138078
\end{aligned}
$$

$4a=100,\ \therefore \Delta M=1.3808\ \pi$ cm.

This is a little larger value than found by formulæ (17) and (21), and we shall see later that it is more nearly correct than either of the other values.

EXAMPLE 17. ROSA'S FORMULÆ (23) AND (24). FOR COILS OF EQUAL RADII AND SQUARE SECTION.

$$A=a=25,\quad b=c=2,\quad d=10$$

$$\begin{array}{rcccl}
\log_e\dfrac{8a}{d}-1 = & 2.9957-1 & = & 1.9957 & \\
\dfrac{17b^2}{240d^2} = & \dfrac{68}{24{,}000} & = & \underline{.0028} & 1.9985 \\
\dfrac{-a^2b^2}{5d^4} = & -\dfrac{2500}{50{,}000} & = & -\ .0500 & \\
\dfrac{-3d^2}{16a^2}\left(\log_e\dfrac{8a}{d}-\dfrac{4}{3}\right) = & -\dfrac{300\times 1.6624}{10{,}000} & = & -\ \underline{.0499} & -\ \underline{.0999} \\
 & & & & 1.8986
\end{array}$$

$$\frac{b^2}{6a}=\frac{4}{150}\quad \therefore\ \Delta M=.05063\pi$$

The approximate formula (24) would have given .0519 (agreeing with formulæ 14 and 20), which would be amply accurate for any experimental purpose. When the section is larger these small terms are, however, more important.

EXAMPLE 18. SECOND EXAMPLE BY FORMULA (23).

$$A=a=25,\quad b=c=5,\quad d=10$$

$$\begin{array}{rll}
\log_e\dfrac{8a}{d}-1 & = 1.9957 & \\
\dfrac{17b^2}{240d^2} & = \underline{.0177} & 2.0134 \\
\dfrac{-a^2b^2}{5d^4} & = -.3125 & \\
-\dfrac{3d^2}{16a^2}\left(\log_e\dfrac{8a}{d}-\dfrac{4}{3}\right) & = -\underline{.0499} & -\underline{.3624} \\
 & & 1.6510
\end{array}$$

$$\frac{b^2}{6a}=\frac{25}{150}$$

$$\begin{aligned}
\therefore\ \Delta M &= 0.2752\pi \\
\underline{M_0} &= \underline{107.4885\pi}\ \text{(see example 12.)} \\
M &= 107.7637\pi\ \text{cm.}
\end{aligned}$$

This is a very simple formula for computing ΔM, and within a considerable range (i. e., d not larger than a and yet the coils not in contact) it is very accurate.

EXAMPLE 19. ROSA-WEINSTEIN FORMULA (25). FOR COILS OF EQUAL RADII AND EQUAL SECTION.

$$A=a=25, \quad b=4, \quad c=1, \quad d=10$$

$$a_1=15.0000533 \qquad \sin^2\gamma=\frac{2500}{2600}=\frac{25}{26}$$

$$a_2=\ 0.0020267 \qquad \cos^2\gamma=\frac{100}{2600}=\frac{1}{26}$$

$$a_3=\ 0.217 \qquad \frac{c^2}{24a^2}=.0000667$$

$$a_1-a_2-a_3+(2a_2-3a_3)\cos^2\gamma+8a_3\cos^4\gamma\ =14.7587120$$

$$a_1+\frac{a_2}{2}+2a_3+(2a_2+3a_3)\cos^2\gamma+8a_3\cos^4\gamma=15.4628292$$

$$A\ =\ 0.0004730 \qquad \text{Also } F\ =\ 3.0351168$$

$$B\ =\ 0.0123901 \qquad E\ =\ 1.0489686$$

$$\left(F-E\right)\left(A+\frac{c^2}{24a^2}\right)=\ 0.0010719$$

$$EB\ =\ 0.0129968$$

$$\text{Sum}\ =\ 0.0140687$$

$$4\pi a\sin\gamma=100\pi\sqrt{\frac{25}{26}}\ \therefore\ \Delta M\ =\ 1.3795\pi\text{ cm.}$$

This is not as simple to calculate as (22) and when d is less than $a/2$ is less accurate than (22). But for $d=a$ or greater it is more accurate than (22), and indeed the most accurate of all the formulæ.

EXAMPLE 20. FORMULA (26.) MUTUAL INDUCTANCE IN TERMS OF SELF-INDUCTANCE. FOR COILS RELATIVELY NEAR.

For $a=25$, $b_1=1$, $c=1$, we have, n being the number of turns in one of the two equal coils,

$$L_1=4\pi a n^2\ (4.103816)$$

For $b=2$, $c=1$,

$$L_2=4\pi a n^2\ (4\times 3.698695)$$

For $b=3$, $c=1$,

$$L=4\pi a n^2\ (9\times 3.411766)$$

Then the mutual inductance of 1 on 3 is by formula (26)

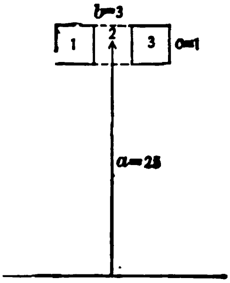

Fig. 44.

$$M=4\pi a n^2\left[\frac{L+L_1-2L_2}{2}\right]$$

$$=4\pi a n^2\left[\frac{30.705894+4.103816-29.589560}{2}\right]$$

$$=4\pi a n^2 \times 2.610075$$

$$=819.979\ n^2 \text{ cm.}$$

If $n=100$,

$$M=8.19979 \text{ millihenrys,}$$

as the mutual inductance of coil 1 on coil 3, Fig. 44.

EXAMPLE 21. FORMULA (27). MUTUAL INDUCTANCE BY GEOMETRICAL MEAN DISTANCE.

$A=25.1$
$a=25.0$
$b=c=0.1$ cm
$d=0.1$ cm

The geometrical mean distance of two coils, corner to corner, as in Fig. 9, is 0.997701, and $\log\frac{r}{R}=0.002302$

$$\therefore\ \Delta M=100\times 0.002302\ (1.002)\pi$$

$$=0.23307\pi \text{ cm.}$$

3. MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

EXAMPLE 22. MAXWELL'S FORMULA (28) AND COHEN'S (35).

Two solenoids, Fig. 45, of equal length, 200 cm, each wound with a single layer coil.

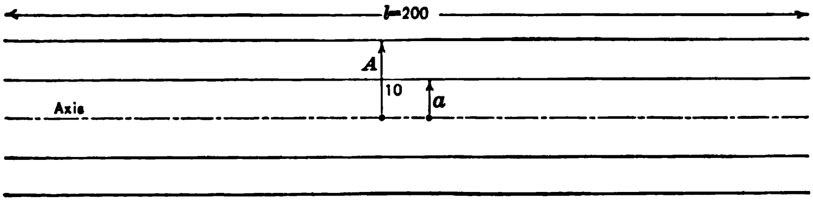

Fig. 45.

$A=10=$ radius of outer.
$a=\ \ 5=$ radius of inner.

Substituting in (28) for α we have the following:

$$\alpha = 0.487508 - \frac{1}{16}\frac{a^2}{A^2}(0.999875) - \frac{1}{64}\frac{a^4}{A^4}(0.500001) - \frac{35}{2048}\frac{a^6}{A^6}\left(\frac{1}{7}\right)$$

$$= 0.487508 - .015610 - .000488 - .000038$$

$$= 0.471372$$

$$\therefore M = 4\pi^2 a^2 n^2 (200 - 9.42744)$$

$$M = 19057.25\pi^2 n^2$$

If $n = 10$ turns per cm, $M = \dfrac{100\,\pi^2 \times 19057.25}{10^9}$ henry

$$= 0.018809 \text{ henry.}$$

By the approximate formula of Maxwell (29)

$$2\alpha = 1 - \frac{1}{8.4} - \frac{1}{64.16} - \frac{1}{1024.64} - \dots$$

$$= 0.96773$$

$$\therefore M = 0.018784 \text{ henry.}$$

This example by Heaviside's extension of Maxwell's formula (see p. 23) has exactly the same value of M; that is, the added terms do not amount to as much as a millionth of a henry in this particular case.

To show that the result by Maxwell's formula (28) is very accurate for this case we may now calculate M by Cohen's absolute formula:

$$M = 4\pi n^2 (V - V_1)$$

Substituting in (35) for V we have the following terms:

$$V = 7863.79 + 4200532.04 - 4169106.25 - 23561.95$$
$$= 15727.63$$
$$V_1 = 1392.18 - 632.16 = 760.02$$
$$\therefore M = 4\pi n^2 (15727.63 - 760.02)$$
$$M = 0.0188088 \text{ henry.}$$

This agrees with the result by Maxwell's formula to within 1 part in 175000.

The example by Cohen's formula illustrates the disadvantage of that formula for numerical calculations. Aside from the fact that it is complicated, and involves the use of both complete and incomplete elliptic integrals, the value of M depends on the difference between very large positive and negative terms, so that in order to get a value of M correct to 1 part in 100000 it is necessary in the above example to calculate the large terms to 1 part in 25000000. As a means of testing other formulæ, however, this absolute formula is of great value.

EXAMPLE 23. RÒITI'S FORMULA (30) COMPARED WITH SEARLE AND AIREY'S (33).

We will now calculate the example, Fig. 46 (originally given by Searle and Airey[82]), by Ròiti's formula, and also by the formula of Searle and Airey.

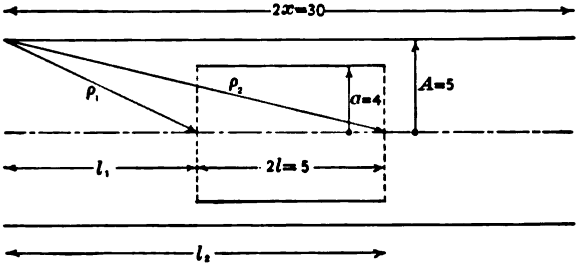

Fig. 46.

$2x = 30$ cm = length of outer solenoid.
$2l = 5$ " = " " inner "
$A = 5$ " = radius " outer "
$a = 4$ " = " " inner "

$$N_1 = 300 \text{ turns} \therefore n_1 = \frac{300}{30} = 10 \text{ per cm}$$

$$N_2 = 200 \quad \text{"} \quad n_2 = \frac{200}{5} = 40 \text{ per cm}$$

$$\begin{array}{ll} l_1 = 12.5 & \rho_1 = \sqrt{12.5^2 + 25} = 13.462912 \\ l_2 = 17.5 & \rho_2 = \sqrt{17.5^2 + 25} = \underline{18.200275} \\ & \therefore \rho_2 - \rho_1 = 4.737363 \end{array}$$

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[82] Electrician (London), **56**, p. 319; 1905.

$$\frac{A^2a^2}{8}\left(\frac{1}{\rho_1^{3}}-\frac{1}{\rho_2^{3}}\right)=+\ .012200$$

$$-\frac{A^4a^2}{16}\left(\frac{1}{\rho_1^{5}}-\frac{1}{\rho_2^{5}}\right)=-\ .000704$$

$$\left[\frac{5A^4a^4}{64}+\frac{5A^2a^6}{128}\right]\left(\frac{1}{\rho_1^{7}}-\frac{1}{\rho_2^{7}}\right)=+\ .000181$$

$$-\frac{35A^2a^6}{256}\left(\frac{1}{\rho_1^{9}}-\frac{1}{\rho_2^{9}}\right)=-\ .000022$$

$$+\frac{105A^6a^6}{1024}\left(\frac{1}{\rho_1^{11}}-\frac{1}{\rho_2^{11}}\right)=+\ .000002$$

$$\text{Sum}=\ \overline{4.749020}$$

$$4\pi^2a^2n_1n_2=25600\ \pi^2$$

$$\therefore\ M=\frac{25600\ \pi^2\times 4.749020}{10^9}\ \text{henry}$$

$$\text{or } M=0.001199896 \qquad \text{"}$$

Searle and Airey's formula (33) gives

$$M=1{,}198{,}480\ (1+0.001150+0.000034)$$
$$=1{,}199{,}900\ \text{cm}$$
$$=0.00119990\ \text{henry.}$$

The difference is inappreciable.

The same problem by Russell's formula (36) (extended to include six terms in each part of the formula) gives

$$M=0.00119989\ \text{henry.}$$

Thus these three formulæ all agree to within less than one part in 100,000. Searle and Airey's is the most rapidly convergent, and therefore most convenient. In other words, it is the most accurate for the same number of terms.

EXAMPLE 24. GRAY'S FORMULA (32) COMPARED WITH RÒITI'S (30).

Let the winding be 20 turns per cm on each coil; $n_1=n_2=20$.

$A=25$ cm $\qquad N_1=n_1\,A\sqrt{3}$

$\qquad\qquad\qquad\qquad\qquad \therefore\ N_1N_2=3n_1n_2Aa$

$a=10$ cm $\qquad N_2=n_2\,a\sqrt{3}$

$$d=\sqrt{x^2+A^2}=\frac{A}{2}\sqrt{7}$$

$$\therefore \quad M=\frac{2\pi^2a^2N_1N_2}{d}=4\pi^2a^2n_1n_2\left[\frac{3a}{\sqrt{7}}\right]$$

$$M=.0179057 \text{ henry.}$$

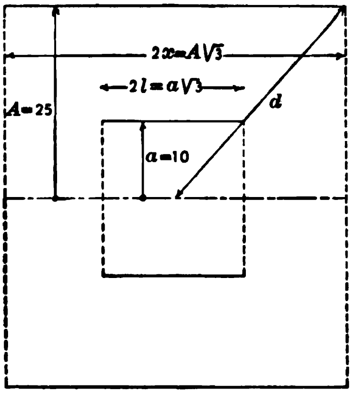

Fig. 47.

We have also calculated the mutual inductance for these coils by Ròiti's formula (30).

The value is, $M=.0179058$, which is practically identical with the value by Gray's formula.

When $A=25$ cm and $a=10$ cm, $N_1=20A\sqrt{3}=866.025$ and $N_2=20a\sqrt{3}=346.4$. As there must be an integral number of turns, let us suppose the winding is exactly 20 turns per cm on each coil and the lengths therefore 43.3 cm and 17.3 cm, respectively. Then $d=\sqrt{x^2+A^2}=\sqrt{625+\left(\frac{43.3}{2}\right)^2}=33.0715$ cm. This does not exactly conform to the condition imposed in deriving the simple formula (32) of Gray used above. Hence (32) will not be as exact with these slightly altered dimensions, and we must calculate at least one correction term to get an accurate value of M.

By Gray's formula (32), $M=\frac{2\pi^2 100\times 866\times 346}{33.0715\times 10^9}=.0178842$ henry.

The first correction term in (34) increases this value to .0178854 henry.

We will now calculate the mutual inductance of these coils by Ròiti's formula (30):

$A=25$	$2x=43.3$	$l_1=13.0$ cm	$\rho_1=$	28.17800
$a=10$	$2l\ =17.3$	$l_2=30.3$ "	$\rho_2=$	39.28218
			$\rho_2-\rho_1=$	11.10418
			2nd term = +	.22030
			3rd = −	.01781
			4th = +	.01952
			5th = +	.00156
			6th = −	.00453
			7th = +	.00274
			Sum =	11.32596

$$M = \frac{4\pi^2 a^2 n_1 n_2 \times 11.32596}{10^9} \text{ henry,}$$

$$= .0178853 \text{ henry.}$$

This differs from the result by Gray's formula by only 1 part in 178000.

In taking the dimensions of coils where an accurate value of the mutual inductance is sought it should be borne in mind that the above formulæ have been derived on the supposition that the current is uniformly distributed over the length of the coaxial solenoids; in other words, these formulæ are all current-sheet formulæ. Hence, for coils made up of many turns of wire we must meet the conditions imposed by current-sheet formulæ. In calculating self-inductances, this requires an accurate determination of the size of the wire and of the distance between the axes of successive wires, from which we can calculate two correction terms to be combined with the value of L given by the current-sheet formulæ.[83]

In the case of mutual inductances, however, there are no correction terms to calculate; but we must take the dimensions of the current sheets that are equivalent to the coils of wire; that is, the radius of each coil will be the mean distance to the center of the wire, and the length of each will be the over-all length, including the insulation, when the coil is wound of insulated wire in contact, or the length from center to center of a winding of $n+1$ turns, where n is the whole number of turns used.[84] It is also very important that the winding on both coils shall be uniform,[85] and that the leads shall be brought out so that there shall be no mutual inductance due to them.

The mutual inductance will of course be different at high frequencies from its value at low frequencies. We assume, however that for all purposes for which an extremely accurate mutual inductance is desired the frequency of the current would be low, say, not more than a few hundred per second. If the value at very high frequency is desired the coil should be wound with stranded wire, each strand of which is separately insulated.

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[83] Rosa, this Bulletin, **2**, p. 181; 1906.

[84] Rosa, this Bulletin, **2**, p. 161, 1906; and vol. 3, p. 1, 1907.

[85] Searle and Airey, Electrician (London), **56**, p. 318; 1905.

4 MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SOLENOID.

EXAMPLE 25. ROSA'S FORMULA (42) COMPARED WITH JONES'S FORMULA (40).

Professor Jones gave the calculations by formula (40) of the constant of the Lorenz apparatus made for McGill University, obtaining the values given below, the second value being that obtained after the plate had been reground and again measured.

A calculation[86] of the same two cases by formula (42) gives very closely agreeing results.

	1st Value.	2nd Value, disc slightly smaller.
By formula (40)	$M = 18{,}056.36$	18,042.52
" " (42)	$M = 18{,}056.34$	18,042.62
Difference	.02	—.10

These differences, amounting to 1 part in a million in the first case and 5 parts in a million in the second case, are wholly negligible in the most refined experimental work.

EXAMPLE 26. FORMULA (42) COMPARED WITH JONES'S FIRST FORMULA.

Take as a second example the case given by Jones[87] to illustrate his first formula.

$A = 10$ inches $a = 5$ inches $x = 2$ inches

$$d^2 = 104 \qquad \frac{a^2A^2}{d^4} = \frac{2500}{10816} \qquad \log \frac{a^2A^2}{d^4} = \bar{1}.36387 33$$

1st term $= 1.0000000$

2 " $= .0866771$

$X_2 = 2.8400$ 3 " $= .0118537$

$X_4 = 2.1064$ 4 " $= .0017781$

$X_6 = 1.5208$ 5 " $= .0002670$

$X_8 = 1.0173$ 6 " $= .0000379$

$X_{10} = 0.5815$ 7 " $= .0000060$

Sum $= 1.1006198$

$$\frac{2\pi^2 a^2}{d} = 48.38972$$

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[86] This Bulletin, 3, p. 218; 1907.

[87] Phil. Mag., 27, p. 61; 1889. In this example, P_0 should be 0.654870 instead of 0.54870, as printed in Jones's article.

$\therefore M = 53.25868\, N$, N being the number of turns of wire on the coil.

Jones gives $M = 53.25879\, N$.

The difference between these values is 2 parts in a million.

EXAMPLE 27. CALCULATION OF CONSTANT OF AYRTON-JONES CURRENT BALANCE BY FORMULÆ (40) AND (42).

As a further test of the formulæ let us calculate the constant of an electro-dynamometer or current balance of the Ayrton-Jones type, of which AB, Fig. 48, is the upper fixed coil and ED is the moving coil, the circle S at the upper end lying in the plane through the middle of AB and the circle R at the lower end of ED lying in the middle plane of the lower fixed coil BC.

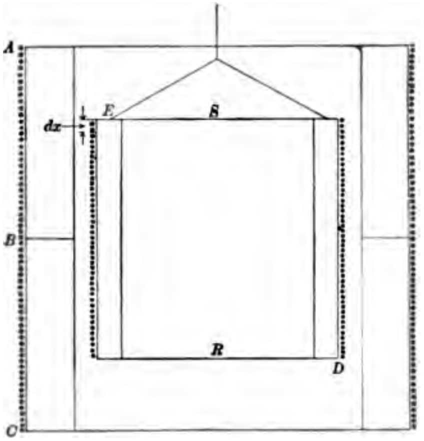

Fig. 48.

Assume the dimensions as follows:

$A = 16$ cm = radius of fixed coil, Fig. 16.

$a = 10$ cm = radius of moving coil.

$x_1 = 8$ cm = half length of AB = O_1A

$x_2 = 24$ cm = 1.5 times AB = O_2A

$n_1 = 10$ = number of turns per cm

$N_1 = 80$ = number of turns in distance $O_1A = x_1$, Fig. 16.

$N_2 = 240$ = number of turns in distance $O_2A = x_2$

$d_1 = \sqrt{A^2 + x^2_1} = 8\sqrt{5}$ = diagonal AP_1, Fig. 16.

$d_2 = \sqrt{A^2 + x^2_2} = 8\sqrt{13}$ = diagonal AP_2

We have to determine two mutual inductances, namely, M_S between the coil O_1A of 80 turns on the circle S, and M_R between the coil O_2A of 240 turns on the circle R. In each case the circle is in the plane passing through the lower end of the coil.

Formula (12) will be used, taking N_1, x_1, and d_1 in the first case and N_2, x_2, and d_2 in the second case.

	For M_S		*For* M_R
A	16 cm		16 cm
a	10		10
x	8		24
A^2	256		256
x^2	64		576
$N=nx$	80		240
d^2	320		832
log d^2	2.5051500		2.9201233
$\frac{a^2A^2}{d^4}$	$\bar{1}$.3979400		$\bar{2}$.5679934
$\frac{x^2}{A^2}$	$\bar{1}$.3979400		0.1760913
X_2	+2.000		− 6.00
X_4	+0.250		+ 0.25
X_6	−0.9375		+23.5
X_8	−1.203		−45.7
X_{10}	−0.562		−49.0
1st term	1.0000000		1.0000000
2d “	+ .0937500		+ .0138683
3d “	+ .0097656		− .0006411
4th “	+ .0002670		+ .0000009
5th “	− .0002253		+ .0000027
6th “	− .0000662		− .0000002
7th “	- .0000036		.0000000
Sum $=S$	1.1034875		1.0132306
log S_1 =	0.0427674	log S_2 =	0.0057083
“ $2\pi^2$ =	1.2953298	“ $2\pi^2$ =	1.2953298
“ $a^2(=100)$ =	2.0000000	“ $a^2(=100)$ =	2.0000000
“ $N_1(=80)$ =	1.9030900	“ $N_2(=240)$ =	2.3802112
	5.2411872		5.6812493
“ d_1 =	1.2525750	“ d_2 =	1.4600616
log M_S =	3.9886122	log M_R =	4.2211877
∴ M_S =	9741.19	M_R =	16641.32

THE SAME EXAMPLE BY JONES'S FORMULA.

We will now calculate M_S and M_R by Jones's second formula given above, using also the following equation to find $F-\Pi$:

$$\frac{k'^2 \sin\beta \cos\beta (F-\Pi)}{c} = F(k)E(k',\beta) + E(k)F(k',\beta) - F(k)F(k',\beta) - \frac{\pi}{2}$$

	For M_S	*For* M_R
A	16 cm	16 cm
a	10	10
x	8	24
$\Theta = 2\pi N$	160 π	480 π
$c = \frac{2\sqrt{Aa}}{A+a}$	0.9730085	0.9730085
$c' = \sqrt{1-c^2}$	0.2307692	0.2307692
$k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+x^2}}$	0.9299812	0.7149701
$k' = \sqrt{1-k^2}$	0.3676073	0.6991550
$\log \sin \beta \left(\sin \beta = \frac{c'}{k'}\right)$	9.7977938	9.5186043
$F(k)$	2.4373371	1.8636661
$E(k)$	1.1323456	1.3449927
$\frac{F-E}{k^2}$	1.5088957	1.0146546
$F(k', \beta)$	0.6852557	0.3394833
$E(k', \beta)$	0.6721988	0.3333201
$\frac{k'^2 \sin \beta \cos \beta (F-\Pi)}{c}$	−0.8266738	−1.1256799
$\frac{c'^2}{c^2}(F-\pi)$	−0.6851799	−0.4045298
$\log\left\{\frac{F-E}{k^2}+\frac{c'^2}{c^2}(\mathrm{F}-\Pi)\right\}$	$\bar{1}.9157773$	$\bar{1}.7854187$
$\log(\Theta(A+a)ck)$	4.0728340	4.4357689
$\log M$	3.9886113	4.2211876
	$M_S = 9741.17$ cm	$M_R = 16641.32$

M_S differs from the value obtained by formula (12) by 2 parts in a million, M_R is identical.

M_S is the mutual inductance of the winding O_1A on S. The inductance M_1 of the whole coil AB on S is twice as much, that is

$$M_1 = 19482.34$$

The inductance of AB on R is M_R above, minus the inductance of O_2B on R which is the same as that of O_1A on S, that is, M_S. Therefore,

$$M_2 = 16641.32 - 9741.17 = 6900.15$$

Hence $M_1 - M_2 = 12582.19$ cm.

The force of attraction of the one winding AB in dynes is

$$\frac{1}{2}f = i_1 i_2 n_2 \, (M_1 - M_2).$$

The force due to the second winding BC is equal to this. Suppose $i_1 = i_2 = 1$ ampere $= 0.1$ c.g.s. unit of current and $n_2 = 10$ turns per cm. Then

$$i_1 i_2 n_2 = 0.10$$

$$\begin{aligned} \therefore f &= 0.20 \times 12582.19 \text{ dynes} \\ &= 2516.438 \text{ dynes} \\ 2f &= 5032.876 \text{ dynes} = \text{change of force on reversal of current} \\ &= 5.1356 \text{ gms where } g = 980. \end{aligned}$$

If there are two sets of coils, one on each side of the balance, as in the ampere balance built for the National Physical Laboratory, the force would be doubled again.

In calculating the mutual inductance of the disk and surrounding solenoid in the Lorenz apparatus the series (12) will be more convergent when the winding is long, and of course more convergent when the disk is not of too great diameter.

EXAMPLE 28. MUTUAL INDUCTANCE OF CAMPBELL'S FORM OF STANDARD BY FORMULAE (41) AND (42).

A cylinder 20 cm in diameter has two coils of 50 turns each wound as shown in Fig. 49, each covering 5 cm (=AB) with an interval of 10 cm between (=AA′). A secondary coil of 1000 turns of finer wire is wound in a channel S, with a mean radius of 14.5 cm. The magnetic field near S, due to the double solenoid, is very weak, and is zero at some point; at this place M will be a maximum, and variations in M due to small changes in A will be very small. To calculate M for the solenoid AB and the coil S, we take two cases, as in the preceding example. First, M for S and a winding O_2B of 100 turns; second, M for S and O_2A of 50 turns. The difference will be M for S and the actual winding AB. Or, supposing AB to have 100 turns, M will be the same as for AB of 50 and A′B′ of 50. Using formula (41) we have the following values:

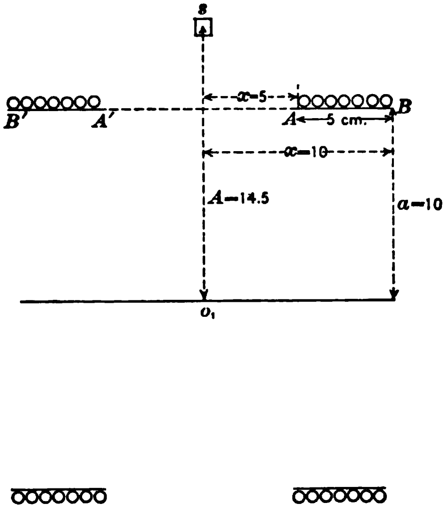

Fig. 49.

	For M_1	For M_2
$a=$	10	$=$ 10
$A=$	14.5	$=$ 14.5
$x=b=$	10	$=$ 5.0
$\log k=$	1.9590874	$=$ 1.98366715
$\gamma=$	$65^\circ 31' 7''.32$	$=$ $74^\circ 23' 38''.88$
$k'=$	$\sqrt{0.1717243}$	$=$ $\sqrt{0.0723711}$
$\beta=$	$26^\circ 18' 36''.85$	$=$ $43^\circ 3' 33''.02$
$\gamma'=$	$24^\circ 28' 52''.71$	$=$ $15^\circ 36' 21''.19$
$F=$	2.3267717	$=$ 2.7312000
$E=$	1.1590043	$=$ 1.0812388
$\frac{c}{k}(F-E)=$	1.2612955	$=$ 1.6839704
$F(k,\beta)=$	0.4618972	$=$ 0.7561693
$E(k,\beta)=$	0.4565314	$=$ 0.7469284
$\psi=$	-1.0479406	$=-0.7784355$

$$\begin{array}{rll}
\frac{A-a}{b}\psi = & -0.4715733 & = -0.7005920 \\
\frac{c}{k}(F-E)+\frac{A-a}{b}\psi = & 0.7897222 & = 0.9833784 \\
n_1 n_2 = & 200{,}000 & = 100{,}000 \\
M_1 = & 24{,}313{,}660 \text{ cm} & M_2 = 15{,}137{,}960 \text{ cm} \\
= & 24.31366 \text{ millihenrys} & = 15.13796 \text{ millihenrys} \\
M = & M_1 - M_2 = 9.1757 \text{ millihenrys.} &
\end{array}$$

Campbell gives[25] the value of M as 9.1762 millihenrys, but for want of any particulars of his calculation we do not know wherein the difference lies.

We have worked this problem out also by formula (42) with the following results:

$$\begin{array}{rl}
M_1 = & 24.31369 \text{ millihenrys} \\
M_2 = & 15.13917 \quad \text{" \quad "} \\
M = & 9.1745 \quad \text{" \quad "}
\end{array}$$

The value of M_1 agrees with that found by (41) to about one part in a million. M_2 is, however, a little larger, making M smaller. This is due to the fact that formula (42) is not as convergent for $x=5$ in this problem as for $x=10$, and hence the terms neglected after the seventh are appreciable. Hence, for so short a coil as this, formula (40) or (41) will give a more accurate result than (42).

5. CIRCULAR RINGS OF CIRCULAR SECTION.

EXAMPLE 29. COMPARISON OF FIVE FORMULÆ FOR THE SELF INDUCTANCE OF CIRCLES.

For a circle of radius $a=25$ cm and $\rho=0.05$ cm we obtain from the five formulæ the following values of L:

By Kirchhoff's formula (45)	$L=654.40496\pi$ cm
By Maxwell's formula (47)	$L=654.40533\pi$ cm
By Wien's formula (49)	$L=654.40537\pi$ cm
By Rayleigh and Niven's (51)	$L=654.40548\pi$ cm
By Wien's second formula (50)	$L=654.40617\pi$ cm

Thus for so small a value of $\frac{\rho}{a}$ as 1/500 any of these formulæ is sufficiently accurate, the greatest difference being less than 1 in a million, except in the case of formula (50).

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[25] A. Campbell, Proc. Roy. Soc., **79**, p. 428; 1907.

EXAMPLE 30. SECOND COMPARISON OF FIVE FORMULÆ FOR CIRCLES.

For a circle of radius $a=25$ cm, $\rho=0.5$ cm, $\frac{\rho}{a}$ being 1/50.

By Kirchhoff's formula (45)	$L=424.1464\pi$ cm
By Maxwell's formula (47)	$L=424.1734\pi$ cm
By Wien's formula (49)	$L=424.1761\pi$ cm
By Rayleigh and Niven's formula (51)	$L=424.1781\pi$ cm
By Wien's second formula (50)	$L=424.2326\pi$ cm

EXAMPLE 31. THIRD COMPARISON OF FIVE FORMULÆ FOR CIRCLES.

For a circle of radius $a=10$ cm, $\rho=1.0$, $\frac{\rho}{a}=1/10$.

By Kirchhoff's formula (45)	$L=105.281\pi$ cm
By Maxwell's formula (47)	$L=105.476\pi$ cm
By Wien's formula (49)	$L=105.497\pi$ cm
By Rayleigh and Niven's formula (51)	$L=105.517\pi$ cm
By Wien's second formula (50)	$L=105.902\pi$ cm

It will be seen that for the smallest ring of radius 10 cm and diameter of section 2 cm Maxwell's formula gives a result 1 part in 5,000 too small and Rayleigh and Niven's a value as much too large, while the simple approximate formula of Kirchhoff is in error by 1 in 500. For the larger ring the differences are much smaller.

Wien's second formula gives appreciably larger values than the others, as it should do.

6. SINGLE LAYER SOLENOIDS.

EXAMPLE 32. RAYLEIGH AND NIVEN'S FORMULA (54) AND CORRECTION FORMULA (59) COMPARED WITH THE SUMMATION FORMULA (60).

$a=25$ cm, $b=1$ cm, $n=10$ turns. Suppose the bare wire is 0.8 mm diameter, the covered wire 1.0 mm.

By formula (54)

$$L_s=4\pi\times 25\times 100\left\{\log_e 200-\frac{1}{2}+\frac{1}{20,000}\left(\log_e 200+\frac{1}{4}\right)\right\}$$

$$=10,000\pi\times 4.798595$$

$$=47,985.95\ \pi\ \text{cm}$$

which is the value of L for a current sheet.

The correction ΔL by formula (59) is

$$\Delta L = 1000\ \pi\ (A+B)$$

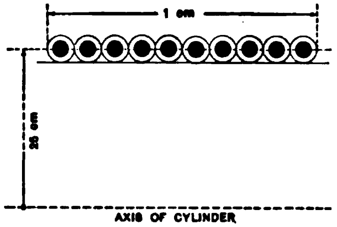

Fig. 50.

Since $D = 1.0$ mm and $d = 0.8$ mm,

$$d/D = 0.8$$

By Table VII, $A = 0.3337$
" " VIII, $B = 0.2664$

$$A+B = 0.6001$$

$$\therefore\quad \Delta L = 600.1\ \pi \text{ cm.}$$

$$\therefore\quad L = 47{,}985.95\ \pi - 600.1\ \pi$$

or,

$$L = 47{,}385.85\ \pi \text{ cm.}$$

The value of L may also be calculated by the summation formula (60), using Wien's formula (49) for L_1 and Maxwell's formula (9) for the M's. The following are the values of the ten terms of (60) and the resulting value of L:

$$
\begin{aligned}
10\ L &= 6767.20\ \pi \text{ cm.}\\
18\ M_{12} &= 10081.66\ \pi\\
16\ M_{13} &= 7852.54\ \pi\\
14\ M_{14} &= 6303.44\ \pi\\
12\ M_{15} &= 5057.87\ \pi\\
10\ M_{16} &= 3991.89\ \pi\\
8\ M_{17} &= 3047.79\ \pi\\
6\ M_{18} &= 2193.46\ \pi\\
4\ M_{19} &= 1408.98\ \pi\\
2\ M_{110} &= 680.99\ \pi\\
\text{Sum} = L &= 47385.82\ \pi \text{ cm.}
\end{aligned}
$$

The difference of less than one in a million between the results obtained by formulæ (54) and (59) combined and formula (60) is a good check on the corrections of (59), which amount in this case to more than one per cent of the value of the self-inductance. Formula (54) for as short a coil as this is very accurate, the next term, the fourth term of (56), being inappreciable.

EXAMPLE 33.

As an extreme case to test the use of formulæ (54) and (59) we may calculate the self-inductance of a single turn of wire. Let us take the particular case already calculated by Maxwell's and Wien's formulæ, (47) and (49), example 29. The radius $a=25$ cm, the diameter of the bare wire $=1$ mm. We may now assume that the wire is covered and that the diameter D is 2 mm. Then $\frac{d}{D}=0.5$. In using Rayleigh's current sheet formula we take the length of the equivalent current sheet as equal to D. We thus have

$$L_s=4\pi a\left\{\log_e\frac{200}{0.2}-\frac{1}{2}+\frac{0.04}{20{,}000}\left(\log_e\frac{200}{0.2}+\frac{1}{4}\right)\right\}$$

$$=100\pi\left\{6.907755-0.5+\frac{7.16}{500{,}000}\right\}$$

$$=640.777\pi \text{ cm.}$$

From Tables VII and VIII $A=-0.1363$ and $B=0$. Thus, since $n=1$, $\Delta L=4\pi a\times(-0.1363)=-13.63\pi$, and being negative is added to L_s. Hence

$$L=(640.777+13.63)\pi$$
$$=654.407\pi.$$

This is practically identical with the value (654.405π cm) given by the other formulæ, example 29, the slight difference being due to the fact that the correction term A is carried only to four places of decimals.

If we had taken the bare wire of diameter 0.1 cm as equivalent to a current sheet 0.1 cm long in the above formulæ for L_s, we should have obtained a different value for L_s, but in that case $\frac{d}{D}$ would be unity and A would be $+.5568$. The resulting value of L would, however, be the same as above.

EXAMPLE 34. COFFIN'S FORMULA (56) COMPARED WITH LORENZ'S (58).

We will use for this case a single layer coil wound on an accurately measured marble cylinder belonging to the Bureau of Standards.

Length of winding, $l=30.5510$ cm $=b$ in formula (58)
Radius " " $a=27.0862$ cm
Number of turns, $n=440$.

By (56)

$$L=4\pi\overline{440}^2\times 27.0862\Big\{1.4590689+0.0878241-0.0020427$$
$$+.0001651-0.0000204\Big\}$$

$$=4\pi\overline{440}^2\times 27.0862\times 1.5449950$$
$$=101810000\text{ cm}=0.1018100\text{ henry.}$$

By (58)

$$d^2=\ a^2+b^2=3868.0128$$
$$4a^2-b^2=2001.2858$$
$$\gamma=60^\circ\ 34'\ 43.''61$$
$$\log F=0.3369388$$
$$\text{"}\ \ E=0.0811833$$

Then

$$L=\frac{4\pi.\overline{440}^2}{3\,(30.551)^2}.\Big\{150050.14+126105.38-158977.00\Big\}$$

or, $L=101810200\text{ cm}=0.1018102\text{ henry.}$

The agreement is very close indeed, and a like agreement could be depended upon for all coils having the ratio of length to radius as small as in this case. For longer coils the difference rapidly increases.

EXAMPLE 35. STRASSER'S FORMULA (61) COMPARED WITH (54) AND (59) AND WITH (60.)

Take the coil of 10 turns used in example 32.

$a=25,\quad d=0.10\quad \rho=0.04,\quad n=10.$

From Table V, $A=97.92,\ B=4187.55$

Substituting in (61),

$$L=100\pi\Big[\ 10(\log_e\frac{200}{.04}-1.75)+90(\log_e\frac{200}{0.1}-2)-97.92$$
$$+\frac{100}{5000}\Big\{(3\log_e\frac{200}{0.1}-1)\frac{9900}{12}-4187.55\Big\}\Big]$$

or, $L=100\pi\Big[473.8329+0.0276\Big]=47386.05\ \pi\text{ cm.}$

This very close agreement with the results by the other two methods (see example 32) is a confirmation of the accuracy of the constants A and B of Table V. Of course, a close agreement with (60) is to be expected, for (61) is derived directly from (60).

EXAMPLE 36. FORMULÆ (62) AND (63) FOR TOROIDAL COILS.

Professor Frölich's standard of self-inductance had the following dimensions:

$r_2 = 35.05377$ cm = outer mean radius.
$r_1 = 24.97478$ cm = inner mean radius.
$h = 20.08455$ cm = height, center to center of wire.
$\rho = 0.011147$ cm = radius of wire.
$n = 2738$ = whole number of turns.

These values substituted in (62) give

$$L_s = 0.1020893 \text{ henry}.$$

The correction $\varDelta L = -2nl\,(A+B)$ to be substituted in (63) to give the true value of L is found as follows:

The mean spacing of the winding is $D = \pi \dfrac{r_1 + r_2}{n} = 0.0689$

The diameter of the bare wire $d = 2\rho \qquad = .0223$

$\therefore \quad d/D \qquad = 0.324$

From Table VII,

$$\begin{aligned} A &= -0.57 \\ B &= +0.33^{88} \\ \hline \therefore A+B &= -0.24 \end{aligned}$$

$2nl = 2 \times 2738 \times 60.327 = 330300$ cm = whole length of wire in winding.

$$-2nl(A+B) = +79{,}300 \text{ cm}$$

$$\begin{aligned} &= 0.0000793 \text{ henry} \\ L_s &= 0.1020893 \quad \text{"} \\ \hline L &= 0.1021686 \quad \text{"} \end{aligned}$$

Thus, the correction increases the value of the self-inductance. If the insulation were thinner and the wire thicker (with the same pitch) the correction might be of opposite sign. Thus, if ρ were .02

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[88] This Bulletin, 4, p. 141; 1907. This value applies to any toroidal coils.

and hence d/D were 0.58, A would be $+0.012$ and ΔL would then be $-.0001130$ and $L=0.1019763$ henry, considerably less than the preceding value.

7. CIRCULAR COILS OF RECTANGULAR SECTION.

EXAMPLE 37. MAXWELL'S APPROXIMATE FORMULÆ (64), (65) AND PERRY'S APPROXIMATE FORMULA (66) COMPARED WITH WEINSTEIN'S FORMULA (68).

Suppose a coil of mean radius 4 cm, with 100 turns of insulated wire, wound in a square channel 1 × 1 cm.

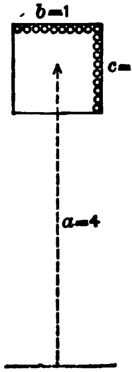

Fig. 51.

Substituting in (64) $a=4$, $n=10$, $R=0.44705$ (the g. m. d. of a square 1 cm on a side) we have

$$L=4\pi\ 100\left[\log_e\frac{32}{.44705}-2\right]$$

$$=1.141 \text{ millihenrys.}$$

This is a first approximation to the self-inductance of the coil.

Formula (65) gives a second approximation as follows:

$$L=4\pi\ 100\left[\log_e\frac{32}{0.44705}\left(1+\frac{3\times\overline{0.447}^2}{256}\right)-\left(2+\frac{\overline{0.447}^2}{256}\right)\right]$$

$$=1.146 \text{ millihenrys.}$$

Perry's approximate formula, which applies only to relatively short coils, happens to give a very close approximation for this case. Substituting in (66), the above values, and also $b=c=1$,

$$L=\frac{4\pi\ 100\times 16}{0.9268+0.44+0.39}$$

$$=1.144 \text{ millihenrys.}$$

Substituting in the more accurate formula (68) of Weinstein we shall obtain a value with which to compare the above approximations.

$$L=1600\pi\left[\left(1+\frac{1}{384}\right)\log_e\frac{32}{1}+0.03657\times\frac{1}{16}-1.194914\right]$$

$$=1.147 \text{ millihenrys.}$$

For $a=4$, $b=2$, $c=1$ $n=200$

Formula (64) gives 3.750 millihenrys
" (65) " 3.787 "
" (66) " 3.661 "
" (68) " 3.805 "

For $a=10$, $b=1$, $c=1$, $n=100$

Formula (64) gives 4.005 millihenrys
" (65) " 4.007 "
" (66) " 3.993 "
" (68) " 4.008 "

It will be seen that formula (66) does not give as close approximations as the others, except in the case of the first example, where it happens to give a value very close to that given by (68). All the values, those of (68) included, are subject to correction by (72) when the coil is wound with round insulated wire.

EXAMPLE 38. FORMULÆ (68) AND (69) COMPARED WITH CURRENT-SHEET FORMULÆ.

As a test of these formulæ we may calculate the self-inductance of a single turn of wire, using the case already calculated in example 33; that is, a circle of radius $a=25$ cm, and the diameter of the bare wire is 1 mm. Substituting these values in (68) we have

$$L=100\pi\left[\left(1+\frac{.01}{15000}\right)\log_e 2000+\frac{.03657}{(250)^2}-1.194914\right]$$

$$=640.5995\ \pi\ \text{cm}.$$

Substituting in (69),

$$L=100\pi\left[\left(1+\frac{.01}{15000}\right)\log_e\frac{200}{\sqrt{.02}}-0.848340+\frac{.01\times.8162}{10000}\right]$$

$$=640.5995\ \pi\ \text{cm},$$

agreeing with the value by (68).

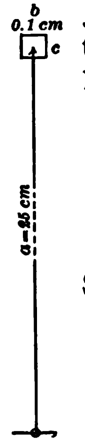

Fig. 52.

These values are for a conductor of square cross section. To reduce to a circular section of same diameter (0.1 cm) we must apply the second correction term of (72); that is, add to the above value

$$\Delta L = 4\pi a \times 0.138060$$

$$\text{Thus, } L = (640.5995 + 13.8060)\pi$$

$$= 654.4055\pi \text{ cm},$$

which agrees with the value found for the self-inductance of a round wire 0.1 cm diameter, bent into a circle of 25 cm radius, by formula (49), example 29, and formulæ (54) and (59), example 33.

EXAMPLE 39. STEFAN'S FORMULA (69) COMPARED WITH (54) BY MEANS OF ROSA'S CORRECTION FORMULA (70).

Suppose a coil of mean radius 10 cm, wound with 100 turns in a square channel 1 × 1 cm. Assuming the current uniformly distributed we obtain from (69), in which $y_1 = 0.848340$, $y_2 = 0.8162$,

$$\log_e \frac{8a}{\sqrt{b^2+c^2}} = \log_e \frac{80}{\sqrt{2}} = 4.03545$$

$$L_u = 4\pi \times 100{,}000\left[\left(1 + \frac{4}{9600}\right) 4.03545 - 0.84834 + 0.00051\right]$$

$$= 4\pi \times 318{,}930 \text{ cm}$$

$$= 4.00779 \text{ millihenrys.}$$

By formula (54) we have for the self-inductance of a current sheet for which $a = 10$, $b = 1$, $n = 1$,

$$L_s = 4\pi \times 38.83475 \text{ cm}$$

This is larger than the value for the coil of section 1 × 1 by ΔL, the value of the latter being given by formula (70).

By Table IX, $A = 0.6942$. More closely, it is 0.69415.[89]

By Table X, $B = 0$. In this case $n' = 1$. Hence,

$$\Delta_1 L = 4\pi \times 10 \times 0.69415 = 4\pi \times 6.9415 \text{ cm}$$

$$\therefore L_1 = 4\pi\,(38.83475 - 6.9415) = 400.782 \text{ cm.}$$

This is the value of the self-inductance for one turn only, the current being uniformly distributed. For 100 turns L is 10^4 times as great.

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[89] This Bulletin, 4, p. 369; 1907.

$$\therefore\ L_u = 4.00782 \text{ millihenrys.}$$

This value agrees with the above value by Stefan's formula within less than 1 part in 100,000.

For a coil of the same radius, but of length $b = 10$ cm, $c = 1$ cm, wound with 10 layers of 100 turns each, we have the following values:

By Stefan's formula, $y_1 = 0.59243$, $y_2 = 0.1325$

$$\begin{aligned} L_u &= 4\pi \times 10 \times \overline{1000}^2 \times 1.55536 \\ &= 195.452 \text{ millihenrys.} \end{aligned}$$

By (54) the current sheet value of L for 10 turns is

$$\begin{aligned} L_{10} &= 4\pi \times 10 \times 100 \times 1.65095 \\ &= 4\pi \times 1650.95. \end{aligned}$$

The correction for depth of section by (70) is, since by Tables IX and X, $A = 0.6942$, $B = 0.2792$, and therefore $A + B = 0.9734$

$$\begin{aligned} \Delta_1 L &= 4\pi 10 \times 10 \times 0.9734 \\ &= 4\pi \times 97.34 \\ \therefore\ L_u = L_{10} - \Delta_1 L &= 4\pi(1650.95 - 97.34) \\ &= 4\pi \times 1553.61 \text{ cm. for 10 turns.} \end{aligned}$$

For $n = 1000$ turns the self-inductance will be $\overline{100}^2$ times as great.

$$\begin{aligned} L_u &= 4\pi \times 15.5361 \times 10^6 \text{ cm} \\ &= 195.232 \text{ millihenrys.} \end{aligned}$$

This value is about 1 part in 900 smaller than the above value, showing that Stefan's formula gives too large results by that amount for a coil of this length. If the coil were twice as long, the error would be about ten times as great.

It is interesting to obtain by this method an estimate of the error by Stefan's formula for coils longer than those for which it is intended. For short coils it is seen to be very accurate, subject always to the corrections of formula (72), and for longer coils it gives a good approximation. The method of (70), however, applies to coils of any length.

EXAMPLE 40. STEFAN'S FORMULA (69) COMPARED WITH (60) AND WITH STRASSER'S (61) FOR COILS OF FEW TURNS, USING THE CORRECTION FORMULA (72).

Coil of 2 turns of wire, 0.4 mm diameter, wound in a circle of 1.46 cm radius with a pitch of 2 mm. Stefan's formula assumes a uniform distribution over a rectangular section. Suppose a section as shown in Fig. 53, 4×2 mm, with one turn of wire in the center of each square. For the rectangular section, with the current uniformly distributed, the self-inductance by Stefan's formula is with $a = 1.46$, $c/b = 0.5$, $y_1 = 0.7960$, $y_2 = 0.3066$, $L_u = 4\pi a n^2 \times 2.4763 = 4\pi a n \times 4.9526$, n being 2. To reduce this to the case of a winding of 2 turns of wire as shown we must apply the corrections given by (72) thus:

$$\begin{array}{lll} \log D/d = \log_e 5 & = & 1.60944 \\ \text{second term} & = & 0.13806 \\ \text{third term } E & = & \underline{0.00653} \\ & & 1.7540 \end{array}$$

$$\begin{aligned} \therefore \Delta L &= 4\pi a n \times 1.754 \\ L &= L_u + \Delta L = 4\pi a n \times 6.7066 \\ &= 246.1 \text{ cm.} \end{aligned}$$

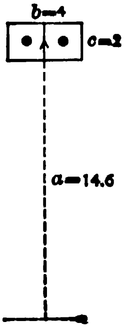

Fig. 53.

By the summation formula (60) we have in this case

$$\begin{aligned} L &= 2L_1 + 2M_{12} \\ &= 4\pi a[9.2400 + 4.1606] \\ &= 245.86 \text{ cm.} \end{aligned}$$

The value by Strasser's formula is the same as by the summation formula to which it is equivalent. We have also used formulæ (54) and (59) for this case and have obtained 246.0.

This is one of several problems calculated by Drude[90] by Stefan's formula. Drude concluded that Stefan's formula was inapplicable to such coils, as it gave results from 10 to 25 per cent too large. His trouble was, however, due to taking the length of the coil as the distance between the center of the first wire and the center of the last (instead of n times the pitch) and neglecting the correction terms of formula (72). As we have seen above, Stefan's formula when properly used can be depended upon to give accurate results for short coils, and results within less than 1 per cent for coils of length equal to the radius of the coil.

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[90] Wied. Annal., 9, p. 601; 1902.

We have calculated several other cases given by Drude and give below the results, together with his experimental values. The radius is the same in each case, and the numbers in the first column are the number of turns in the several coils.

n	By Stefan's Formula (69) and (72)	By Rayleigh's Formula (54) and (59)	By Strasser's Formula (61) or (60)	Drude's Observed Values (Values of L in Centimeters)
2	246.1	246.0	245.9	238.5
4	711.9	711.1	710.8	697.9
6	1298.7	1297.7	1297.8	1271.4
9	2318.0	2313.0	2315.7	2300.1

It will be seen that the values by the different formulæ agree very closely, and that the experimental values agree as closely as could be expected for such small inductances.

EXAMPLE 41. COHEN'S FORMULA (71) COMPARED WITH (70).

A solenoid of length $l=50$ cm, mean radius 5 cm, depth of winding 0.4 cm, is wound with 4 layers of wire of 500 turns each. Substituting these values in (68) we have ($n=10$)

$$L_s = 16\pi^2 n^2 (1144.3+3336.0-10.84-2.07)$$
$$=70.551 \text{ millihenrys.}$$

By the second method we first find L_s by (54), then $\Delta_1 L$ by (70), and $\Delta_2 L$ by (72)

$$\begin{aligned} L_s &= 72.648 \text{ millihenrys} \\ -\Delta_1 L &= -2.167 \quad \text{"} \\ \Delta_2 L &= \underline{\ \ 0.048} \quad \text{"} \\ L &= 70.529 \quad \text{"} \end{aligned}$$

This shows a very close agreement between (68) and (67).

In calculating L_s we may use Table IV. Since $d/l=0.2$

$$Q=3.6324, \quad an^2 = 5\times\overline{2000}^2 = 20,000,000$$
$$L_s = 3.6324\times 20,000,000 \text{ cm}$$

or,

$$L_s = 72.648 \text{ millihenrys.}$$

EXAMPLE 42. FORMULÆ (54) AND (59) COMPARED WITH (69) AND (72) FOR COIL OF 20 TURNS WOUND WITH A SINGLE LAYER.

$$a=25, \quad b=2 \text{ cm}, \quad c=0.1 \text{ cm}, \quad n=20.$$

Diameter of bare wire 0.6 mm, of covered wire 1.0 mm.

In the last case we obtained the self-inductance of the coil by two distinct methods, the first being the method of summation, the second by assuming the current uniformly distributed over the section, and then applying the three corrections C, F, E. In this problem we may first calculate L by use of the current sheet formula (54), and then apply the corrections for section, A and B formula (59); and, second, by Stefan's formula for uniform distribution, and apply the three corrections C, F, E, which give the value for a winding of round insulated wires.

Rayleigh's formula for this example gives:

$$L=4\pi a n^2\left\{\log_e 100-0.5+\frac{4}{20{,}000}\left(\log_e 100+\frac{1}{4}\right)\right\}$$

$$\begin{aligned}
\log_e 100 &= 4.605170 \\
\frac{4}{20{,}000}\left(\log_e 100+\frac{1}{4}\right) &= \underline{0.000971} \\
&\ 4.606141 \\
&\ \underline{-0.500000} \\
&\ 4.106141
\end{aligned}$$

$$4\pi a n^2=40{,}000\pi, \quad \therefore L_s=164{,}245.64\pi \text{ cm}.$$

This is the self-inductance of a winding of 20 turns of infinitely thin tape, each turn being 1 mm wide, with edges touching without making electrical contact, which arrangement fulfills the conditions of a current sheet. To reduce this to the case of round wires we must apply the corrections A and B for self and mutual induction.[91]

$$\begin{aligned}
&\text{By Table VII, for } d/D=0.6, \ A=.0460 \\
&\text{By Table VIII, for } n=20, \quad B=\underline{.2974} \\
&\qquad\qquad\qquad\qquad\qquad A+B=.3434 \\
&\qquad\qquad 4\pi a n= \quad 2{,}000\pi \\
&\Delta L=4\pi a n(A+B)= \quad 686.8\pi \text{ cm} \\
&\qquad L=L_s-\Delta L=163{,}558.84\pi \text{ cm}.
\end{aligned}$$

By Stefan's formula we find, substituting the above values of a, n, b, c, and taking $y_1=.548990$ and $y_2=.1269$

$$L_u=162{,}234.60\pi \text{ cm}.$$

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[91] Rosa, This Bulletin, **2**, p. 161; 1906.

The correction E for a single layer coil of 20 turns is given on page 49. The three corrections are then as follows:

$$C = .13806$$

$$F = .51082 = \log_e \frac{10}{6}$$

$$E = \underline{.01357}$$

$$.66245$$

$$\therefore \varDelta L = 4\pi an(C+F+E) = 1324.90\pi \text{ cm.}$$

$$\therefore L = L_u + \varDelta L = 163{,}559.50\pi \text{ cm.}$$

This value of L is greater than the value found by the other method by only four parts in a million. Thus we see that the method of calculating L_u by Stefan's or Weinstein's formula and applying the corrections C, F, E gives practically identical results with the method of summation and also with the current sheet method *for short coils*. When, however, the coils are longer, the agreement is not so good, for the reason that the formula of Weinstein (and Stefan's, derived from it) is not as accurate when the section of the coil is greater. Thus if the coil in the above problem had been 5 cm long and 2.5 mm deep and wound with 20 turns of heavier wire, the difference would have been 1 part in 25,000 (still very good agreement), and if it were 10 cm long and 0.5 cm deep (the radius being 25 cm) it would have been 1 part in 2,200. For most experimental work, therefore, Stefan's formula is amply accurate.

8. LINEAR CONDUCTORS.

EXAMPLE 43. FORMULÆ (73), (74), (75), AND (76).

A straight copper wire 200 cm long and 0.2 cm diameter will have a self-inductance by formula (74) of

$$L = 200\left(\log_e \frac{200}{0.1} - \frac{3}{4}\right) = 1370.18 \text{ cm.}$$

If it were twice as long

$$L = 400\left(\log_e \frac{400}{0.1} - \frac{3}{4}\right) = 3017.62 \text{ cm.}$$

The more exact formula (73) gives practically the same result where ρ is so small compared with l.

If the wire were of iron with a permeability of 1000, we should have in the first case for $l = 100$

$$L = 200\ (\log_e 2000 - 1 + 250) = 51320 \text{ cm.}$$

For sufficiently rapid oscillations so that the current may be considered to be confined to the surface of the wire

$$L = 200\ (\log_e 2000 - 1) = 1320.18 \text{ cm.}$$

If the length of the conductor were 10 meters and the diameter 0.2 cm as before the self-inductance by (74) would be

$$\begin{aligned} L &= 2000\left(\log_e 20000 - \frac{3}{4}\right) = 18307.0 \text{ cm} \\ &= 18.307 \text{ microhenrys.} \end{aligned}$$

EXAMPLE 44. FORMULÆ (77) AND (78).

Two parallel copper wires of length 100 cm and distance apart 200 cm will have a mutual inductance of

$$\begin{aligned} M &= 2\left[100 \log_e \frac{100 + 100\sqrt{5}}{200} - 100\sqrt{5} + 200\right] \\ &= 200\left[\log_e \frac{1+\sqrt{5}}{2} - \sqrt{5} + 2\right] \\ &= 200\,(\log_e 1.61803 - 0.2361) \\ &= 49.02 \text{ cm.} \end{aligned}$$

If the length of each conductor were 200 cm and the distance apart 100 cm, then

$$\begin{aligned} M &= 400\left[\log_e \frac{2+\sqrt{5}}{1} - \frac{\sqrt{5}}{2} + \frac{1}{2}\right] \\ &= 330.24 \text{ cm.} \end{aligned}$$

The approximate formula (78) is only applicable when the length of the conductors is great compared with their distance apart. Suppose two conductors 10 meters long are 10 cm apart, then by (78)

$$\begin{aligned} M &= 2000\left[\log_e \frac{2000}{10} - 1 + \frac{10}{1000}\right] \\ &= 2000\ [5.2983 - 0.990] \\ &= 8616.6 \text{ cm} = 8.6166 \text{ microhenrys.} \end{aligned}$$

EXAMPLE 45. FORMULÆ (79) AND (80).

Suppose a return circuit of two parallel wires, each 10 meters long and 0.2 cm diameter, distant apart 10 cm, center to center. The self-inductance of the circuit, neglecting the ends, is by (80)

$$L=4000\left[\log_e \frac{10}{0.1}+\frac{1}{4}-\frac{10}{1000}\right]$$
$$=4000\times 4.8452$$
$$=19380.8 \text{ cm}=19.3808 \text{ microhenrys.}$$

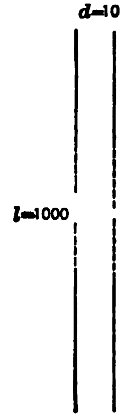

Fig. 54.

We have already calculated (example 43) the self-inductance of one of these two wires by itself. Doubling the value we have 36.6140 microhenrys as the self-inductance of two wires in series. In example 44 we calculated the mutual inductance of these two wires. Doubling the value for M we have 17.2332 microhenrys. The resultant self-inductance of the circuit (neglecting the ends) is

$$L=2L_1-2M=36.6140-17.2332$$
$$=19.3808 \text{ microhenrys.}$$

as found above by formula (78).

Taking account of the ends neglected above, we should find that $2L_1$ for the two ends by (74) is 181.9 cm and $2M$ by (77) is practically zero. Hence the self-inductance of the circuit is, including the ends,

$$L=19.5627 \text{ microhenrys.}$$

EXAMPLE 46. FORMULA (81) FOR THE MUTUAL INDUCTANCE OF ADJACENT CONDUCTORS IN THE SAME STRAIGHT LINE.

When the two conductors are of equal length, $l=m$, and (81) becomes

$$M=2\,l \log_e 2=2\,l\times 0.69315 \text{ cm.}$$

If $l=1000$ cm, $M=1386.3$ cm.
If $m=1000\,l$, (81) gives

$$M=l\log_e 1001+1000\,l\log 1.001$$
$$=l\log_e 1001+l \text{ approximately.}$$

If $l=1$ cm, we have

$$M=\log_e 1001+1000 \log_e 1.001$$
$$=6.909+0.999=7.908.$$

The self-inductance of the short wire AB, suppose 1 cm long and of 1 mm radius, is

$$L=2\left(\log_e \frac{2}{0.1}-.75\right)=2\ (2.9957-.75)=4.4915 \text{ cm},$$

which is a little more than one-half of the mutual inductance of AB and BC, BC being 1000 times the length of AB.

In closed circuits, all the magnetic lines due to a circuit are effective in producing self-inductance, and hence the self-inductance is always greater than the mutual inductance of that circuit with any other, assuming one turn in each. But with open circuits, as in this case, we may have a mutual inductance between two single conductors greater than the self-inductance of one of them.

EXAMPLE 47. FORMULA (83) FOR THE SELF-INDUCTANCE OF A RECTANGULAR BAR.

In formula (83), substituting $l=1000$, and $\alpha+\beta=2$ for a square bar 1000 cm long and 1 square cm section, we have, neglecting the small last term,

$$L=2000\left[\log_e \frac{2000}{2}+\frac{1}{2}\right]$$
$$=2000\ (6.908+0.5)=14816 \text{ cm}$$
$$=14.816 \text{ microhenrys.}$$

This would also be the self-inductance for any section having $\alpha+\beta=2$ cm.

EXAMPLE 48. FORMULÆ (84) AND (85) FOR THE SELF-INDUCTANCE OF A SQUARE MADE UP OF A ROUND WIRE.

If the side of the square is one meter, $a=100$ cm, $\rho=0.1$ cm, we have from (84)

$$L=800\ (\log_e 1000-0.524)$$
$$=5107 \text{ cm}=5.107 \text{ microhenrys.}$$

If $\rho=.05$ cm,

$$L=5662 \text{ cm}=5.662 \text{ microhenrys.}$$

That is, the self-inductance of such a rectangle of round wire is about 11 per cent greater for a wire 1 mm in diameter than for one 2 mm in diameter.

If l/ρ is constant, L is proportional to l, that is, if the thickness of the wire is proportional to the length of the wire in the square, the self-inductance of the square is proportional to its linear dimensions.

EXAMPLE 49. FORMULA (86) FOR THE SELF-INDUCTANCE OF A RECTANGLE OF ROUND WIRE.

Suppose a rectangle 2 meters long and 1 meter broad.

Substituting $a=200$ cm, $b=100$, $\rho=0.1$, in (86) we have

$$L=8017.1 \text{ cm}=8.017 \text{ microhenrys.}$$

We can obtain the same result from the values of self and mutual inductances calculated in examples 43 and 44. That is, the resultant self-inductance of the rectangle is the sum of the self-inductances of the four sides, minus twice the mutual inductances of the two pairs of opposite sides. Thus

$$L=(L_1+L_3)+(L_2+L_4)-2M_{13}-2M_{24}$$

$$\begin{array}{lrrr}
\text{By example 43,} & L_1+L_3 & =6035.24 & \\
 & L_2+L_4 & =\underline{2740.36} & 8775.60 \\
\text{By example 44,} & 2M_{13} & =660.48 & \\
 & 2M_{24} & =\underline{98.04} & \underline{758.52} \\
 & & \therefore L= & 8017.08 \text{ cm} \\
 & & = & 8.0171 \text{ microhenrys.}
\end{array}$$

The agreement of this result with that obtained from formula (86) serves as a check on the latter formula, and also illustrates how the values of the self and mutual inductances of open circuits may be combined to give the self-inductance of a closed circuit.

EXAMPLE 50. FORMULÆ (87), (88), AND (89) FOR THE SELF-INDUCTANCE OF A RECTANGLE OR SQUARE MADE UP OF A BAR OF RECTANGULAR SECTION.

$$\text{Let } a=200,\ b=100,\ \alpha=\beta=1.0 \text{ cm.}$$

Substituting these values in (87) we obtain

$$\begin{aligned} L&=4\ (2971.05-1209.76-577.95-150+447.21+0.99)\\ &=5926.16 \text{ cm.}\end{aligned}$$

For a square 10 meters on a side, made of square bar 1 sq. cm cross section we have $a=1000$, $\alpha=1$; substituting in (89)

$$L=8000\,(6.908+.033)$$
$$=8000\times 6.941 \text{ cm}=55.53 \text{ microhenrys.}$$

For a circular section, diameter 1 cm, $\rho=0.5$; substituting in (84)

$$L=8000\left(\log_e 2000+\frac{1}{2000}-0.524\right)$$
$$=8000\times 7.076 \text{ cm}=56.61 \text{ microhenrys,}$$

a little more than for a square section, as would be expected.

EXAMPLE 51. FORMULA (91) FOR THE MUTUAL INDUCTANCE OF PARALLEL SQUARES.

Suppose two parallel squares each 1 meter on a side, 10 centimeters distant from one another.

$a=100$, $d=10$. Substituting in (91),

$$M=8\left[100\log_e\left(\frac{1+\sqrt{1.01}}{1+\sqrt{2.01}}\cdot\frac{\sqrt{101}}{1}\right)+\sqrt{20100}-2\sqrt{10100}+10\right]$$
$$=800\left[\log_e\left(\frac{10.1+\sqrt{101}}{1+\sqrt{2.01}}\right)+\sqrt{2.01}-2\sqrt{1.01}+0.1\right]$$
$$=1142.5 \text{ cm}=1.1425 \text{ microhenrys.}$$

EXAMPLE 52. FORMULÆ (92), (93) AND (94) FOR THE SELF AND MUTUAL INDUCTANCE OF THIN STRAIGHT STRIPS OR TAPES.

Let the tape of thin copper be 10 meters long and 1 cm wide. Substituting $l=1000$ and $b=1$ in (92) we have

$$L=2000\left(\log_e 2000+\frac{1}{2}\right)$$
$$=2000\times 8.1009=16202 \text{ cm}$$
$$=16.202 \text{ microhenrys,}$$

as the self-inductance when the conducting strip is very thin. If the tape is 2 mm thick we may allow for the effect of the thickness by using (93) and we find

$$L=2000\times 7.9009 \text{ cm}=15.802 \text{ microhenrys,}$$

which differs slightly from the preceding value.

Two such tapes edge to edge in one plane will have a mutual inductance by (91) of

$$\begin{aligned} M &= 2000\ (\log_e 2000 - 0.8863) \\ &= 2000 \times 6.7146 \text{ cm} \\ &= 13.429 \text{ microhenrys.} \end{aligned}$$

EXAMPLE 53. FORMULA (96) FOR THE SELF-INDUCTANCE OF A RETURN CIRCUIT OF TWO PARALLEL SHEETS; NON-INDUCTIVE SHUNTS.

Suppose the dimensions of a thin manganin sheet which has been doubled on itself be as follows:

$$l = 30 \text{ cm}, \quad b = 10 \text{ cm}, \quad d = 1 \text{ cm}.$$

By (111) $\log R_2 = 1.0787$

$$\log R_1 = \log_e 10 - \frac{3}{2} = 0.8026$$

$$\begin{aligned} L &\doteq 4l\ (\log R_2 - \log R_1) \\ &= 120 \times 0.2761 \\ &= 33.13 \text{ cm} \\ &= .0331 \text{ microhenrys.} \end{aligned}$$

EXAMPLE 54. FORMULA (101), 3 CONDUCTORS IN MULTIPLE.

Suppose three cylindrical conductors, each 10 meters long and 4 mm diameter, the distance apart of their centers being 1 cm. Substitute in (101) as follows:

$l = 1000$ cm, $\rho = 2$ mm, $d = 1$ cm. Then

$(r_g a^2)^{\frac{1}{3}} = 0.538$ cm and

$$L = 2000 \left(\log_e \frac{2000}{0.538} - 1 \right)$$

$$= 2000 \times 7.221 \text{ cm} = 14.442 \text{ microhenrys.}$$

If the whole current flowed through a single one of the three conductors the self-inductance would be

$$L = 2000 \left(\log_e \frac{2000}{0.2} - \frac{3}{4} \right) = 17.92 \text{ microhenrys,}$$

or about 25 per cent more than when divided among the three.

APPENDIX.

TABLES OF CONSTANTS AND FUNCTIONS USEFUL IN THE CALCULATION OF MUTUAL AND SELF-INDUCTANCE.

TABLE I.

Maxwell's Table of Values of $\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}=\left[\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right]$

(For use with Formula 1.)

	$\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}$	Δ_1		$\text{Log}\,\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
60° 0′	$\bar{1}$.499 4780	2 7868	65° 0′	$\bar{1}$.637 6633	2 7508
6′	$\bar{1}$.502 2648	2 7854	6′	$\bar{1}$.640 4141	2 7508
12′	$\bar{1}$.505 0502	2 7840	12′	$\bar{1}$.643 1649	2 7507
18′	$\bar{1}$.507 8342	2 7828	18′	$\bar{1}$.645 9156	2 7507
24′	$\bar{1}$.510 6170	2 7816	24′	$\bar{1}$.648 6663	2 7507
30′	$\bar{1}$.513 3986	2 7803	30′	$\bar{1}$.651 4170	2 7509
36′	$\bar{1}$.516 1789	2 7790	36′	$\bar{1}$.654 1679	2 7510
42′	$\bar{1}$.518 9579	2 7778	42′	$\bar{1}$.656 9189	2 7512
48′	$\bar{1}$.521 7357	2 7765	48′	$\bar{1}$.659 6701	2 7514
54′	$\bar{1}$.524 5122	2 7753	54′	$\bar{1}$.662 4215	2 7516
61° 0′	$\bar{1}$.527 2875	2 7743	66° 0′	$\bar{1}$.665 1731	2 7519
6′	$\bar{1}$.530 0618	2 7734	6′	$\bar{1}$.667 9250	2 7522
12′	$\bar{1}$.532 8352	2 7725	12′	$\bar{1}$.670 6772	2 7524
18′	$\bar{1}$.535 6077	2 7715	18′	$\bar{1}$.673 4296	2 7528
24′	$\bar{1}$.538 3792	2 7705	24′	$\bar{1}$.676 1824	2 7532
30′	$\bar{1}$.541 1497	2 7694	30′	$\bar{1}$.678 9356	2 7535
36′	$\bar{1}$.543 9191	2 7683	36′	$\bar{1}$.681 6891	2 7539
42′	$\bar{1}$.546 6874	2 7672	42′	$\bar{1}$.684 4430	2 7543
48′	$\bar{1}$.549 4546	2 7663	48′	$\bar{1}$.687 1973	2 7548
54′	$\bar{1}$.552 2209	2 7654	54′	$\bar{1}$.689 9521	2 7553
62° 0′	$\bar{1}$.554 9863	2 7645	67° 0′	$\bar{1}$.692 7074	2 7561
6′	$\bar{1}$.557 7508	2 7637	6′	$\bar{1}$.695 4635	2 7567
12′	$\bar{1}$.560 5145	2 7629	12′	$\bar{1}$.698 2202	2 7573
18′	$\bar{1}$.563 2774	2 7622	18′	$\bar{1}$.700 9775	2 7580
24′	$\bar{1}$.566 0396	2 7615	24′	$\bar{1}$.703 7355	2 7587
30′	$\bar{1}$.568 8011	2 7607	30′	$\bar{1}$.706 4942	2 7595
36′	$\bar{1}$.571 5618	2 7598	36′	$\bar{1}$.709 2537	2 7603
42′	$\bar{1}$.574 3216	2 7589	42′	$\bar{1}$.712 0140	2 7610
48′	$\bar{1}$.577 0805	2 7582	48′	$\bar{1}$.714 7750	2 7619
54′	$\bar{1}$.579 8387	2 7575	54′	$\bar{1}$.717 5369	2 7628
63° 0′	$\bar{1}$.582 5962	2 7570	68° 0′	$\bar{1}$.720 2997	2 7637
6′	$\bar{1}$.585 3532	2 7567	6′	$\bar{1}$.723 0634	2 7647
12′	$\bar{1}$.588 1099	2 7563	12′	$\bar{1}$.725 8281	2 7656
18′	$\bar{1}$.590 8662	2 7559	18′	$\bar{1}$.728 5937	2 7667
24′	$\bar{1}$.593 6221	2 7555	24′	$\bar{1}$.731 3604	2 7679
30′	$\bar{1}$.596 3776	2 7549	30′	$\bar{1}$.734 1283	2 7689
36′	$\bar{1}$.599 1325	2 7543	36′	$\bar{1}$.736 8972	2 7701
42′	$\bar{1}$.601 8868	2 7537	42′	$\bar{1}$.739 6673	2 7713
48′	$\bar{1}$.604 6405	2 7533	48′	$\bar{1}$.742 4386	2 7725
54′	$\bar{1}$.607 3938	2 7530	54′	$\bar{1}$.745 2111	2 7737
64° 0′	$\bar{1}$.610 1468	2 7527	69° 0′	$\bar{1}$.747 9848	2 7749
6′	$\bar{1}$.612 8995	2 7524	6′	$\bar{1}$.750 7597	2 7763
12′	$\bar{1}$.615 6519	2 7521	12′	$\bar{1}$.753 5360	2 7778
18′	$\bar{1}$.618 4040	2 7519	18′	$\bar{1}$.756 3138	2 7791
24′	$\bar{1}$.621 1559	2 7516	24′	$\bar{1}$.759 0929	2 7806
30′	$\bar{1}$.623 9075	2 7514	30′	$\bar{1}$.761 8735	2 7821
36′	$\bar{1}$.626 6589	2 7513	36′	$\bar{1}$.764 6556	2 7836
42′	$\bar{1}$.629 4102	2 7512	42′	$\bar{1}$.767 4392	2 7853
48′	$\bar{1}$.632 1614	2 7510	48′	$\bar{1}$.770 2245	2 7871
54′	$\bar{1}$.634 9124	2 7509	54′	$\bar{1}$.773 0116	2 7888
65° 0′	$\bar{1}$.637 6633	2 7508	70° 0′	$\bar{1}$.775 8004	2 7904

	$\log \frac{M}{4\pi\sqrt{Aa}}$	Δ_1		$\log \frac{M}{4\pi\sqrt{Aa}}$	Δ_1
70° 0′	$\bar{1}$.775 8004	2 7904	75° 0′	$\bar{1}$.918 5141	2 9472
6′	$\bar{1}$.778 5908	2 7920	6′	$\bar{1}$.921 4613	2 9522
12′	$\bar{1}$.781 3828	2 7938	12′	$\bar{1}$.924 4135	2 9572
18′	$\bar{1}$.784 1766	2 7956	18′	$\bar{1}$.927 3707	2 9623
24′	$\bar{1}$.786 9722	2 7975	24′	$\bar{1}$.930 3330	2 9676
30′	$\bar{1}$.789 7697	2 7995	30′	$\bar{1}$.933 3006	2 9729
36′	$\bar{1}$.792 5692	2 8017	36′	$\bar{1}$.936 2735	2 9783
42′	$\bar{1}$.795 3709	2 8037	42′	$\bar{1}$.939 2518	2 9838
48′	$\bar{1}$.798 1746	2 8056	48′	$\bar{1}$.942 2356	2 9895
54′	$\bar{1}$.800 9802	2 8078	54′	$\bar{1}$.945 2251	2 9951
71° 0′	$\bar{1}$.803 7880	2 8100	76° 0′	$\bar{1}$.948 2202	3 0007
6′	$\bar{1}$.806 5980	2 8124	6′	$\bar{1}$.951 2209	3 0066
12′	$\bar{1}$.809 4104	2 8148	12′	$\bar{1}$.954 2275	3 0127
18′	$\bar{1}$.812 2252	2 8172	18′	$\bar{1}$.957 2402	3 0188
24′	$\bar{1}$.815 0424	2 8195	24′	$\bar{1}$.960 2590	3 0251
30′	$\bar{1}$.817 8619	2 8220	30′	$\bar{1}$.963 2841	3 0316
36′	$\bar{1}$.820 6839	2 8245	36′	$\bar{1}$.966 3157	3 0380
42′	$\bar{1}$.823 5084	2 8270	42′	$\bar{1}$.969 3537	3 0446
48′	$\bar{1}$.826 3354	2 8297	48′	$\bar{1}$.972 3983	3 0514
54′	$\bar{1}$.829 1651	2 8323	54′	$\bar{1}$.975 4497	3 0583
72° 0′	$\bar{1}$.831 9974	2 8349	77° 0′	$\bar{1}$.978 5080	3 0652
6′	$\bar{1}$.834 8323	2 8377	6′	$\bar{1}$.981 5731	3 0723
12′	$\bar{1}$.837 6700	2 8406	12′	$\bar{1}$.984 6454	3 0795
18′	$\bar{1}$.840 5106	2 8435	18′	$\bar{1}$.987 7249	3 0869
24′	$\bar{1}$.843 3541	2 8464	24′	$\bar{1}$.990 8118	3 0944
30′	$\bar{1}$.846 2005	2 8494	30′	$\bar{1}$.993 9062	3 1020
36′	$\bar{1}$.849 0499	2 8525	36′	$\bar{1}$.997 0082	3 1099
42′	$\bar{1}$.851 9024	2 8556	42′	0.000 1181	3 1178
48′	$\bar{1}$.854 7580	2 8588	48′	0.003 2359	3 1259
54′	$\bar{1}$.857 6168	2 8620	54′	0.006 3618	3 1341
73° 0′	$\bar{1}$.860 4788	2 8653	78° 0′	0.009 4959	3 1426
6′	$\bar{1}$.863 3441	2 8688	6′	0.012 6385	3 1511
12′	$\bar{1}$.866 2129	2 8723	12′	0.015 7896	3 1598
18′	$\bar{1}$.869 0852	2 8759	18′	0.018 9494	3 1687
24′	$\bar{1}$.871 9611	2 8795	24′	0.022 1181	3 1778
30′	$\bar{1}$.874 8406	2 8831	30′	0.025 2959	3 1871
36′	$\bar{1}$.877 7237	2 8869	36′	0.028 4830	3 1964
42′	$\bar{1}$.880 6106	2 8907	42′	0.031 6794	3 2061
48′	$\bar{1}$.883 5013	2 8946	48′	0.034 8855	3 2159
54′	$\bar{1}$.886 3959	2 8986	54′	0.038 1014	3 2258
74° 0′	$\bar{1}$.889 2945	2 9025	79° 0′	0.041 3272	3 2360
6′	$\bar{1}$.892 1970	2 9066	6′	0.044 5633	3 2465
12′	$\bar{1}$.895 1036	2 9108	12′	0.047 8098	3 2570
18′	$\bar{1}$.898 0144	2 9151	18′	0.051 0668	3 2679
24′	$\bar{1}$.900 9295	2 9194	24′	0.054 3347	3 2789
30′	$\bar{1}$.903 8489	2 9239	30′	0.057 6136	3 2901
36′	$\bar{1}$.906 7728	2 9284	36′	0.060 9037	3 3016
42′	$\bar{1}$.909 7012	2 9329	42′	0.064 2053	3 3132
48′	$\bar{1}$.912 6341	2 9376	48′	0.067 5185	3 3252
54′	$\bar{1}$.915 5717	2 9424	54′	0.070 8437	3 3375
75° 0′	$\bar{1}$.918 5141	2 9472	80° 0′	0.074 1812	3 3500

	Log $\frac{M}{4\pi\sqrt{Aa}}$	Δ_1		Log $\frac{M}{4\pi\sqrt{Aa}}$	Δ_1
80° 0′	0.074 1812	3 3500	85° 0′	0.265 4154	4 6004
6′	0.077 5312	3 3628	6′	0.270 0156	4 6499
12′	0.080 8940	3 3760	12′	0.274 6655	4 7015
18′	0.084 2700	3 3892	18′	0.279 3670	4 7553
24′	0.087 6592	3 4027	24′	0.284 1223	4 8109
30′	0.091 0619	3 4165	30′	0.288 9332	4 8689
36′	0.094 4784	3 4307	36′	0.293 8021	4 9293
42′	0.097 9091	3 4452	42′	0.298 7314	4 9924
48′	0.101 3543	3 4601	48′	0.303 7238	5 0585
54′	0.104 8144	3 4752	54′	0.308 7823	5 1274
81° 0′	0.108 2896	3 4906	86° 0′	0.313 9097	5 1995
6′	0.111 7802	3 5064	6′	0.319 1092	5 2751
12′	0.115 2866	3 5226	12′	0.324 3843	5 3544
18′	0.118 8092	3 5392	18′	0.329 7387	5 4375
24′	0.122 3484	3 5561	24′	0.335 1762	5 5250
30′	0.125 9045	3 5735	30′	0.340 7012	5 6172
36′	0.129 4780	3 5912	36′	0.346 3184	5 7143
42′	0.133 0692	3 6094	42′	0.352 0327	5 8168
48′	0.136 6786	3 6280	48′	0.357 8495	5 9254
54′	0.140 3066	3 6470	54′	0.363 7749	6 0404
82° 0′	0.143 9536	3 6667	87° 0′	0.369 8154	6 1624
6′	0.147 6203	3 6869	6′	0.375 9777	6 2923
12′	0.151 3072	3 7076	12′	0.382 2700	6 4306
18′	0.155 0148	3 7287	18′	0.388 7006	6 5786
24′	0.158 7435	3 7503	24′	0.395 2792	6 7370
30′	0.162 4938	3 7722	30′	0.402 0162	6 9072
36′	0.166 2660	3 7949	36′	0.408 9234	7 0904
42′	0.170 0609	3 8183	42′	0.416 0138	7 2884
48′	0.173 8792	3 8425	48′	0.423 3022	7 5031
54′	0.177 7217	3 8673	54′	0.430 8053	7 7373
83° 0′	0.181 5890	3 8926	88° 0′	0.438 5417	7 9921
6′	0.185 4816	3 9185	6′	0.446 5341	8 2723
12′	0.189 4001	3 9452	12′	0.454 8064	8 5816
18′	0.193 3453	3 9728	18′	0.463 3880	8 9247
24′	0.197 3181	4 0013	24′	0.472 3127	9 3079
30′	0.201 3194	4 0308	30′	0.481 6206	9 7389
36′	0.205 3502	4 0606	36′	0.491 3595	10 2275
42′	0.209 4108	4 0915	42′	0.501 5870	10 7868
48′	0.213 5023	4 1236	48′	0.512 3738	11 4341
54′	0.217 6259	4 1565	54′	0.523 8079	12 1932
84° 0′	0.221 7824	4 1904	89° 0′	0.536 0011	13 0958
6′	0.225 9728	4 2255	6′	0.549 0969	14 1917
12′	0.230 1983	4 2617	12′	0.563 2886	15 5520
18′	0.234 4600	4 2991	18′	0.578 8406	17 2914
24′	0.238 7591	4 3379	24′	0.596 1320	19 6050
30′	0.243 0970	4 3778	30′	0.615 7370	22 8537
36′	0.247 4748	4 4192	36′	0.638 5907	27 7976
42′	0.251 8940	4 4621	42′	0.666 3883	36 3882
48′	0.256 3561	4 5065	48′	0.702 7765	55 9176
54′	0.260 8626	4 5526	54′	0.758 6941	
85° 0′	0.265 4154	4 6004			

The above table has been recalculated and some of the values corrected in the last place. The values given are sufficiently accurate to give *M* within one part in a million.

TABLE II.

Giving the Values of Log F and Log E as Functions of tan γ. (*See p. 41.*)

tan γ	Log F	F	Log E	E
0.1	0.1971 996	1.5747 065	0.1950 415	1.5669 007
0.2	0.2003 678	1.5862 361	0.1918 928	1.5555 817
0.3	0.2054 261	1.6048 192	0.1869 144	1.5378 514
0.4	0.2120 849	1.6296 146	0.1804 536	1.5151 429
0.5	0.2200 096	1.6596 236	0.1729 048	1.4890 346
0.6	0.2288 634	1.6938 051	0.1646 557	1.4610 185
0.7	0.2383 385	1.7311 652	0.1560 492	1.4323 502
0.8	0.2481 728	1.7708 135	0.1473 640	1.4039 900
0.9	0.2581 561	1.8119 912	0.1388 116	1.3766 121
1.0	0.2681 272	1.8540 745	0.1305 409	1.3506 441
1.5	0.3147 473	2.0641 787	0.0955 992	1.2462 329
2.0	0.3535 711	2.2572 057	0.0713 258	1.1784 897
2.5	0.3852 192	2.4278 352	0.0547 850	1.1344 491
3.0	0.4112 984	2.5780 917	0.0432 738	1.1047 748
4.0	0.4518 237	2.8302 429	0.0289 324	1.0688 885
5.0	0.4821 752	3.0351 154	0.0207 426	1.0489 205
7.5	0.5341 061	3.4206 300	0.0109 567	1.0255 497
10.0	0.5682 672	3.7005 581	0.0068 338	1.0158 598
12.5	0.5932 708	3.9198 622	0.0047 004	1.0108 819

TABLE III.

Values of the Constant K as Functions of x/A and a/A.

(For use in Formula 43.)

x/A a/A	— .50	.75	1	1.25	1.50	1.75	2
0.50	9.39283	12.30385	14.27982	15.62795	16.56549	17.23299	17.71973
0.55	9.52044	12.40135	14.34594	15.67140	16.59411	17.25215	17.73283
0.60	9.66358	12.50816	14.41766	15.71837	16.62503	17.27286	17.74701
0.65	9.82296	12.62412	14.49474	15.76867	16.65813	17.29504	17.76221
0.70	9.99921	12.74897	14.57688	15.82212	16.69330	17.31865	17.77841
0.75	10.19272	12.88232	14.66377	15.87850	16.73039	17.34357	17.79554

TABLE IV.

Values of the Constant Q in Formula (58a), $L_s=n^2aQ$.

For the self-inductance of a single-layer winding on a solenoid; n is the whole number of turns of wire in the winding and a is the mean radius. The corrections by Tables VII and VIII must be made to get L from L_s as usual. (See p. 42.)

In the following table $2a$ is the diameter, b is the length, and therefore $2a/b=\tan \gamma$. (See Fig. 21.)

$2\frac{a}{b}=\tan\gamma$	Q	$2\frac{a}{b}=\tan\gamma$	Q
0.20	3.63240	1.80	19.57938
0.30	5.23368	2.00	20.74631
0.40	6.71017	2.20	21.82049
0.50	8.07470	2.40	22.81496
0.60	9.33892	2.60	23.74013
0.70	10.51349	2.80	24.60482
0.80	11.60790	3.00	25.41613
0.90	12.63059	3.20	26.18009
1.00	13.58892	3.40	26.90177
1.20	15.33799	3.60	27.58548
1.40	16.89840	3.80	28.23494
1.60	18.30354	4.00	28.85335

For an explanation of the above formula see p. 41.

TABLE V.

Constants A and B for Strasser's Formula (61).

n	A	B	n	A	B
1	....	....	16	354.4	35 694
2	....	....	17	415.8	46 757
3	1.386	8.315	18	482.8	60 427
4	4.970	43.296	19	555.5	76 662
5	11.33	140.82	20	634.2	96 910
6	20.90	366.95	21	718.9	119 330
7	34.06	794.73	22	809.7	146 517
8	51.11	1499.55	23	906.6	178 140
9	72.32	2590.62	24	1009.8	217 338
10	97.92	4187.55	25	1119.4	259 868
11	128.17	6572.94	26	1235.4	305 044
12	163.14	9769.47	27	1357.9	359 767
13	202.1	14042.1	28	1487.0	421 783
14	248.2	19532.2	29	1618.1	491 819
15	298.6	26740.1	30	1765.4	570 515

TABLE VI.

Table of Constants for Stefan's Formula (69).

b/c or c/b	y_1	y_2	b/c or c/b	y_1	y_2
0.00	0.50000	0.1250	0.55	0.80815	0.3437
0.05	.54899	.1269	0.60	.81823	.3839
0.10	.59243	.1325	0.65	.82648	.4274
0.15	.63102	.1418	0.70	.83311	.4739
0.20	.66520	.1548	0.75	.83831	.5234
0.25	.69532	.1714	0.80	.84225	.5760
0.30	.72172	.1916	0.85	.84509	.6317
0.35	.74469	.2152	0.90	.84697	.6902
0.40	.76454	.2423	0.95	.84801	.7518
0.45	.78154	.2728	1.00	.84834	.8162
0.50	.79600	.3066			

TABLE VII.

Values of Correction Term A, Depending on the Ratio $\frac{d}{D}$ of the Diameters of Bare and Covered Wire on the Single Layer Coil.

(For use in Formula 59.)

$\frac{d}{D}$	A	Δ_1	$\frac{d}{D}$	A	Δ_1
1.00	0.5568	100	0.70	0.2001	144
.99	.5468	101	.69	.1857	146
.98	.5367	103	.68	.1711	148
.97	.5264	104	.67	.1563	150
.96	.5160	105	.66	.1413	152
.95	.5055	106	.65	.1261	155
.94	.4949	107	.64	.1106	157
.93	.4842	108	.63	.0949	160
.92	.4734	109	.62	.0789	163
.91	.4625	110	.61	.0626	166
.90	.4515	112	.60	.0460	168
.89	.4403	113	.59	.0292	171
.88	.4290	114	.58	.0121	174
.87	.4176	116	.57	−.0053	177
.86	.4060	117	.56	−.0230	180
.85	.3943	118	.55	−.0410	184
.84	.3825	120	.54	−.0594	187
.83	.3705	121	.53	−.0781	190
.82	.3584	123	.52	−.0971	194
.81	.3461	124	.51	−.1165	198
.80	.3337	126	.50	−.1363	
.79	.3211	127	.50	−.1363	1053
.78	.3084	129	.45	−.2416	1178
.77	.2955	131	.40	−.3594	1335
.76	.2824	133	.35	−.4928	1542
.75	.2691	134	.30	−.6471	1823
.74	.2557	136	.25	−.8294	2232
.73	.2421	138	.20	−1.0526	2877
.72	.2283	140	.15	−1.3403	4054
.71	.2143	142	.10	−1.7457	
.70	.2001				

TABLE VIII.

Values of the Correction Term B, Depending on the Number of Turns of Wire on the Single Layer Coil.

(For use in Formula 59.)

Number of Turns	B	Number of Turns	B
1	0.0000	50	0.3186
2	.1137	60	.3216
3	.1663	70	.3239
4	.1973	80	.3257
5	.2180	90	.3270
6	.2329	100	.3280
7	.2443	125	.3298
8	.2532	150	.3311
9	.2604	175	.3321
10	.2664	200	.3328
15	.2857	300	.3343
20	.2974	400	.3351
25	.3042	500	.3356
30	.3083	600	.3359
35	.3119	700	.3361
40	.3148	800	.3363
45	.3169	900	.3364
50	.3186	1000	.3365

TABLE IX.

Value of the Constant A_s as a Function of t/a, t being the Depth of the Winding and a the Mean Radius.

(For use in Formula 70.)

t/a	A_s
0.	0.6949
0.10	0.6942
0.15	0.6933
0.20	0.6922
0.25	0.6909

TABLE X.

Values of the Correction Term B_s depending on the Number of Turns of Square Conductor on Single Layer Coil.

(For use in Formula 70.)

Number of Turns	B_s	Number of Turns	B_s
1	0.0000	16	0.3017
2	.1202	17	.3041
3	.1753	18	.3062
4	.2076	19	.3082
5	.2292	20	.3099
6	.2446	21	.3116
7	.2563	22	.3131
8	.2656	23	.3145
9	.2730	24	.3157
10	.2792	25	.3169
11	.2844	26	.3180
12	.2888	27	.3190
13	.2927	28	.3200
14	.2961	29	.3209
15	.2991	30	.3218

TABLE XI.

Table of Napierian Logarithms to nine decimal places for Numbers from 1 to 100.

1	0.000 000 000	51	3.931 825 633
2	0.693 147 181	52	3.951 243 719
3	1.098 612 289	53	3.970 291 914
4	1.386 294 361	54	3.988 984 047
5	1.609 437 912	55	4.007 333 185
6	1.791 759 469	56	4.025 351 691
7	1.945 910 149	57	4.043 051 268
8	2 079 441 542	58	4.060 443 011
9	2.197 224 577	59	4.077 537 444
10	2.302 585 093	60	4.094 344 562
11	2.397 895 273	61	4.110 873 864
12	2.484 906 650	62	4.127 134 385
13	2.564 949 357	63	4.143 134 726
14	2.639 057 330	64	4.158 883 083
15	2.708 050 201	65	4.174 387 270
16	2.772 588 722	66	4.189 654 742
17	2.833 213 344	67	4.204 692 619
18	2.890 371 758	68	4.219 507 705
19	2.944 438 979	69	4.234 106 505
20	2.995 732 274	70	4.248 495 242
21	3.044 522 438	71	4.262 679 877
22	3.091 042 453	72	4.276 666 119
23	3.135 494 216	73	4.290 459 441
24	3.178 053 830	74	4.304 065 093
25	3.218 875 825	75	4.317 488 114
26	3.258 096 538	76	4.330 733 340
27	3.295 836 866	77	4.343 805 422
28	3.332 204 510	78	4.356 708 827
29	3.367 295 830	79	4.369 447 852
30	3.401 197 382	80	4.382 026 635
31	3.433 987 204	81	4.394 339 155
32	3.465 735 903	82	4.406 719 247
33	3.496 507 561	83	4.418 840 608
34	3.526 360 525	84	4.430 816 799
35	3.555 348 061	85	4.442 651 256
36	3.583 518 938	86	4.454 347 296
37	3.610 917 913	87	4.465 908 119
38	3.637 586 160	88	4.477 336 814
39	3.663 561 646	89	4.488 636 370
40	3.688 879 454	90	4.499 809 670
41	3.713 572 067	91	4.510 859 507
42	3.737 669 618	92	4.521 788 577
43	3.761 200 116	93	4.532 599 493
44	3.784 189 634	94	4.543 294 782
45	3.806 662 490	95	4.553 876 892
46	3.828 641 396	96	4.564 348 191
47	3.850 147 602	97	4.574 710 979
48	3.871 201 011	98	4.584 967 479
49	3.891 820 298	99	4.595 119 850
50	3.912 023 005	100	4.605 170 186

log 1525 = log 25 + log 61
log 9.8 = log 98 − log 10
etc.

TABLE XII.

Values of F and E.

The following table of elliptic integrals of the first and second kind is taken from Legendre's *Traité des Fonctions Elliptiques*, Vol. 2, Table VIII:

	F	Δ_1	Δ_2		E	Δ_1	Δ_2
0°	1.570 796	120	239	0°	1.570 796	— 120	−239
1	1.570 916	359	240	1	1.570 677	— 359	−239
2	1.571 275	599	240	2	1.570 318	— 598	−239
3	1.571 874	839	241	3	1.569 720	— 836	−238
4	1.572 712	1 080	22	4	1.568 884	−1 075	−238
5	1.573 792	1 321	243	5	1.567 809	−1 312	−237
6	1.575 114	1 564	244	6	1.566 497	−1 549	−236
7	1.576 678	1 808	246	7	1.564 948	−1 785	−235
8	1.578 486	2 054	247	8	1.563 162	−2 020	−234
9	1.580 541	2 302	249	9	1.561 142	−2 255	−233
10	1.582 843	2 551	252	10	1.558 887	−2 487	−232
11	1.585 394	2 803	254	11	1.556 400	−2 719	−230
12	1.588 197	3 057	257	12	1.553 681	−2 949	−228
13	1.591 254	3 314	260	13	1.550 732	−3 177	−227
14	1.594 568	3 574	263	14	1.547 554	−3 404	−225
15	1.598 142	3 836	266	15	1.544 150	−3 629	−223
16	1.601 978	4 103	270	16	1.540 521	−3 852	−221
17	1.606 081	4 373	274	17	1.536 670	−4 073	−218
18	1.610 454	4 647	278	18	1.532 597	−4 291	−216
19	1.615 101	4 925	283	19	1.528 306	−4 507	−214
20	1.620 026	5 208	288	20	1.523 799	−4 721	−211
21	1.625 234	5 495	293	21	1.519 079	−4 932	−208
22	1.630 729	5 788	298	22	1.514 147	−5 140	−205
23	1.636 517	6 087	304	23	1.509 007	−5 345	−202
24	1.642 604	6 391	311	24	1.503 662	−5 547	−199
25	1.648 995	6 702	317	25	1.498 115	−5 746	−196
26	1.655 697	7 019	324	26	1.492 368	−5 942	−192
27	1.662 716	7 343	332	27	1.486 427	−6 134	−189
28	1.670 059	7 675	340	28	1.480 293	−6 323	−185
29	1.677 735	8 015	349	29	1.473 970	−6 508	−181
30	1.685 750	8 364	358	30	1.467 462	−6 689	−177
31	1.694 114	8 722	367	31	1.460 774	−6 866	−173
32	1.702 836	9 089	377	32	1.453 908	−7 039	−168
33	1.711 925	9 466	388	33	1.446 869	−7 207	−164
34	1.721 391	9 854	400	34	1.439 662	−7 371	−159
35	1.731 245	10 254	412	35	1.432 291	−7 531	−155
36	1.741 499	10 666	425	36	1.424 760	−7 685	−150
37	1.752 165	11 091	439	37	1.417 075	−7 835	−145
38	1.763 256	11 530	453	38	1.409 240	−7 980	−140
39	1.774 786	11 983	469	39	1.401 260	−8 120	−134
40	1.786 770	12 452	486	40	1.393 140	−8 254	−129
41	1.799 222	12 938	504	41	1.384 886	−8 382	−123
42	1.812 160	13 442	523	42	1.376 504	−8 505	−117
43	1.825 602	13 965	543	43	1.367 999	−8 622	−111
44	1.839 567	14 508	565	44	1.359 377	−8 733	−105
45	1.854 075	15 073	588	45	1.350 644	−8 838	— 98

TABLE XII—Continued.

	F	Δ_1	Δ_2		E	Δ_1	Δ_2
45°	1.854 075	15 073	588	45°	1.350 644	−8 838	−98
46	1.869 148	15 661	613	46	1.341 806	−8 936	−92
47	1.884 809	16 274	640	47	1.332 870	−9 028	−85
48	1.901 083	16 914	669	48	1.323 842	−9 113	−78
49	1.917 997	17 584	700	49	1.314 729	−9 190	−71
50	1.935 581	18 284	735	50	1.305 539	−9 261	−63
51	1.953 865	19 017	770	51	1.296 278	−9 324	−56
52	1.972 882	19 787	809	52	1.286 954	−9 380	−48
53	1.992 670	20 597	852	53	1.277 574	−9 427	−40
54	2.013 266	21 449	898	54	1.268 147	−9 467	−31
55	2.034 715	22 347	949	55	1.258 680	−9 498	−22
56	2.057 062	23 296	1 004	56	1.249 182	−9 520	−14
57	2.080 358	24 300	1 064	57	1.239 661	−9 534	− 4
58	2.104 658	25 364	1 130	58	1.230 127	−9 538	+ 5
59	2.130 021	26 494	1 203	59	1.220 589	−9 533	+15
60	2.156 516	27 698	1 284	60	1.211 056	−9 518	+25
61	2.184 213	28 982	1 373	61	1.201 538	−9 492	36
62	2.213 195	30 355	1 472	62	1.192 046	−9 457	47
63	2.243 549	31 827	1 583	63	1.182 589	−9 410	58
64	2.275 376	33 410	1 708	64	1.173 180	−9 351	70
65	2.308 787	35 118	1 848	65	1.163 828	−9 281	82
66	2.343 905	36 965	2 006	66	1.154 547	−9 199	95
67	2.380 870	38 971	2 186	67	1.145 348	−9 104	109
68	2.419 842	41 158	2 393	68	1.136 244	−8 995	123
69	2.460 999	43 551	2 631	69	1.127 250	−8 872	138
70	2.504 550	46 181	2 907	70	1.118 378	−8 734	153
71	2.550 731	49 088	3 230	71	1.109 643	−8 581	169
72	2.599 820	52 318	3 611	72	1.101 062	−8 412	187
73	2.652 138	55 930	4 066	73	1.092 650	−8 225	205
74	2.708 068	59 996	4 614	74	1.084 425	−8 020	224
75	2.768 063	64 609	5 283	75	1.076 405	−7 796	245
76	2.832 673	69 892	6 112	76	1.068 610	−7 550	268
77	2.902 565	76 004	7 156	77	1.061 059	−7 282	292
78	2.978 569	83 160	8 497	78	1.053 777	−6 990	318
79	3.061 729	91 657	10 261	79	1.046 786	−6 672	347
80	3.153 385	101 918	12 647	80	1.040 114	−6 325	379
81	3.255 303	114 565	15 989	81	1.033 789	−5 946	415
82	3.369 868	130 554	20 879	82	1.027 844	−5 531	455
83	3.500 422	151 433	28 453	83	1.022 313	−5 076	502
84	3.651 856	179 886	41 130	84	1.017 237	−4 573	558
85	3.831 742	221 016	64 880	85	1.012 664	−4 016	626
86	4.052 758	285 896	118 167	86	1.008 648	−3 389	715
87	4.338 654	404 063	288 129	87	1.005 259	−2 675	842
88	4.742 717	692 193		88	1.002 584	−1 832	1081
89	5.434 910			89	1.000 752	− 752	
90				90	1.000 000		

TABLE XIII.

Values of log F and log E.

[See Note page 131.]

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
45°.0	0.2681 2722	3 4688	105	0.1305 4086	2 8279	52
45.1	0.2684 7411	3 4793	105	0.1302 5807	2 8331	52
45.2	0.2688 2204	3 4898	105	0.1299 7476	2 8383	52
45.3	0.2691 7102	3 5004	106	0.1296 9094	2 8434	52
45.4	0.2695 2106	3 5110	106	0.1294 0659	2 8486	51
45.5	0.2698 7216	3 5216	106	0.1291 2174	2 8537	51
45.6	0.2702 2431	3 5322	106	0.1288 3636	2 8589	51
45.7	0.2705 7753	3 5428	107	0.1285 5048	2 8640	51
45.8	0.2709 3181	3 5535	107	0.1282 6408	2 8691	51
45.9	0.2712 8716	3 5642	107	0.1279 7717	2 8742	51
46.0	0.2716 4358	3 5749	108	0.1276 8975	2 8793	51
46.1	0.2720 0108	3 5857	108	0.1274 0182	2 8844	51
46.2	0.2723 5965	3 5965	108	0.1271 1338	2 8894	50
46.3	0.2727 1930	3 6073	108	0.1268 2444	2 8945	50
46.4	0.2730 8003	3 6181	109	0.1265 3499	2 8995	50
46.5	0.2734 4184	3 6290	109	0.1262 4504	2 9045	50
46.6	0.2738 0474	3 6399	109	0.1259 5459	2 9095	50
46.7	0.2741 6873	3 6508	110	0.1256 6364	2 9145	50
46.8	0.2745 3381	3 6618	110	0.1253 7218	2 9195	50
46.9	0.2748 9999	3 6728	110	0.1250 8023	2 9245	50
47.0	0.2752 6727	3 6838	110	0.1247 8778	2 9295	49
47.1	0.2756 3565	3 6948	111	0.1244 9483	2 9344	49
47.2	0.2760 0513	3 7059	111	0.1242 0139	2 9393	49
47.3	0.2763 7572	3 7170	111	0.1239 0746	2 9443	49
47.4	0.2767 4741	3 7281	112	0.1236 1303	2 9492	49
47.5	0.2771 2023	3 7393	112	0.1233 1811	2 9541	49
47.6	0.2774 9415	3 7505	112	0.1230 2271	2 9589	49
47.7	0.2778 6920	3 7617	112	0.1227 2681	2 9638	49
47.8	0.2782 4537	3 7729	113	0.1224 3043	2 9687	48
47.9	0.2786 2266	3 7842	113	0.1221 3357	2 9735	48
48.0	0.2790 0109	3 7955	113	0.1218 3622	2 9783	48
48.1	0.2793 8064	3 8069	114	0.1215 3838	2 9831	48
48.2	0.2797 6133	3 8183	114	0.1212 4007	2 9879	48
48.3	0.2801 4315	3 8297	114	0.1209 4128	2 9927	48
48.4	0.2805 2612	3 8411	115	0.1206 4201	2 9975	48
48.5	0.2809 1023	3 8526	115	0.1203 4226	3 0022	47
48.6	0.2812 9548	3 8641	115	0.1200 4204	3 0070	47
48.7	0.2816 8189	3 8756	116	0.1197 4134	3 0117	47
48.8	0.2820 6945	3 8872	116	0.1194 4017	3 0164	47
48.9	0.2824 5817	3 8988	116	0.1191 3854	3 0211	47
49.0	0.2828 4805	3 9104	117	0.1188 3643	3 0258	47
49.1	0.2832 3909	3 9221	117	0.1185 3385	3 0304	46
49.2	0.2836 3130	3 9338	117	0.1182 3081	3 0351	46
49.3	0.2840 2467	3 9455	118	0.1179 2730	3 0397	46
49.4	0.2844 1923	3 9573	118	0.1176 2333	3 0443	46
49.5	0.2848 1495	3 9691	118	0.1173 1890	3 0489	46
49.6	0.2852 1186	3 9809	119	0.1170 1401	3 0535	46
49.7	0.2856 0996	3 9928	119	0.1167 0866	3 0581	46
49.8	0.2860 0924	4 0047	119	0.1164 0286	3 0626	45
49.9	0.2864 0971	4 0167	120	0.1160 9660	3 0671	45
50.0	0.2868 1137	4 0286	120	0.1157 8988	3 0717	45

TABLE XIII—Continued.

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
50°0	0.2868 1137	4 0286	120	0.1157 8988	3 0717	45
50.1	0.2872 1424	4 0406	121	0.1154 8271	3 0762	45
50.2	0.2876 1830	4 0527	121	0.1151 7510	3 0807	45
50.3	0.2880 2357	4 0648	121	0.1148 6703	3 0851	45
50.4	0.2884 3005	4 0769	122	0.1145 5852	3 0896	44
50.5	0.2888 3774	4 0891	122	0.1142 4956	3 0940	44
50.6	0.2892 4665	4 1013	122	0.1139 4016	3 0985	44
50.7	0.2896 5677	4 1135	123	0.1136 3032	3 1028	44
50.8	0.2900 6812	4 1258	123	0.1133 2003	3 1072	44
50.9	0.2904 8070	4 1381	123	0.1130 0931	3 1116	43
51.0	0.2908 9451	4 1504	124	0.1126 9815	3 1159	43
51.1	0.2913 0955	4 1628	124	0.1123 8656	3 1203	43
51.2	0.2917 2584	4 1753	125	0.1120 7453	3 1246	43
51.3	0.2921 4336	4 1877	125	0.1117 6207	3 1289	43
51.4	0.2925 6214	4 2002	125	0.1114 4919	3 1332	43
51.5	0.2929 8216	4 2128	126	0.1111 3587	3 1374	42
51.6	0.2934 0344	4 2254	126	0.1108 2213	3 1417	42
51.7	0.2938 2597	4 2380	127	0.1105 0796	3 1459	42
51.8	0.2942 4977	4 2506	127	0.1101 9337	3 1501	42
51.9	0.2946 7483	4 2634	127	0.1098 7836	3 1543	42
52.0	0.2951 0117	4 2761	128	0.1095 6294	3 1584	41
52.1	0.2955 2878	4 2889	128	0.1092 4709	3 1626	41
52.2	0.2959 5767	4 3017	129	0.1089 3083	3 1667	41
52.3	0.2963 8784	4 3146	129	0.1086 1416	3 1708	41
52.4	0.2968 1930	4 3275	130	0.1082 9707	3 1749	41
52.5	0.2972 5205	4 3405	130	0.1079 7958	3 1790	41
52.6	0.2976 8610	4 3535	130	0.1076 6168	3 1831	40
52.7	0.2981 2144	4 3665	131	0.1073 4338	3 1871	40
52.8	0.2985 5810	4 3796	131	0.1070 2467	3 1911	40
52.9	0.2989 9606	4 3927	132	0.1067 0556	3 1951	40
53.0	0.2994 3533	4 4059	132	0.1063 8605	3 1991	40
53.1	0.2998 7592	4 4191	133	0.1060 6614	3 2030	39
53.2	0.3003 1783	4 4324	133	0.1057 4584	3 2070	39
53.3	0.3007 6107	4 4457	134	0.1054 2514	3 2109	39
53.4	0.3012 0564	4 4591	134	0.1051 0406	3 2148	39
53.5	0.3016 5155	4 4725	134	0.1047 8258	3 2186	38
53.6	0.3020 9880	4 4859	135	0.1044 6072	3 2225	38
53.7	0.3025 4739	4 4994	135	0.1041 3847	3 2263	38
53.8	0.3029 9733	4 5130	136	0.1038 1584	3 2301	38
53.9	0.3034 4863	4 5265	136	0.1034 9283	3 2339	38
54.0	0.3039 0128	4 5402	137	0.1031 6944	3 2377	37
54.1	0.3043 5530	4 5539	137	0.1028 4567	3 2414	37
54.2	0.3048 1069	4 5676	138	0.1025 2153	3 2451	37
54.3	0.3052 6745	4 5814	138	0.1021 9702	3 2488	37
54.4	0.3057 2559	4 5952	139	0.1018 7214	3 2525	37
54.5	0.3061 8511	4 6091	139	0.1015 4689	3 2562	36
54.6	0.3066 4602	4 6230	140	0.1012 2127	3 2598	36
54.7	0.3071 0833	4 6370	140	0.1008 9529	3 2634	36
54.8	0.3075 7203	4 6511	141	0.1005 6895	3 2670	36
54.9	0.3080 3714	4 6652	141	0.1002 4226	3 2705	35
55.0	0.3085 0365	4 6793	142	0.0999 1520	3 2741	35

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
55°.0	0.3085 0365	4 6793	142	0.0999 1520	3 2741	35
55.1	0.3089 7158	4 6935	142	0.0995 8779	3 2776	35
55.2	0.3094 4093	4 7077	143	0.0992 6003	3 2811	35
55.3	0.3099 1170	4 7220	143	0.0989 3193	3 2846	34
55.4	0.3103 8391	4 7364	144	0.0986 0347	3 2880	34
55.5	0.3108 5754	4 7508	145	0.0982 7467	3 2914	34
55.6	0.3113 3262	4 7652	145	0.0979 4553	3 2948	34
55.7	0.3118 0915	4 7798	146	0.0976 1605	3 2982	33
55.8	0.3122 8712	4 7943	146	0.0972 8623	3 3015	33
55.9	0.3127 6655	4 8089	147	0.0969 5607	3 3049	33
56.0	0.3132 4745	4 8236	147	0.0966 2559	3 3082	33
56.1	0.3137 2981	4 8384	148	0.0962 9477	3 3114	32
56.2	0.3142 1365	4 8532	149	0.0959 6363	3 3147	32
56.3	0.3146 9896	4 8680	149	0.0956 3216	3 3179	32
56.4	0.3151 8577	4 8829	150	0.0953 0037	3 3211	32
56.5	0.3156 7406	4 8979	150	0.0949 6826	3 3243	31
56.6	0.3161 6385	4 9129	151	0.0946 3583	3 3274	31
56.7	0.3166 5514	4 9280	151	0.0943 0309	3 3305	31
56.8	0.3171 4794	4 9432	152	0.0939 7003	3 3336	31
56.9	0.3176 4226	4 9584	153	0.0936 3667	3 3367	30
57.0	0.3181 3809	4 9736	153	0.0933 0300	3 3397	30
57.1	0.3186 3545	4 9890	154	0.0929 6903	3 3428	30
57.2	0.3191 3435	5 0044	155	0.0926 3475	3 3457	30
57.3	0.3196 3479	5 0198	155	0.0923 0018	3 3487	29
57.4	0.3201 3677	5 0353	156	0.0919 6531	3 3516	29
57.5	0.3206 4030	5 0509	156	0.0916 3014	3 3545	29
57.6	0.3211 4539	5 0666	157	0.0912 9469	3 3574	28
57.7	0.3216 5204	5 0823	158	0.0909 5895	3 3603	28
57.8	0.3221 6027	5 0980	158	0.0906 2292	3 3631	28
57.9	0.3226 7008	5 1139	159	0.0902 8662	3 3659	28
58.0	0.3231 8146	5 1298	160	0.0899 5003	3 3686	27
58.1	0.3236 9444	5 1458	160	0.0896 1317	3 3714	27
58.2	0.3242 0902	5 1618	161	0.0892 7603	3 3741	27
58.3	0.3247 2520	5 1779	162	0.0889 3862	3 3767	26
58.4	0.3252 4299	5 1941	162	0.0886 0095	3 3794	26
58.5	0.3257 6240	5 2104	163	0.0882 6301	3 3820	26
58.6	0.3262 8344	5 2267	164	0.0879 2481	3 3846	26
58.7	0.3268 0611	5 2431	165	0.0875 8635	3 3871	25
58.8	0.3273 3041	5 2595	165	0.0872 4764	3 3897	25
58.9	0.3278 5637	5 2761	166	0.0869 0867	3 3922	25
59.0	0.3283 8397	5 2927	167	0.0865 6945	3 3946	24
59.1	0.3289 1324	5 3094	168	0.0862 2999	3 3971	24
59.2	0.3294 4418	5 3261	168	0.0858 9028	3 3995	24
59.3	0.3299 7679	5 3429	169	0.0855 5033	3 4018	23
59.4	0.3305 1108	5 3598	170	0.0852 1015	3 4042	23
59.5	0.3310 4707	5 3768	171	0.0848 6973	3 4065	23
59.6	0.3315 8475	5 3939	171	0.0845 2908	3 4088	22
59.7	0.3321 2414	5 4110	172	0.0841 8820	3 4110	22
59.8	0.3326 6524	5 4282	173	0.0838 4710	3 4132	22
59.9	0.3332 0806	5 4455	174	0.0835 0578	3 4154	21
60.0	0.3337 5261	5 4629	175	0.0831 6424	3 4176	21

TABLE XIII—Continued.

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
60°0	0.3337 5261	5 4629	175	0.0831 6424	3 4176	21
60.1	0.3342 9890	5 4803	175	0.0828 2248	3 4197	21
60.2	0.3348 4694	5 4979	176	0.0824 8051	3 4217	20
60.3	0.3353 9673	5 5155	177	0.0821 3834	3 4238	20
60.4	0.3359 4827	5 5332	178	0.0817 9596	3 4258	20
60.5	0.3365 0159	5 5510	179	0.0814 5338	3 4278	19
60.6	0.3370 5669	5 5688	179	0.0811 1060	3 4297	19
60.7	0.3376 1357	5 5868	180	0.0807 6763	3 4316	19
60.8	0.3381 7225	5 6048	181	0.0804 2446	3 4335	18
60.9	0.3387 3274	5 6229	182	0.0800 8111	3 4354	18
61.0	0.3392 9503	5 6412	183	0.0797 3758	3 4372	18
61.1	0.3398 5915	5 6595	184	0.0793 9386	3 4389	17
61.2	0.3404 2509	5 6778	185	0.0790 4997	3 4407	17
61.3	0.3409 9288	5 6963	186	0.0787 0590	3 4424	17
61.4	0.3415 6251	5 7149	187	0.0783 6167	3 4440	16
61.5	0.3421 3400	5 7336	188	0.0780 1727	3 4456	16
61.6	0.3427 0735	5 7523	188	0.0776 7270	3 4472	15
61.7	0.3432 8258	5 7712	189	0.0773 2798	3 4488	15
61.8	0.3438 5970	5 7901	190	0.0769 8310	3 4503	15
61.9	0.3444 3871	5 8091	191	0.0766 3807	3 4518	14
62.0	0.3450 1962	5 8283	192	0.0762 9290	3 4532	14
62.1	0.3456 0245	5 8475	193	0.0759 4758	3 4546	14
62.2	0.3461 8720	5 8668	194	0.0756 0212	3 4560	13
62.3	0.3467 7388	5 8863	195	0.0752 5652	3 4573	13
62.4	0.3473 6250	5 9058	196	0.0749 1079	3 4586	12
62.5	0.3479 5308	5 9254	197	0.0745 6494	3 4598	12
62.6	0.3485 4562	5 9451	198	0.0742 1895	3 4610	12
62.7	0.3491 4014	5 9650	199	0.0738 7285	3 4622	11
62.8	0.3497 3664	5 9849	200	0.0735 2664	3 4633	11
62.9	0.3503 3513	6 0050	202	0.0731 8030	3 4644	10
63.0	0.3509 3563	6 0251	203	0.0728 3387	3 4654	10
63.1	0.3515 3814	6 0454	204	0.0724 8732	3 4664	10
63.2	0.3521 4268	6 0658	205	0.0721 4068	3 4674	9
63.3	0.3527 4925	6 0862	206	0.0717 9394	3 4683	9
63.4	0.3533 5787	6 1068	207	0.0714 4711	3 4692	8
63.5	0.3539 6856	6 1275	208	0.0711 0019	3 4700	8
63.6	0.3545 8131	6 1483	209	0.0707 5319	3 4708	8
63.7	0.3551 9614	6 1693	210	0.0704 0610	3 4716	7
63.8	0.3558 1307	6 1903	212	0.0700 5895	3 4723	7
63.9	0.3564 3211	6 2115	213	0.0697 1172	3 4729	6
64.0	0.3570 5325	6 2328	214	0.0693 6442	3 4736	6
64.1	0.3576 7653	6 2542	215	0.0690 1706	3 4741	5
64.2	0.3583 0195	6 2757	216	0.0686 6965	3 4747	5
64.3	0.3589 2952	6 2974	218	0.0683 2218	3 4752	4
64.4	0.3595 5926	6 3191	219	0.0679 7466	3 4756	4
64.5	0.3601 9117	6 3410	220	0.0676 2710	3 4760	4
64.6	0.3608 2527	6 3630	221	0.0672 7950	3 4764	3
64.7	0.3614 6158	6 3852	223	0.0669 3186	3 4767	3
64.8	0.3621 0009	6 4075	224	0.0665 8420	3 4769	2
64.9	0.3627 4084	6 4299	225	0.0662 3650	3 4772	2
65.0	0.3633 8383	6 4524	227	0.0658 8879	3 4773	1

TABLE XIII—Continued.

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
65°0	0.3633 8383	6 4524	227	0.0658 8879	3 4773	1
65.1	0.3640 2907	6 4751	228	0.0655 4106	3 4774	1
65.2	0.3646 7658	6 4979	229	0.0651 9331	3 4775	+0
65.3	0.3653 2637	6 5209	231	0.0648 4556	3 4775	—0
65.4	0.3659 7846	6 5439	232	0.0644 9781	3 4775	1
65.5	0.3666 3286	6 5672	234	0.0641 5005	3 4775	1
65.6	0.3672 8957	6 5905	235	0.0638 0231	3 4773	2
65.7	0.3679 4863	6 6141	237	0.0634 5457	3 4772	2
65.8	0.3686 1003	6 6377	238	0.0631 0686	3 4769	3
65.9	0.3692 7380	6 6615	239	0.0627 5916	3 4767	3
66.0	0.3699 3995	6 6855	241	0.0624 1150	3 4764	4
66.1	0.3706 0850	6 7096	242	0.0620 6386	3 4760	4
66.2	0.3712 7946	6 7338	244	0.0617 1626	3 4756	5
66.3	0.3719 5284	6 7582	246	0.0613 6870	3 4751	5
66.4	0.3726 2866	6 7828	247	0.0610 2119	3 4746	6
66.5	0.3733 0694	6 8075	249	0.0606 7373	3 4740	6
66.6	0.3739 8768	6 8324	250	0.0603 2633	3 4734	7
66.7	0.3746 7092	6 8574	252	0.0599 7899	3 4727	7
66.8	0.3753 5666	6 8826	254	0.0596 3172	3 4720	8
66.9	0.3760 4492	6 9080	255	0.0592 8453	3 4712	8
67.0	0.3767 3572	6 9335	257	0.0589 3741	3 4703	9
67.1	0.3774 2907	6 9592	259	0.0585 9037	3 4695	9
67.2	0.3781 2499	6 9851	260	0.0582 4343	3 4685	10
67.3	0.3788 2349	7 0111	262	0.0578 9658	3 4675	11
67.4	0.3795 2460	7 0373	264	0.0575 4983	3 4664	11
67.5	0.3802 2833	7 0637	266	0.0572 0318	3 4653	12
67.6	0.3809 3471	7 0903	268	0.0568 5665	3 4642	12
67.7	0.3816 4373	7 1170	269	0.0565 1023	3 4629	13
67.8	0.3823 5544	7 1440	271	0.0561 6394	3 4617	13
67.9	0.3830 6984	7 1711	273	0.0558 1777	3 4603	14
68.0	0.3837 8695	7 1984	275	0.0554 7174	3 4589	15
68.1	0.3845 0679	7 2259	277	0.0551 2585	3 4575	15
68.2	0.3852 2938	7 2536	279	0.0547 8011	3 4559	16
68.3	0.3859 5475	7 2815	281	0.0544 3451	3 4544	16
68.4	0.3866 8290	7 3096	283	0.0540 8908	3 4527	17
68.5	0.3874 1386	7 3379	285	0.0537 4380	3 4510	18
68.6	0.3881 4765	7 3664	287	0.0533 9870	3 4493	18
68.7	0.3888 8429	7 3951	289	0.0530 5377	3 4475	19
68.8	0.3896 2380	7 4240	291	0.0527 0903	3 4456	19
68.9	0.3903 6620	7 4531	293	0.0523 6447	3 4436	20
69.0	0.3911 1152	7 4825	296	0.0520 2010	3 4416	21
69.1	0.3918 5977	7 5120	298	0.0516 7594	3 4396	21
69.2	0.3926 1097	7 5418	300	0.0513 3198	3 4375	22
69.3	0.3933 6515	7 5718	302	0.0509 8824	3 4353	23
69.4	0.3941 2234	7 6020	305	0.0506 4471	3 4330	23
69.5	0.3948 8254	7 6325	307	0.0503 0141	3 4307	24
69.6	0.3956 4579	7 6632	309	0.0499 5834	3 4283	24
69.7	0.3964 1211	7 6941	312	0.0496 1551	3 4259	25
69.8	0.3971 8152	7 7253	314	0.0492 7292	3 4233	26
69.9	0.3979 5405	7 7567	317	0.0489 3059	3 4208	26
70.0	0.3987 2972	7 7883	319	0.0485 8851	3 4181	27

TABLE XIII—Continued.

γ	Log F	Δ_1	Δ_2	Log E	Δ_1	Δ_2
70°0	0.3987 2972	7 7883	319	0.0485 8851	3 4181	27
70.1	0.3995 0855	7 8202	322	0.0482 4670	3 4154	28
70.2	0.4002 9058	7 8524	324	0.0479 0516	3 4126	29
70.3	0.4010 7582	7 8848	327	0.0475 6390	3 4098	29
70 4	0.4018 6430	7 9175	329	0.0472 2292	3 4068	30
70.5	0.4026 5605	7 9504	332	0.0468 8224	3 4039	31
70.6	0.4034 5109	7 9836	335	0.0465 4185	3 4008	31
70.7	0.4042 4945	8 0171	337	0.0462 0177	3 3977	32
70.8	0.4050 5116	8 0508	340	0.0458 6201	3 3945	33
70.9	0.4058 5625	8 0849	343	0.0455 2256	3 3912	33
71.0	0.4066 6474	8 1192	346	0.0451 8344	3 3879	34
71.1	0.4074 7666	8 1538	349	0.0448 4465	3 3844	35
71.2	0.4082 9204	8 1887	352	0.0445 0621	3 3810	36
71.3	0.4091 1090	8 2239	355	0.0441 6812	3 3774	36
71.4	0.4099 3329	8 2594	358	0.0438 3038	3 3738	37
71.5	0.4107 5923	8 2952	361	0.0434 9300	3 3700	38
71.6	0.4115 8875	8 3313	364	0.0431 5600	3 3663	39
71.7	0.4124 2187	8 3677	367	0.0428 1937	3 3624	39
71.8	0.4132 5864	8 4044	371	0.0424 8313	3 3585	40
71.9	0.4140 9909	8 4415	374	0.0421 4729	3 3544	41
72.0	0.4149 4324	8 4789	377	0.0418 1184	3 3504	42
72.1	0.4157 9112	8 5166	381	0.0414 7681	3 3462	42
72.2	0.4166 4279	8 5547	384	0.0411 4219	3 3419	43
72.3	0.4174 9826	8 5931	388	0.0408 0799	3 3376	44
72.4	0.4183 5757	8 6319	391	0.0404 7423	3 3332	45
72.5	0.4192 2076	8 6710	395	0.0401 4091	3 3287	46
72.6	0.4200 8786	8 7105	399	0.0398 0804	3 3241	46
72.7	0.4209 5891	8 7503	402	0.0394 7563	3 3195	47
72.8	0.4218 3394	8 7906	406	0.0391 4368	3 3148	48
72.9	0.4227 1300	8 8312	410	0.0388 1220	3 3099	49
73.0	0.4235 9612	8 8722	414	0.0384 8121	3 3050	50
73.1	0.4244 8334	8 9136	418	0.0381 5070	3 3001	51
73.2	0.4253 7470	8 9554	422	0.0378 2070	3 2950	52
73.3	0.4262 7023	8 9976	426	0.0374 9120	3 2898	52
73.4	0.4271 6999	9 0402	430	0.0371 6221	3 2846	53
73.5	0.4280 7401	9 0832	435	0.0368 3375	3 2793	54
73.6	0.4289 8233	9 1267	439	0.0365 0582	3 2739	55
73.7	0.4298 9499	9 1706	443	0.0361 7843	3 2684	56
73.8	0.4308 1205	9 2149	448	0.0358 5160	3 2628	57
73.9	0.4317 3354	9 2597	452	0.0355 2532	3 2571	58
74.0	0.4326 5950	9 3049	457	0.0351 9961	3 2513	59
74.1	0.4335 9000	9 3506	462	0.0348 7448	3 2455	60
74.2	0.4345 2506	9 3968	467	0.0345 4993	3 2395	60
74.3	0.4354 6474	9 4435	472	0.0342 2598	3 2335	61
74.4	0.4364 0909	9 4906	477	0.0339 0263	3 2273	62
74.5	0.4373 5815	9 5583	482	0.0335 7989	3 2211	63
74.6	0.4383 1198	9 5865	487	0.0332 5778	3 2148	64
74.7	0.4392 7063	9 6352	492	0.0329 3630	3 2084	65
74.8	0.4402 3414	9 6844	498	0.0326 1546	3 2019	66
74.9	0.4412 0258	9 7341	503	0.0322 9528	3 1952	67
75.0	0.4421 7599	9 7844	509	0.0319 7575	3 1885	68

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
75°0	0.4421 7599	9 7844	509	0.0319 7575	3 1885	68
75.1	0.4431 5444	9 8353	514	0.0316 5690	3 1817	69
75.2	0.4441 3797	9 8867	520	0.0313 3872	3 1748	70
75.3	0.4451 2664	9 9387	526	0.0310 2124	3 1678	71
75.4	0.4461 2051	9 9913	532	0.0307 0446	3 1607	72
75.5	0.4471 1965	10 0446	538	0.0303 8839	3 1535	73
75.6	0.4481 2410	10 0984	544	0.0300 7304	3 1462	74
75.7	0.4491 3394	10 1528	551	0.0297 5842	3 1388	75
75.8	0.4501 4922	10 2079	557	0.0294 4454	3 1313	76
75.9	0.4511 7001	10 2637	564	0.0291 3141	3 1237	77
76.0	0.4521 9638	10 3201	571	0.0288 1904	3 1159	78
76.1	0.4532 2839	10 3771	578	0.0285 0745	3 1081	79
76.2	0.4542 6610	10 4349	585	0.0281 9664	3 1002	80
76.3	0.4553 0959	10 4934	592	0.0278 8663	3 0921	82
76.4	0.4563 5893	10 5526	599	0.0275 7742	3 0839	83
76.5	0.4574 1419	10 6126	607	0.0272 6902	3 0757	84
76.6	0.4584 7545	10 6733	615	0.0269 6145	3 0673	85
76.7	0.4595 4278	10 7347	622	0.0266 5472	3 0588	86
76.8	0.4606 1625	10 7970	630	0.0263 4884	3 0502	87
76.9	0.4616 9594	10 8600	639	0.0260 4382	3 0415	88
77.0	0.4627 8195	10 9239	647	0.0257 3967	3 0327	89
77.1	0.4638 7433	10 9886	656	0.0254 3640	3 0237	91
77.2	0.4649 7319	11 0541	664	0.0251 3403	3 0147	92
77.3	0.4660 7860	11 1206	673	0.0248 3257	3 0055	93
77.4	0.4671 9066	11 1879	682	0.0245 3202	2 9962	94
77.5	0.4683 0945	11 2561	692	0.0242 3240	2 9868	95
77.6	0.4694 3506	11 3253	701	0.0239 3372	2 9772	97
77.7	0.4705 6760	11 3954	711	0.0236 3600	2 9676	98
77.8	0.4717 0714	11 4665	721	0.0233 3925	2 9578	99
77.9	0.4728 5379	11 5386	731	0.0230 4347	2 9479	100
78.0	0.4740 0766	11 6118	742	0.0227 4868	2 9378	102
78.1	0.4751 6884	11 6860	753	0.0224 5490	2 9277	103
78.2	0.4763 3743	11 7612	764	0.0221 6213	2 9174	104
78.3	0.4775 1355	11 8376	775	0.0218 7039	2 9070	105
78.4	0.4786 9731	11 9150	786	0.0215 7969	2 8964	107
78.5	0.4798 8881	11 9937	798	0.0212 9005	2 8858	108
78.6	0.4810 8818	12 0735	810	0.0210 0148	2 8750	109
78.7	0.4822 9553	12 1545	823	0.0207 1398	2 8640	111
78.8	0.4835 1098	12 2368	835	0.0204 2758	2 8529	112
78.9	0.4847 3466	12 3203	848	0.0201 4229	2 8417	113
79.0	0.4859 6669	12 4052	862	0.0198 5811	2 8304	115
79.1	0.4872 0721	12 4914	876	0.0195 7507	2 8189	116
79.2	0.4884 5635	12 5789	890	0.0192 9318	2 8073	118
79.3	0.4897 1424	12 6679	904	0.0190 1246	2 7955	119
79.4	0.4909 8103	12 7583	919	0.0187 3291	2 7836	120
79.5	0.4922 5687	12 8503	934	0.0184 5454	2 7716	122
79.6	0.4935 4189	12 9437	950	0.0181 7739	2 7594	123
79.7	0.4948 3626	13 0387	966	0.0179 0145	2 7470	125
79.8	0.4961 4013	13 1353	983	0.0176 2675	2 7345	126
79.9	0.4974 5367	13 2336	1000	0.0173 5330	2 7219	128
80.0	0.4987 7703	13 3336	1018	0.0170 8111	2 7091	129

TABLE XIII—Continued.

γ	Log F	Δ₁	Δ₂	Log E	Δ₁	Δ₂
80°.0	0.4987 7703	13 3336	1018	0.0170 8111	2 7091	129
80.1	0.5001 1040	13 4354	1036	0.0168 1020	2 6962	131
80.2	0.5014 5394	13 5390	1054	0.0165 4058	2 6831	132
80.3	0.5028 0783	13 6444	1073	0.0162 7227	2 6698	134
80.4	0.5041 7227	13 7517	1093	0.0160 0529	2 6564	136
80.5	0.5055 4744	13 8610	1113	0.0157 3965	2 6429	137
80.6	0.5069 3354	13 9724	1134	0.0154 7536	2 6291	139
80.7	0.5083 3078	14 0858	1156	0.0152 1245	2 6153	140
80.8	0.5097 3936	14 2014	1178	0.0149 5092	2 6012	142
80.9	0.5111 5949	14 3192	1201	0.0146 9080	2 5870	144
81.0	0.5125 9141	14 4393	1225	0.0144 3210	2 5726	145
81.1	0.5140 3534	14 5617	1249	0.0141 7484	2 5581	147
81.2	0.5154 9151	14 6867	1274	0.0139 1903	2 5433	149
81.3	0.5169 6018	14 8141	1300	0.0136 6470	2 5285	151
81.4	0.5184 4159	14 9441	1327	0.0134 1185	2 5134	152
81.5	0.5199 3600	15 0769	1355	0.0131 6052	2 4981	154
81.6	0.5214 4369	15 2124	1384	0.0129 1070	2 4827	156
81.7	0.5229 6493	15 3508	1414	0.0126 6243	2 4671	158
81.8	0.5245 0001	15 4922	1445	0.0124 1572	2 4513	160
81.9	0.5260 4923	15 6366	1477	0.0121 7058	2 4354	162
82.0	0.5276 1289	15 7843	1510	0.0119 2704	2 4192	163
82.1	0.5291 9132	15 9352	1544	0.0116 8512	2 4029	165
82.2	0.5307 8485	16 0896	1579	0.0114 4483	2 3863	167
82.3	0.5323 9381	16 2476	1616	0.0112 0620	2 3696	169
82.4	0.5340 1857	16 4092	1655	0.0109 6924	2 3527	171
82.5	0.5356 5949	16 5747	1694	0.0107 3397	2 3356	173
82.6	0.5373 1696	16 7441	1736	0.0105 0041	2 3183	175
82.7	0.5389 9137	16 9177	1779	0.0102 6859	2 3007	177
82.8	0.5406 8313	17 0955	1823	0.0100 3851	2 2830	179
82.9	0.5423 9268	17 2778	1870	0.0098 1021	2 2651	181
83.0	0.5441 2047	17 4648	1918	0.0095 8371	2 2469	184
83.1	0.5458 6695	17 6566	1968	0.0093 5902	2 2285	186
83.2	0.5476 3260	17 8534	2021	0.0091 3616	2 2100	188
83.3	0.5494 1795	18 0555	2076	0.0089 1517	2 1912	190
83.4	0.5512 2350	18 2631	2133	0.0086 9605	2 1721	193
83.5	0.5530 4980	18 4764	2193	0.0084 7884	2 1529	195
83.6	0.5548 9744	18 6956	2255	0.0082 6355	2 1334	197
83.7	0.5567 6700	18 9211	2320	0.0080 5021	2 1137	199
83.8	0.5586 5912	19 1532	2389	0.0078 3884	2 0937	202
83.9	0.5605 7443	19 3921	2460	0.0076 2947	2 0735	204
84.0	0.5625 1364	19 6381	2535	0.0074 2211	2 0531	207
84.1	0.5644 7745	19 8916	2614	0.0072 1680	2 0324	209
84.2	0.5664 6661	20 1531	2697	0.0070 1356	2 0115	212
84.3	0.5684 8192	20 4228	2784	0.0068 1241	1 9903	214
84.4	0.5705 2420	20 7012	2875	0.0066 1338	1 9689	217
84.5	0.5725 9431	20 9887	2972	0.0064 1649	1 9472	220
84.6	0.5746 9318	21 2859	3073	0.0062 2177	1 9252	222
84.7	0.5768 2177	21 5932	3180	0.0060 2925	1 9029	225
84.8	0.5789 8109	21 9112	3293	0.0058 3896	1 8804	228
84.9	0.5811 7221	22 2405	3413	0.0056 5092	1 8576	231
85.0	0.5833 9626	22 5818	3539	0.0054 6516	1 8345	234

TABLE XIII—Continued.

γ	Log F	Δ1	Δ2	Log E	Δ1	Δ2
85°.0	0.5833 9626	22 5818	3539	0.0054 6516	1 8345	234
85.1	0.5856 5444	22 9357	3673	0.0052 8171	1 8111	237
85.2	0.5879 4801	23 3031	3816	0.0051 0060	1 7874	240
85.3	0.5902 7832	23 6846	3967	0.0049 2185	1 7634	243
85.4	0.5926 4679	24 0813	4127	0.0047 4551	1 7391	246
85.5	0.5950 5492	24 4940	4299	0.0045 7160	1 7145	249
85.6	0.5975 0432	24 9239	4481	0.0044 0015	1 6896	253
85.7	0.5999 9671	25 3720	4676	0.0042 3119	1 6643	256
85.8	0.6025 3391	25 8396	4885	0.0040 6476	1 6387	260
85.9	0.6051 1788	26 3281	5109	0.0039 0089	1 6127	263
86.0	0.6077 5069	26 8390	5349	0.0037 3962	1 5864	267
86.1	0.6104 3459	27 3739	5607	0.0035 8097	1 5598	270
86.2	0.6131 7198	27 9346	5886	0.0034 2499	1 5327	274
86.3	0.6159 6543	28 5231	6186	0.0032 7172	1 5053	278
86.4	0.6188 1775	29 1418	6512	0.0031 2118	1 4775	282
86.5	0.6217 3193	29 7929	6865	0.0029 7343	1 4493	286
86.6	0.6247 1122	30 4794	7248	0.0028 2850	1 4207	290
36.7	0.6277 5916	31 2042	7667	0.0026 8642	1 3917	295
86.8	0.6308 7958	31 9709	8124	0.0025 4725	1 3622	299
86.9	0.6340 7668	32 7834	8626	0.0024 1103	1 3323	304
87.0	0.6373 5501	33 6459	9177	0.0022 7779	1 3020	308
87.1	0.6407 1961	34 5636	9785	0.0021 4759	1 2712	313
87.2	0.6441 7597	35 5422	10459	0.0020 2048	1 2398	318
87.3	0.6477 3019	36 5881	11208	0.0018 9649	1 2080	324
87.4	0.6513 8900	37 7089	12043	0.0017 7569	1 1757	329
87.5	0.6551 5989	38 9132	12980	0.0016 5813	1 1428	335
87.6	0.6590 5121	40 2112	14035	0.0015 4385	1 1093	340
87.7	0.6630 7233	41 6147	15230	0.0014 3292	1 0753	347
87.8	0.6672 3380	43 1377	16590	0.0013 2540	1 0406	353
87.9	0.6715 4757	44 7967	18149	0.0012 2134	1 0053	360
88.0	0.6760 2724	46 6116	19948	0.0011 2081	9693	367
88.1	0.6806 8840	48 6064	22040	0.0010 2387	9327	374
88.2	0.6855 4904	50 8104	24492	0.0009 3060	8953	382
88.3	0.6906 3009	53 2597	27396	0.0008 4107	8571	390
88.4	0.6959 5605	55 9993	30870	0.0007 5536	8181	399
88.5	0.7015 5598	59 0862	35077	0.0006 7355	7782	408
88.6	0.7074 6460	62 5940	40245	0.0005 9573	7374	418
88.7	0.7137 2400	66 6184	46693	0.0005 2199	6956	429
88.8	0.7203 8584	71 2878	54895	0.0004 5242	6527	441
88.9	0.7275 1462	76 7773	65561	0.0003 8715	6087	453
89.0	0.7351 9234	83 3334	79812	0.0003 2628	5633	467
89.1	0.7435 2568	91 3146	99496	0.0002 6995	5166	483
89.2	0.7526 5714	101 2642	127847	0.0002 1829	4683	501
89.3	0.7627 8356	114 0489	170975	0.0001 7146	4181	522
89.4	0.7741 8844	131 1464	241655	0.0001 2965	3660	546
89.5	0.7873 0308	155 3119	370693	0.0000 9305	3114	576
89.6	0.8028 3427	192 3813	650756	0.0000 6192	2538	615
89.7	0.8220 7240	257 4569	1501510	0.0000 3654	1923	670
89.8	0.8478 1809	407 6079		0.0000 1731	1253	774
89.9	0.8885 7889			0.0000 0479	479	
90.0	Inf.			0.0000 0000		

The preceding table of logarithms of the elliptic integrals of the first and second kinds is taken from Legendre's *Traité des Fonctions Elliptiques*, Vol. 2, Table I. The values from 45° to 90° are given for intervals of 0.1°. The values from 0° to 45°, which are comparatively seldom required, have been omitted. For formula and table to be used in interpolation, see page 132.

TABLE XIV.

Binomial Coefficients for Interpolation by Differences.

k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3		k	Coefficients of Δ_2 and Δ_3	
	K_2	K_3		K_2	K_3		K_2	K_3		K_2	K_3
.01	—.005	+.003	.26	—.096	+.056	.51	—.125	+.062	.76	—.091	+.038
.02	—.010	+.006	.27	—.099	+.057	.52	—.125	+.062	.77	—.089	+.036
.03	—.015	+.010	.28	—.101	+.058	.53	—.125	+.061	.78	—.086	+.035
.04	—.019	+.013	.29	—.103	+.059	.54	—.124	+.060	.79	—.083	+.033
.05	—.024	+.015	.30	—.105	+.060	.55	—.124	+.060	.80	—.080	+.032
.06	—.028	+.018	.31	—.107	+.060	.56	—.124	+.059	.81	—.077	+.031
.07	—.033	+.021	.32	—.109	+.061	.57	—.123	+.058	.82	—.074	+.029
.08	—.037	+.024	.33	—.111	+.062	.58	—.122	+.058	.83	—.071	+.028
.09	—.041	+.026	.34	—.112	+.062	.59	—.121	+.057	.84	—.067	+.026
.10	—.045	+.028	.35	—.114	+.063	.60	—.120	+.056	.85	—.064	+.024
.11	—.049	+.031	.36	—.115	+.063	.61	—.119	+.055	.86	—.060	+.023
.12	—.053	+.033	.37	—.117	+.063	.62	—.118	+.054	.87	—.057	+.021
.13	—.057	+.035	.38	—.118	+.064	.63	—.117	+.053	.88	—.053	+.020
.14	—.060	+.037	.39	—.119	+.064	.64	—.115	+.052	.89	—.049	+.018
.15	—.064	+.039	.40	—.120	+.064	.65	—.114	+.051	.90	—.045	+.016
.16	—.067	+.041	.41	—.121	+.064	.66	—.112	+.050	.91	—.041	+.015
.17	—.071	+.043	.42	—.122	+.064	.67	—.111	+.049	.92	—.037	+.013
.18	—.074	+.045	.43	—.123	+.064	.68	—.109	+.048	.93	—.033	+.012
.19	—.077	+.046	.44	—.123	+.064	.69	—.107	+.047	.94	—.028	+.010
.20	—.080	+.048	.45	—.124	+.064	.70	—.105	+.045	.95	—.024	+.008
.21	—.083	+.049	.46	—.124	+.064	.71	—.103	+.044	.96	—.019	+.007
.22	—.086	+.051	.47	—.125	+.064	.72	—.101	+.043	.97	—.015	+.005
.23	—.089	+.052	.48	—.125	+.063	.73	—.099	+.042	.98	—.010	+.003
.24	—.091	+.053	.49	—.125	+.063	.74	—.096	+.040	.99	—.005	+.002
.25	—.094	+.055	.50	—.125	+.063	.75	—.094	+.039	1.00	—.000	+.000

INTERPOLATION FORMULA.

$$f(a+h)=f(a)+k\Delta_1+\frac{k(k-1)}{2\,!}\Delta_2+\frac{k(k-1)(k-2)}{3\,!}\Delta_3$$
$$+\frac{k(k-1)(k-2)(k-3)}{4\,!}\Delta_4+\ldots. \qquad (a)$$

$$\text{or, } f(a+h)=f(a)+k\Delta_1+K_2\Delta_2+K_3\Delta_3+\ldots. \qquad (b)$$

where the constants K_2 and K_3 are given in the above table as functions of k and

$$k=\frac{h}{\delta}$$

where h is the remainder above the value of a for which the function is given in the table, and δ is the increment of a in the table.

ILLUSTRATION.

To find the value of log F for $49^\circ\ 15'\ 36''=49^\circ\!.260$
For 49.2° log $F=0.28363130=f(a)$
$h=.06,$ $\delta=0.1$ $k=0.6$
From Table XIV, $K_2=-.120$
$K_3=+.056$
From Table XIII,
$\Delta_1=39338$
$\Delta_2=\ \ 117$
$\Delta_3=\ \ \ \ 1$

Substituting these values of K_2, K_3, Δ_1, Δ_2, Δ_3 in formula (b) above we have as the value of log F for the given angle

$$\log F=0.28363130+.00023603-.00000014=0.28386719.$$

WASHINGTON, December 17, 1907.

SOME CONTACT RECTIFIERS OF ELECTRIC CURRENTS.

By L. W. Austin.

In a recent number of the Physical Review[1] I have described an aluminum-tellurium contact capable of transforming small alternating currents of any frequency into direct currents by means of what seems to be thermoelectric action. It is not generally known, however, that there is a large number of solid conductors[2] of electricity, metallic and nonmetallic, in which, when brought together so as to form a contact of not too low resistance, electricity appears to pass more easily in one direction than in the other.

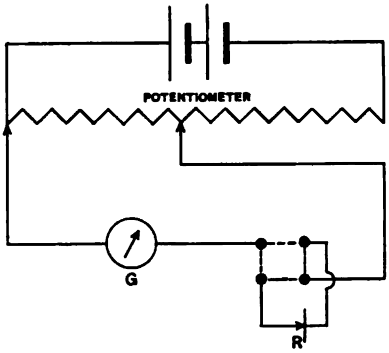

Fig. 1.

Most cases of this kind are too uncertain and capricious in their action to allow definite study. But silicon in contact with almost any of the ordinary metals, carbon-steel, and tellurium-aluminum all show a well marked and fairly regular unilateral conductivity.

In the investigation of these contacts I have used two main methods of experiment. In the first, Fig. 1, a direct electromotive force was applied to the contact R, first in one direction and then in the other, and the resulting currents measured on a galvanometer or ammeter having resistances small in comparison with that of the contact. In the second method, Fig. 2, a known alternating current at 60 cycles was sent through a fall of potential wire and the required voltages taken from sliding contacts and applied to the rectifying contact R, which was in series with the direct current meter.

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[1] Physical Review, **24**, p. 508; 1907.

[2] Possibly the case of carborundum investigated by G. W. Pierce (Phys. Rev., **25**, p. 31; 1907) may be a case of contact rectification.

SILICON.

The fact that a piece of silicon when properly brought in contact with brass or copper, or, in fact, almost any of the common metals, is capable of acting as a detector for electrical oscillations without the use of external electromotive force was first announced by G. W. Pickard[3] who, as it seems to me, erroneously ascribed the phenomenon to thermoelectric action. For while it is possible with much difficulty to adjust the contact so that for small electromotive forces the rectified current flows in the same direction as that produced by a heating of the junction, under ordinary conditions it flows in the opposite direction and may be far greater than could be produced by heating the junction to the melting point of the copper.

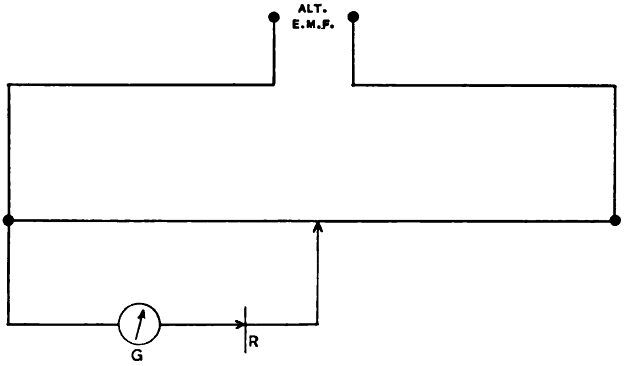

Fig. 2.

The rectifiers used in the experiments were made by embedding a piece of silicon A in solder[4], Fig. 3, and bringing a bit of brass or steel wire P soldered to the end of a flat spring S in contact with it. The contact pressure is adjusted by means of a screw M pressing on the spring from above. As some points on the silicon are far more sensitive than others, the contact point is also made adjustable.

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[3] Electrical World, 48, p. 1003; 1906.

[4] It is often of advantage to polish the contact surface of the rough silicon on an oil stone.

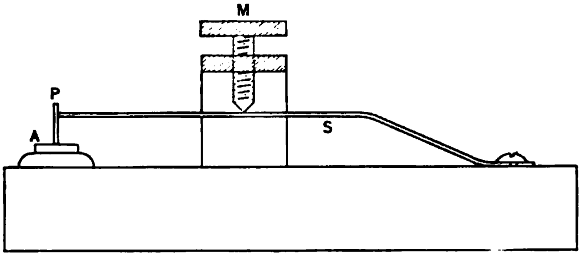

Fig. 3.

Table I shows the unilateral conductivity of a silicon-steel contact for direct currents. The arrangement of the circuit is that shown in Fig. 1. The galvanometer G was a pointer instrument having a sensibility of 2×10^{-6} ampere. The milliammeter used in

TABLE I.

Silicon-Steel. (*Direct Current.*)

Volts		Silicon to Steel		Steel to Silicon		Current Difference 10^{-6} Amp.
		Current 10^{-6} Amp.	Apparent Resistance Ohms	Current 10^{-6} Amp.	Apparent Resistance Ohms	
0.014	Galvanometer 280 ohms.	5.5	2600	5.8	2400	0.3
0.028		10.7	2600	11.9	2350	1.2
0.042		16	2600	18.5	2250	2.5
0.070	Galvanometer $\frac{1}{10}$ shunt.	32	2180	47	1490	15
0.098		45	2180	70	1400	25
0.14		65	2150	118	1190	53
0.28	Galvanometer $\frac{1}{100}$ shunt.	180	1550	600	467	420
0.42		300	1400	1250	336	950
0.56		400	1400	2200	254	1800
0.70	Milliammeter 0.3 ohm.	500	1400	2900	241	2400
0.98		800	1220	5000	196	4200
1.4		1200	1170	10000	140	8800
1.7		2500	680	12800	133	10300
2.0		3200	625	16000	125	12800
2.2		4800	460	20000	110	15200
2.5		6000	420	25000 (Unsteady)	100	19000

place of the galvanometer for the larger currents had a sensibility of 1×10^{-3} ampere per division. This contact was used with the adjustment found most sensitive for small alternating currents. It is seen from the table that the resistance for currents in the direction steel to silicon drops rapidly with increasing voltage. Hence it is desirable to use galvanometers of low resistance with the higher voltages.

Table II contains the rectified currents for various alternating voltages, the contact being adjusted for maximum rectification

TABLE II.

Silicon-Steel. (*Alternating Current.*)

Volts A. C.		Rectified Current 10^{-6} Amp.
0.02	Galvanometer (280 ohms).	0.6
0.04		2
0.055		4
0.065		5.5
0.090		10
0.11		15
0.13		21
0.13	Galvanometer ($\frac{1}{10}$ shunt).	45
0.15		60
0.185		100
0.22		150
0.25		200
0.25	Galvanometer ($\frac{1}{100}$ shunt).	300
0.32		600
0.395		1000
0.495		1500
1	Milliammeter (0.3 ohm).	3500
2		8100
4		17000
6		25000 irregular.

(Contact very hot.)

with the smaller voltages.[5] It is to be noticed that these rectified currents up to about 1.0 volt correspond with some degree of approximation with the differences in the direct currents in Table I.

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[5] A closer contact gives larger rectified currents for the larger voltages.

Below 0.2 volt the phenomenon is extremely reproducible, and contacts may be set and remain constant in their action for weeks at a time, unless violently jarred or exposed to too high voltages. Within this range the rectified currents are approximately proportional to the square of the alternating voltage. Hence a silicon rectifier with a galvanometer is a useful instrument for all kinds of high frequency work with small currents. The effect for larger voltages, at least with the present form of contact, is more or less irregular, which is probably due to excessive heating.

In Table III and curve A of Fig. 4 are shown the relations between very small voltages and rectified current. The galvanometer used was of the ordinary D'Arsonval type, having a resistance of about 250 ohms and a sensibility of 2×10^{-8} ampere per division.

TABLE III.

Silicon-Steel. (*Alternating Current.*)

GALVANOMETER (250 OHMS).

Volts A. C.	Rectified Current 10^{-8} Amp.	
	Obs.	Cal.
0.0014	0.6	0.4
0.0028	1.7	1.6
0.0042	4.1	3.7
0.0056	7.6	6.6
0.0070	10.7	10.3
0.0098	20.3	20.2
0.0140	39.2	41.2
0.021	82.0	92.8
0.025	107.0	131.0
0.028	146.0	165.0
0.042	312.0	371.0
0.056	552.0	659.0

For these very small voltages the direct current is apparently strictly proportional to the square of the voltage within the errors of observation, provided all the resistances in series with the rectifier are negligible. That this is not the case in Table III can be seen if the values calculated from the square law be compared with those observed. These values are given in the third column. In Table IV is given a similar set of observations, taken with the same galvanometer shunted to one-tenth. Here the errors in the agreement lie within the limits of the errors of observation. It is to be noted that for high frequency work the alternating current does not pass through the galvanometer, and hence the square law may be generally taken as true.

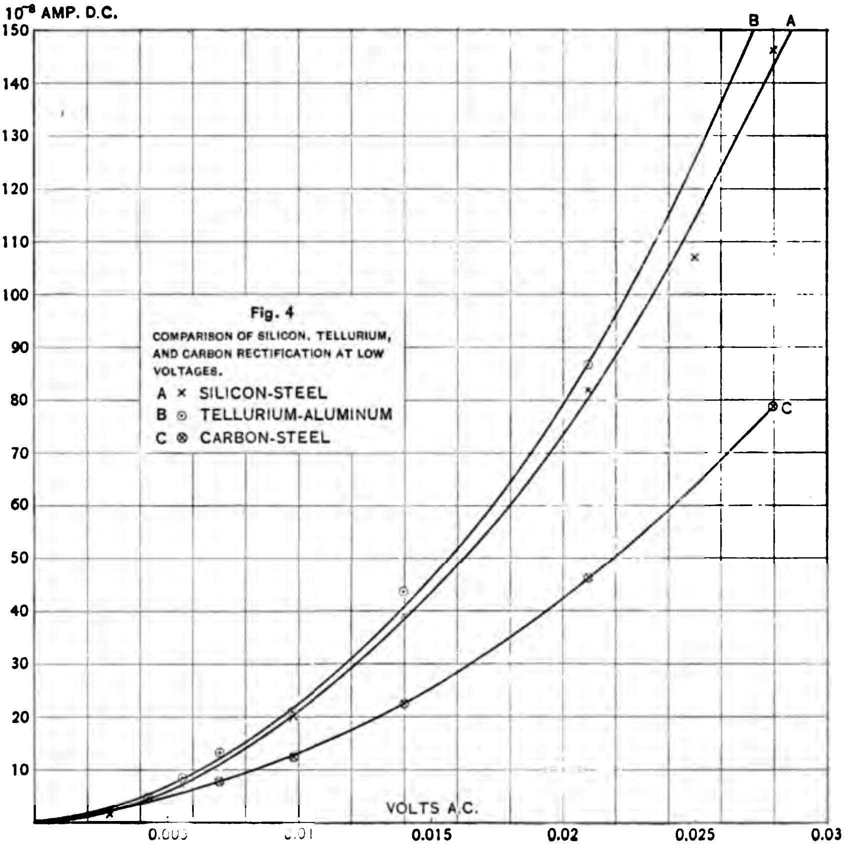

Fig. 4.

With this same rectifier and galvanometer, observations were made, also, with high frequency currents and a comparison of sensitiveness made with the Wollaston wire electrolytic receiver connected to a pair of 1200 ohm telephone receivers of the type ordinarily used in wireless telegraphy. Even with this comparatively nonsensitive galvanometer, all signals of not too short duration capable of being distinctly heard in the telephones produced a readable deflection.

TABLE IV.

Silicon-Steel. (*Alternating Current.*)

GALVANOMETER (250 OHMS 1/10 SHUNT).

Volts A. C.	Rectified Current 10^{-8} Amp.	
	Obs.	Cal.
0.0042	7.0	7.0
0.0070	19.0	19.5
0.0098	39.0	38.3
0.014	77.0	78.2
0.021	170.0	176.0
0.028	311.0	312.0
0.042	690.0	705.0
0.056	1250.0	1250.0
0.070	1960.0	1955.0

For the determination of the absolute sensitiveness of the silicon rectifier at high frequencies, a rectifier of rather low sensibility was compared with an extremely sensitive thermoelement at a frequency of 140,000. They were connected, in turn, in series with a one microfarad paper condenser for stopping the direct current, and with an inductance of 0.0331 microhenry coupled very loosely to a second tuned circuit in which oscillations were excited. The damping of the latter circuit was very small compared with that of the former. The readings on the galvanometer with the thermoelement indicated a mean alternating electromotive force for nine sets of observations, after taking account of the reactance of the coupling coil, of 0.0336 volt. The deflection due to the silicon indicated a mean electromotive force of 0.0350 volt, according to the calibration of the rectifier for low frequency currents. This discrepancy of 4 per cent is less than the average deviation from the mean (5.2 per cent). The experiment shows that there is no change in sensitiveness with frequency within the limits of accuracy of the observations.[6]

While in general the rectified current with silicon rectifiers flows from steel to silicon, an exception has been found in a certain sample which is probably somewhat less pure in its composition than the rest of the silicon specimens used. From this sample, rectifiers of exceptional sensitiveness have been made, but, curiously enough, the rectified current flows from silicon to steel, or in the opposite direction to that usually observed. Still more curiously, however, it is found that the thermoelectric electromotive force[7] at the contact is also reversed, still remaining in opposition to the rectification. That this effect is really a property of this sample is shown by the fact that the reversals persist in the case of other pieces broken from the larger sample and also with freshly broken surfaces. As might be expected, when a contact is made between two specimens of silicon with the opposite rectifying qualities, a rectifier of remarkable sensitiveness is produced.

CARBON-STEEL.

Another substance capable of forming rectifying contacts is carbon. It gives the most satisfactory results when used with steel.[8] The rectifiers were made up in the same general form as those already described, the polished surface of an ordinary steel sewing needle in contact with arc light carbon[9] giving excellent results. By varying the pressure it is possible to produce rectifiers of widely varying resistance.

For high resistance rectifiers particularly good results were obtained by making the contact on the soft center of a cored carbon. The high resistance contacts, while showing regular rectification for alternating currents, do not, so far as I have observed, show regular and satisfactory unilateral conductivity for direct currents.

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[6] In using low frequency calibration for the measurement of high frequency currents, the contact must be close enough so that there is no tendency to coherer action.

[7] The contact was heated by heating the wire above it in a small flame.

[8] A rectifier of some sensitiveness can be produced by bringing a bit of carbon incandescent light filament in contact with arc light carbon.

[9] Graphite seems to be distinctly inferior.

Tables V and VI contain observations on a low resistance rectifier for direct and alternating currents. The differences in the direct currents correspond very closely with the rectified portion of the alternating current up to about 0.3 volt.

TABLE V.

Carbon-Steel. (*Direct Current.*)

MILLIAMMETER (0.3 OHM).

Volts	Carbon to steel		Steel to Carbon		Current Difference 10^{-3} Amp.
	Current 10^{-3} Amp.	Apparent Resistance Ohms	Current 10^{-3} Amp.	Apparent Resistance Ohms	
0.02	1.2	16.6	1.3	15.4	0.1
0.05	3.1	16.1	3.4	14.7	0.3
0.10	6.3	15.9	7.4	13.5	1.1
0.20	13.0	15.4	15.1	13.2	2.1
0.30	20.0	15.0	24.0	12.5	4.0
0.40	33.0	12.1	40.0	10.0	7.0

TABLE VI.

Carbon-Steel. (*Alternating Current.*)

GALVANOMETER (1 OHM).

Volts A. C.	Rectified Current 10^{-3} Amp.
0.02	0.08
0.03	0.15
0.05	0.40
0.10	1.0
0.20	2.4
0.30	3.8
0.40	5.0

A comparison with the silicon observations shows that for the higher voltages the carbon rectification is much less perfect, but as the carbon sensibility curve drops more slowly the difference for very small voltages is less. It is also to be noted that in the case of carbon the effect becomes too irregular for observation at a much lower voltage.

Table VII shows the rectification of a high resistance rectifier for medium voltages, while Table VIII shows the same rectifier used with a more sensitive galvanometer for very low voltages. These results are shown also in curve C in Fig. 4.

TABLE VII.

Carbon-Steel. (*Alternating Current.*)

GALVANOMETER (280 OHMS).

Volts A. C.	Rectified Current 10^{-6} Amp.
0.02	0.4
0.04	1.3
0.06	2.2
0.08	3.2
0.10	4.9
0.12	6.9
0.14	9.0
0.16	12.5
0.18	15.5
0.20	17.8
0.21	20.0

TABLE VIII.

Carbon-Steel. (*Alternating Current.*)

GALVANOMETER (250 OHMS).

Volts A. C.	Rectified Current 10^{-8} Amp.
0.0014	0.8
0.0028	2.0
0.0042	3.2
0.0070	7.6
0.0098	12.2
0.014	22.2
0.021	46.0
0.028	78.6
0.042	172.6
0.056	290.0
0.070	430.0

TELLURIUM.

In addition to the rectifying effect of tellurium-aluminum contacts described in the paper already cited, which appears to be thermoelectric or at least in the same direction as the thermoelectric action and which becomes irregular above low voltages, there is a second rectifying effect in opposition to the first. This in general becomes prominent only at the higher voltages. There is a middle region in the vicinity of 0.5 volt where the two effects appear to be in conflict, and it is interesting as the alternating current is increased to see the galvanometer deflection change sign as one effect passes into the other. This second effect seems entirely analogous to the rectification of silicon and is of the same order of magnitude. Table IX and curve B of Fig. 4 show the action of tellurium-aluminum rectifiers for very small alternating voltages at 60 cycles, and correspond fully to the silicon and carbon observations in Tables III and VIII. It is seen that the sensitiveness is almost exactly the same at that of the silicon-steel contact. For quantitative observations the tellurium-aluminum is somewhat inferior in point of constancy. In Table IX the rectified current is in the same direction as the thermoelectric

TABLE IX.

Tellurium-Aluminum. (*Alternating Current.*)

GALVANOMETER (250 OHMS).

Volts A. C.	Rectified Current 10^{-8} Amp.
0.0028	1.8
0.0042	4.6
0.0056	8.5
0.0070	13.8
0.0098	21.0
0.014	44.0
0.021	87.0
0.028	160.0
0.042	368.0
0.056	690.0

current due to heating the contact. The curve of Fig. 5 shows the course of the phenomenon at somewhat higher voltages and the reversal of current as it passes into the opposite rectifying effect. The voltage at which this transition takes place depends on the pressure between the two metals at the contact, being lower the greater the pressure.

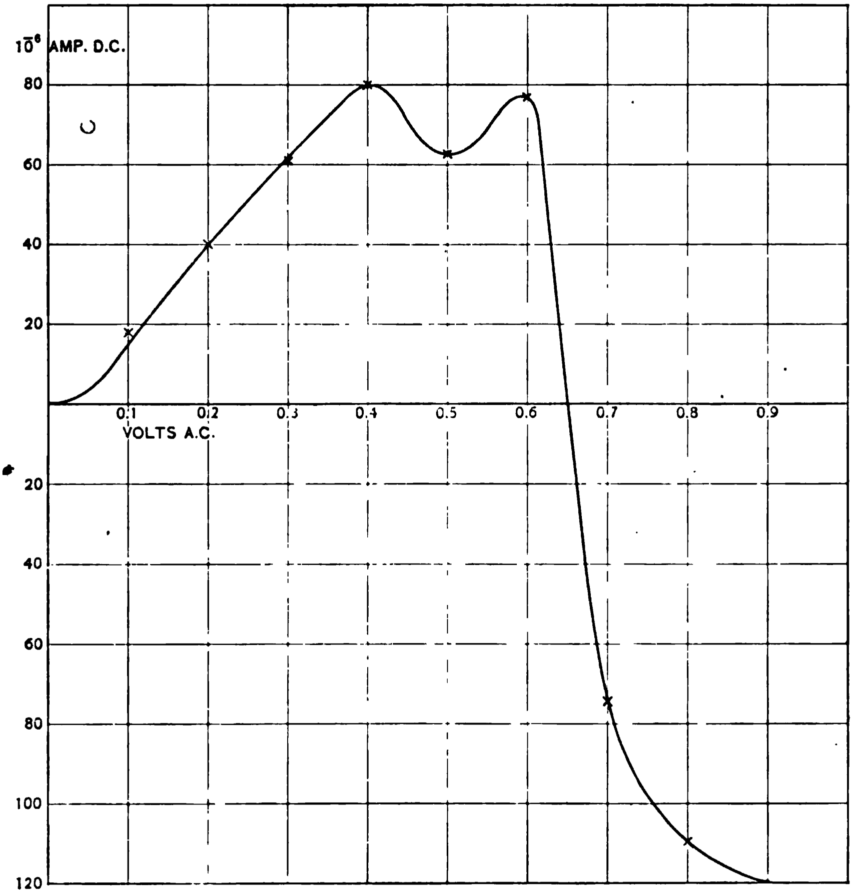

Fig. 5.

Table X shows the relation between alternating current voltage at 60 cycles and rectified current as measured on a milliammeter. In this case the contact was close and regulated to give a maximum effect at these voltages. Experiments have been made also on a contact formed by melting the end of a No. 20 aluminum wire into a block of tellurium, the other contact being formed by inserting a piece of white hot platinum wire in the tellurium. Hot platinum inserted in this way forms a very low resistance contact, and in fact seems to become alloyed with the tellurium. A tellurium-aluminum contact of this kind, while it is not very efficient in rectifying alternating currents, apparently on account of slowness in action, has remarkable unilateral conductivity for direct currents, as is seen in Table XI, and has the additional advantage for purposes of investigation that all the conditions can be kept constant indefinitely. With this form of contact evidences of

TABLE X.

Tellurium-Aluminum. (*Alternating Current, close contact.*)

Volts A. C.	Rectified Current 10^{-3} Amp.
0.42	1
0.52	2
0.63	3
0.70	5
0.98	10
1.32	20

TABLE XI.

Tellurium-Aluminum. (*Direct Current.*)

Volts	Te to Al		Al to Te	
	Amp.	Apparent Resistance Ohms	Amp.	Apparent Resistance Ohms
0.4	0.05	8.0	0.008	50
0.6	0.10	6.0	0.008	75
0.8	0.18	4.4	0.009	89
1.0	0.25	4.0	0.010	100
1.4	0.37	3.8	0.010	140
2.8	1.00	2.8	0.015	186

polarization have been sought by throwing the rectifier from the battery circuit into that of a galvanometer by means of a rapid commutator. If polarization exists, however, it is masked thus far by the large thermoelectric currents due to the unavoidable heating of the rectifying contact.

CONCLUSIONS.

There seems to be no obvious explanation of the phenomenon described above. I will therefore in conclusion only call attention to some of its more general characteristics.

In all the cases noticed, the contact has a resistance of several ohms as though there were a resisting film between the conductors. In all the cases, too, there is a comparatively poor conductor in contact with a good conductor, but the rectification in the cases of carbon-steel and of silicon-steel is from the good to the poor, while in tellurium-aluminum it is in the opposite direction (except for very low voltages where another phenomenon, perhaps thermoelectric, predominates). It may be worth noting that the first effect is in the same direction as the rectified current in the aluminum electrolytic rectifier.

Another peculiar fact is that in the three cases studied the rectified current flows in opposition to the thermoelectric current produced by heating the contact, except in the case of tellurium-aluminum at low voltages, as noted above. This was particularly brought to notice by the reversal of the thermoelectric effect in the silicon specimen in which the rectification was reversed. What possible connection there can be between the two facts is, however, difficult to understand.

From the evidence already given it is clear that we are dealing with a contact phenomenon, depending as it does vitally upon the pressure and area of the contact surfaces. When these surfaces are sufficiently large and the contact sufficiently close the rectification entirely disappears. Even if the rectified currents were not opposed in direction to the ordinary thermoelectric currents, their magnitude would seem to preclude the possibility of a thermoelectric explanation. Nevertheless the fact that the direct currents are in general roughly proportional to the square of the alternating currents suggests heat action.

It may be that there is something of the nature of polarization and a counter electromotive force at the contact, but if this exists no evidence of it has thus far been found.

Of course it is possible that we are here dealing with the still obscure question of the escape of electrons from a conductor and that our rectifying contacts furnish us with conditions under which the electrons pass more readily in one direction than in the other, that is, that it is a case of direct and not secondary rectification.

The rectified currents described have been small but it seems probable that by a proper arrangement of contacts, perhaps in parallel, considerably larger currents could be rectified.

WASHINGTON, April 27, 1908.

A METHOD FOR PRODUCING FEEBLY DAMPED HIGH-FREQUENCY ELECTRICAL OSCILLATIONS FOR LABORATORY MEASUREMENTS.

By L. W. Austin.

The problem of producing high frequency oscillations of sufficient constancy for accurate laboratory experiments has always been a difficult one. Even with large induction coils and high-potential transformers the fluctuation in intensity amounts frequently to from ten to twenty per cent. The problem becomes especially difficult in the experimental testing of sensitive receivers, as in these experiments the use of any except the smallest induction coils is impossible unless they are separated from the receiving apparatus by a very great distance. Besides lack of constancy, another difficulty which is always met with in the case of small coils is the rapid damping of the oscillations.

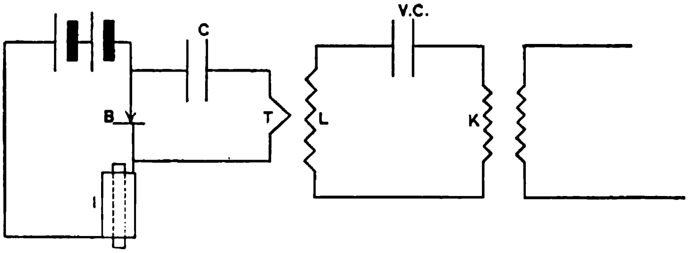

Fig. 1.

With the view of overcoming these difficulties I have experimented with various schemes for producing slightly damped oscillations of small intensity, and have finally decided upon the following arrangement as giving the most satisfactory results:

The general scheme of the apparatus is shown in Fig. 1. Here B is a circuit interrupter, I a reactance coil, C a paper condenser of from 0.1 to 1 microfarad, T a single turn of wire placed around a large inductance L^1, $V\ C$ a variable air condenser, and K a coupling coil for connecting this oscillating circuit with the circuit on which the measurements are to be made. The action is as follows: When the direct circuit is broken at B, the self induction current charges the paper condenser to a sudden high potential. This gives rise to a strongly damped pulse through the single turn of wire T, which starts the second circuit $L\ VC$ to oscillating in its own period, producing waves which are very feebly damped on account of its large inductance and negligible resistance. These can then be made use of for any experiments desired.[2]

If very great constancy in the intensity of the oscillations is not desired, the reactance and interrupter may be in the form of the ordinary buzzer. With this carefully adjusted, the variations in the oscillations may be kept as low as three to six per cent. For stronger oscillations a mercury turbine interrupter with a separate reactance coil may be used. For the greatest constancy of working I have obtained the best results with the vibrating wire mercury interrupter, a plan of which is shown in Fig. 2.[3] For showing

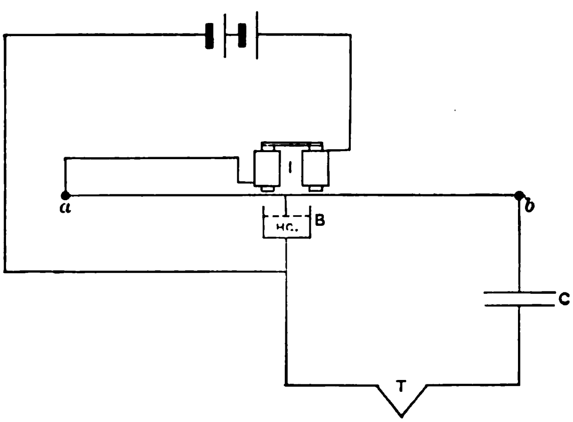

Fig. 2.

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[1] From 0.3 to 1 millihenry.

[2] A somewhat similar arrangement of interrupter and reactance coil is used by the Telefunken Co., and also by the Amalgamated Radio Telegraphic Co. in their wave meter sets, except that there the tuned circuit is connected directly to the interrupter and not excited inductively.

[3] The magnet is from a watch case buzzer, a, b, is a No. 26 steel wire, 10 cm. long. The diameter of the H_g cup is 1 cm.

the degree of constancy easily obtainable with this exciter, I give a series of galvanometer deflections taken exactly ten seconds apart while the vibrator was running continuously.

Galvanometer Connected to Si Rectifier in Third Circuit.

DEFLECTION.
831 mm
830 "
830 "
830 "
830 "
829 "
831 "
833 "
832 "
832 "
830 "
830 "

This form of interrupter is troublesome when used with any but very small currents on account of its tendency to throw out the mercury from the cup. This can be minimized by using a liquid amalgam made by dissolving bits of solder in the mercury, and also by using a cover provided with an opening for the platinum contact. The energy imparted to the oscillating circuit by this type of exciter can amount to at most a few milliwatts. This of course implies, if the necessary loose couplings between the circuits are to be used, that the detecting instrument must be of great sensitiveness.

For telephonic reception any of the receivers ordinarily used in commercial wireless telegraphy are suitable. For damping experiments, or any experiments in which deflection measurements are necessary, I have found the most constant and satisfactory results with the silicon rectifier.[4]

The buzzer form of exciter is of great value in wireless stations for testing the condition of the receiver and for adjusting and making measurements on the receiving circuits. For testing the receivers it is greatly superior to the ordinary form of untuned buzzer, the use of which often leads to incorrect adjustment on account of the difference in behavior of many receivers to single pulses and to slightly damped wave trains.[5]

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[4] This Bulletin, **5**, p. 133.

If it is possible to insert an inductance in addition to the coupling inductance in the antenna, the single turn of the exciter circuit may be placed about this (see Fig. 3), thus exciting waves of the natural antenna frequency. This gives a convenient means of bringing the antenna circuit and the closed circuit to the same tune and of testing the receiver for the actual wave lengths which are to be used. It is obvious that this arrangement may be used also for determination of antenna damping, of wave length, and other measurements if the constants of the second circuit are known. If it is not thought desirable to use an extra inductance in the antenna, a separate circuit can be set up, tuned to one of the wave lengths ordinarily employed, and this may be used as a source of oscillations.

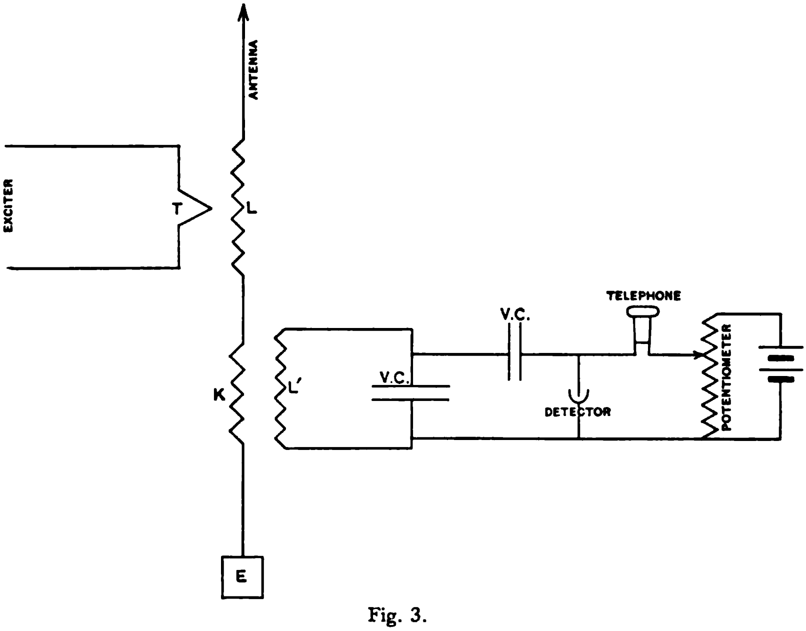

Fig. 3.

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[5] For example, the electrolytic receiver always appears most sensitive to the buzzer when the small electrode is just touching the acid, while for feebly damped waves of considerable wave length a greater immersion gives the best results.

ON THE ADVANTAGES OF A HIGH SPARK FREQUENCY IN RADIO-TELEGRAPHY.

By L. W. Austin.

Since practically all of the long-distance receivers used for radio-telegraphy make use of the telephone for the reception of signals, any circumstance which increases the sensitiveness of the telephone increases the sensitiveness of the receiving apparatus in the same measure.

That the telephone is more sensitive for high frequencies has been noted by a great number of observers, the subject having been especially studied by Lord Rayleigh[1] and M. Wien.[2] Their results are of particular value, since they as far as possible made use of sine waves, free from the overtones which introduce grave errors in experiments of this kind. Most of Lord Rayleigh's observations were made on a 70-ohm telephone of an ordinary type, and his main results are given in tabular form below.

n	**128**	**192**	**256**	**307**	**320**	**384**	**512**	**640**	**768**
Least audible current 10^{-8} amp.	**2800**	**250**	**83**	**49**	**32**	**15**	**7**	**4.4**	**10**

Wien studied a number of different types of telephone, and in his paper discusses at some length the relation between frequency and resistance, inductance, and the natural period of the diaphragm. All of these, together with the sensitiveness of the ear to sounds of different pitch[3], play a part in the phenomenon. As for the purposes of our practical problem only the summation of these effects is of interest, I will merely cite Wien's values for a Siemens' telephone of 187-ohm resistance.

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[1] Phil. Mag. **38**, p. 294; 1894.

[2] Ann. d. Physik, **4**, p. 450; 1901.

[3] Rayleigh, Phil. Mag. **14**, p. 596; 1907.

n	64	128	256	512	720	1024	1500	2400	4000
Least audible current 10^{-8} amp	1800	220	26	1.7	1.5	3.0	6.0	2.0	50

Both Wien's results and Lord Rayleigh's show a remarkably rapid increase in sensitiveness up to a frequency of about 500, above which the change is slight, showing small maxima depending on the natural periods of the diaphragms. This high degree of sensitiveness continues, according to Wien, up to a frequency of about 2500.

It is remarkable in the light of these results that, in the attempts to increase the working range of radio-telegraphy, so little attention has been paid to the advantage of a high spark frequency. One cause of the neglect of this question is apparently the widespread though unfounded belief among commercial workers in wireless telegraphy that the newer types of high-resistance telephone, such as they ordinarily use, do not show any great change of sensitiveness with frequency.

It was therefore thought desirable to carry out a similar investigation on telephones of the type used in radio-telegraphy. For this purpose a pair of Schmidt-Wilckes head telephones of about 800 ohms resistance was chosen for investigation. The Bureau of Standards is supplied with a set of dynamos giving fairly pure sine waves and extending over a range of 60–900 cycles per second. For this experiment the current of 80–100 milliamperes was measured on a sensitive hot-wire instrument and was then shunted through noninductive shunts so as to produce a sufficiently small fall of potential over a slide wire of known resistance. From this the necessary emf. to just produce an audible sound in the telephone was taken.

The table and curve in Fig. 1 give the results of the experiment. The change in volt sensitiveness is seen to be approximately one thousand times between 60 cycles and 900 cycles. I have appended the resistance and inductance at 100 and 900 cycles which were kindly determined for me by Mr. Curtis of the Bureau of Standards. From these values it would undoubtedly be possible to make some approximate estimate of the currents corresponding to the voltages given in the table, but as the inductance and resistance were determined with several milliamperes, while in this experiment we are dealing with from one microampere to less than a thousandth of a microampere, it did not seem certain that the approximation would be very close.

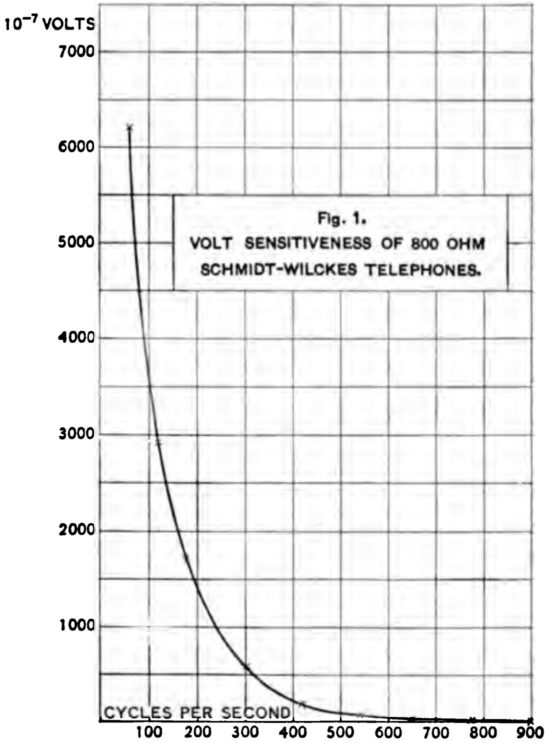

Fig. 1.

Comparing these results with Lord Rayleigh's and Wien's it is seen that the change in sensitiveness is of the same general type, the flat portion of the curve beginning at a somewhat higher frequency, however, and showing no maxima up to the point where the observations ceased. This indicates probably a somewhat higher natural period in the diaphragm.

Considering these results in their bearing on radio-telegraphy, it appears that we can, by increasing the spark frequency at the sending station, increase the effective sensitiveness of the receiving station many hundred times. This can be done, too, without entailing the difficulties connected with an increase in the sensitiveness of the wireless receiver itself. It is well known that the receivers now in use are already so sensitive that at many stations during the summer months evening receiving is made impossible by atmospheric disturbances. An additional advantage in using a high-pitched musical spark is that the ear picks out such signals with ease in the midst of ordinary interference and atmospheric disturbance. The difficulties of installing apparatus with high spark frequency do not seem to be serious, as I am told that efficient alternators giving a frequency of 400, i. e., 800 sparks per second, can be produced without difficulty. The question of cooling the spark gap can, I am sure, be settled by means of some type of rotary gap. A suitable form is shown in Fig. 2. The wheel is rotated rapidly by an independent motor without regard to synchronism with the alternator, the current being conveyed to the wheel by means of brushes. The face of the stationary member of the gap forms an arc of a circle, long enough to a little more than cover the distance between two spokes of the wheel, thus always insuring the proper sparking distance. The rotating wheel itself forms an efficient fan.

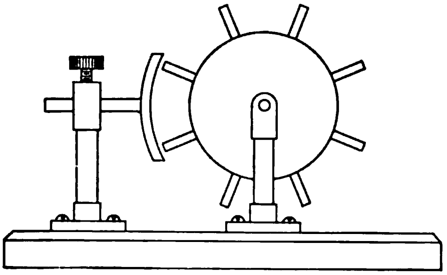
Fig. 2.

In increasing the spark frequency the energy per spark is of course reduced, which is disadvantageous in cases where a receiver of the recording coherer type is used, but experiment shows this reduction in energy per spark to be far more than counterbalanced by the increased sensitiveness of the receiving apparatus, where the telephone is employed.

I have even been able to show in the case of a station using a 60-cycle alternator and high potential transformer that the increase in spark frequency produced by shortening the spark gap and making use of the partial discharges which are usually considered so detrimental, brings about an increase of several times in the loudness of the signals without any increase in power consumed.

There is another quite distinct advantage in spreading a given amount of energy over several sparks instead of concentrating it in one, in that the potential differences are reduced, resulting in a reduction in the condenser losses, which in the average station amount to a considerable share of the total power.

Volt Sensitiveness of a pair of Schmidt-Wilckes 800-ohm Telephones.

Cycles.	Volts to produce audible sound.
60	6200×10^{-7}
120	2900 "
180	1700 "
300	600 "
420	170 "
540	80 "
660	30 "
780	11 "
900	6 "

D. C. Resist. = 813 ohms.
A. C. Resist. = 862 ohms. } 100~
Ind. = 404 m. h. }
A. C. Resist. = 1548 ohms. } 900~
Ind. = 241 m. h. }

WASHINGTON, April 27, 1908.

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DEPARTMENT OF COMMERCE AND LABOR

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S. W. STRATTON, DIRECTOR

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1908

SELECTIVE RADIATION FROM VARIOUS SOLIDS.

By W. W. Coblentz.

I. INTRODUCTION.

Our knowledge of the emission of radiant energy from various substances at different temperatures is extremely limited, being confined to platinum in the case of metallic electrical conductors, to several gases in vacuum tubes, to water-vapor and carbon dioxide in the Bunsen flame, to carbon, to the oxides of copper and iron, and to the radiation from an uniformly heated cavity or theoretically complete radiator. The emission spectra of the Bunsen flame and of gases in vacuum tubes were found to be composed of sharp emission bands superposed upon a weak continuous spectrum. The solids were found to have smooth continuous emission spectra, and it seems to be the general expectation to find that all solids emit continuous spectra.[1]

To Paschen is due the credit for the first systematic study of the spectral distribution of radiant energy from various solids and from the Bunsen flame. Subsequent work by others has been but little more than the establishment of the co-called radiation constants to a greater number of significant figures than was possible by Paschen, with the facilities at his disposal.

The best experimental proof of Kirchhoff's law of the proportionality of emission and absorption is due to Paschen.[2] He found that the intensity of the emission of the CO_2 band at 4.4μ when using a column of gas 7 cm long was as great as for a column of the gas 33 cm long. In other words, the intensity of the radiation of the column of gas 7 cm long was as great as that of a complete radiator for the same wave length and at the same temperature. Unfortunately, the number of radiating substances for which it is possible to determine the temperature, even approximately, is extremely small. Hence the work presented on the following pages can not be more than a qualitative proof of Kirchhoff's law of proportionality between emission and absorption. It has numerous applications, however, particularly in studying substances having sharp emission (and hence sharp absorption) bands. The emission spectra of numerous substances are given here, of which it has not been practicable to study the absorption spectra. Their emission spectra give us some idea of the nature of their absorption, the only difference being that the emission bands are more intense, due to the greater thickness and the higher temperature of the substance examined. Another feature of the results obtained, which is new, is the extreme sharpness of the emission bands. Moreover, the maxima of the emission and absorption bands coincide, although the temperatures at which the two sets of observations were made differ by 500° to 1000° C. The positions of the sharp, well-defined maxima are not affected by change in temperature. This is in marked contrast with the results of Königsberger[3] for the limited region of the visible spectrum, in which he found that for certain selectively absorbing substances the maximum of the absorption band shifts toward the long wave-lengths with rise in temperature, and with the results of Paschen[2] on the emission bands of CO_2 and water vapor, which shifted with rise in temperature, some toward the long, others toward the short wave-lengths. The data may also prove to be useful in determining whether pleochroism is an inter- or intra-molecular phenomenon. For example, the absorption spectrum of Adularia shows a band at 3.2μ, which in the emission spectrum is shifted to 2.9μ. The latter band is characteristic of the silicates, whether in the emission or absorption spectrum. The present work may be considered an examination of the emission spectra of electrical insulators, or so-called "transparent media." The only previous work done on this subject is due to Rubens,[4] who examined the radiation from the Auer mantle, as well as from mantles composed of the oxides of pure cerium and of pure thorium. The difficulty experienced by him was the elimination of the emission spectrum of the hot gases which was superposed upon that of the oxides composing the mantle. Hence, if he had examined a mantle of zirconium oxide, he would not have been able to detect an emission band which occurs at 4.3μ, since the gas flame has a strong emission band at this point.

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[1] Kayser. Spectroscopie, **2**, pp. 135 and 284.

[2] Paschen. Ann. der Phys. (3) **58**, p. 26; 1894.

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[3] Königsberger. Ann. der Phys., 4, p. 796; 1901.

[4] Rubens. Phys. Zs., 6, p. 790; 1905.

The substances used in the present investigation were in the form of solid rods made in an oxy-hydrogen flame, or in the form of thick layers of the substance placed upon a heater. The rods were heated by an electric current from the secondary of a 2000-volt, 300-watt transformer These rods had, of course, to be given an initial heating with an alcohol or blast lamp until they became conducting, just as is necessary with the Nernst glower. A resistance was placed in the primary circuit of the transformer, to regulate the current. The rods were provided with platinum terminals and were securely mounted in incandescent lamp-sockets. After heating them until they became conducting they were securely mounted before the spectrometer slit.

The substances that could not be melted and formed into rods were made into a paste and spread upon the "heater tube" of a Nernst lamp. The "heater tube" consisted of a hollow porcelain tube about 5 cm long and 8 mm diameter, covered with a coil of fine platinum wire, and was used in preference to a platinum strip on account of its rigidity and ease in handling. It gave the same results as the same material on a platinum strip. The spectrometer, the fluorite prism, and the bolometer[4a] were previously used in examining the radiation from the Nernst glower. It is important to notice, however, that by enclosing the optical parts of the instrument it was possible to entirely eliminate the atmospheric absorption bands, which had not been done successfully by previous experimenters. The emission curves are therefore, within experimental errors, an exact portrayal of the distribution of the energy emitted in different parts of the spectrum. Unfortunately it is not possible to determine the temperature of the radiating body. A thermocouple can not be applied on account of the loss of heat by conduction. It is of course absurd to attempt to measure the temperature of these substances with an optical pyrometer. For example, the rod of oligoclase was a perfectly transparent glass and emitted no light other than that due to sparking of the platinum terminals. Nevertheless, a substance such as iron oxide at the same temperature would have emitted light, while both emit strongly in the infra-red. Another illustration is the rod of topaz which was an opaque, white mass. On withdrawing it from the oxy-hydrogen flame it was accidentally stroked with an iron forceps, when the parts so stroked emitted a dull red light, due to the greater emissivity of the iron oxide, while the untouched parts retained their usual white color.

[4a] In the paper on "Instruments and Methods Used in Radiometry," this Bulletin 4, pp. 420 and 454, the term "*bolometer current*" is accidentally used for "*battery current*," which was 0.04 ampere. The "*bolometer current*" was therefore only 0.02 ampere, but could have been increased to 0.04 ampere, which would have doubled the sensibility.

Instead of temperatures, the energy consumptions are given; also the dimensions of the rods, the lengths being the distances between the platinum terminals. The diameters are in some cases not very accurate, owing to the irregularity of some of the filaments. The voltage was measured with an electrostatic voltmeter joined to the terminals of the rod. The current was measured with a milliammeter of a suitable range to insure accuracy.

II. RADIATION FROM ELECTRICALLY HEATED SOLIDS.

Prominent among this group of substances are a series of silicates, which have an emission band in common at 2.9μ, characteristic of SiO_2, which is as sharp as any yet found in gases. The absorption spectra of many of these compounds are recorded in a previous paper by the author.[5]

In order to give the reader some idea of the conditions under which the data were obtained, a rough estimation is made of the temperature at which a complete radiator would emit light of a color similar to that given out by the substance under investigation. The length of the rods used depended upon the melting point. The ends of the rods were shielded from the spectrometer slit.

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[5] Carnegie Publication No. 65.—Investigations of Infra-Red Spectra, Carnegie Institution of Washington, Dec., 1906.

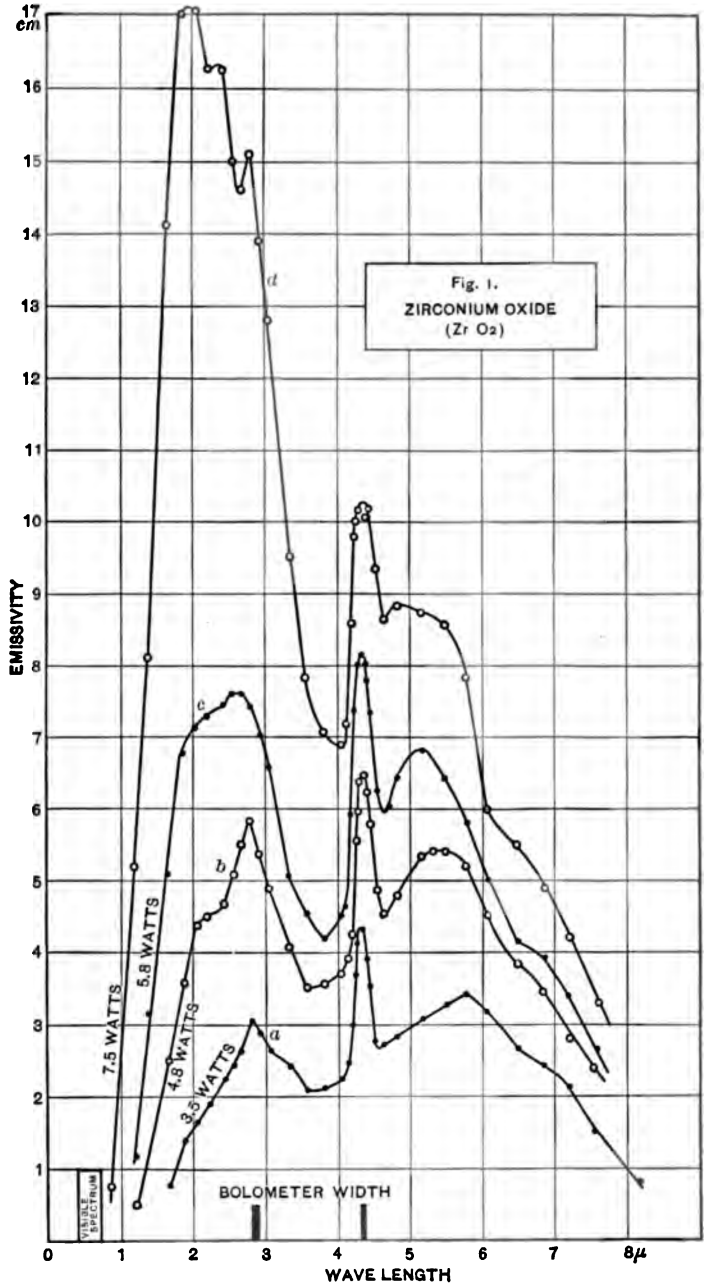

Fig. 1.
ZIRCONIUM OXIDE
(Zr O_2)

All the curves are reduced to the normal spectrum by dividing the observed galvanometer deflections by the slit-width expressed in wave-lengths. The unsteadiness of the bolometer prevented an accurate mapping of small emission bands occurring beyond 6μ.

1. *Zirconium Oxide* (ZrO_2).—[Rod 10 × 2 × 1.4 mm. Energy consumption, 3.5 (900°), 4.8, 5.6, 8.5 watts. Curves *a, b, c, d*, Fig. 1; and 9.6, 12, and 13.6 (1400°) watts. Curves *a, b, c*, Fig. 2]. The specimen examined was a fragment from a furnace. It probably contained some "binder," although from the curve for the pure material, Fig. 8, it appears that the foreign substance contributed but little to the emission bands. The fine platinum terminals were wound around the ends of the rod, which was about 10 mm long, the distance between the terminals being 5 mm. The curves are conspicuous for their sharp emission band at 4.3μ, which remains superposed upon the continuous background even at a bright yellow heat, corresponding to a temperature of about 1400°. This series of energy curves is one of the best illustrations yet recorded of the gradual shift of the maximum of intensity of emission toward the short wave-lengths, with rise of temperature. For an energy consumption of 3.5 watts the maximum of the envelope of these emission bands lies at 4 to 5μ, and shifts steadily toward the short wave-lengths, being at about 1.8μ for an energy consumption of 13.6 watts. The pure oxide is not an efficient radiator of white light, and only becomes so when a small amount of cerium, thorium, or yttrium oxide is added, which combination is the Nernst glower previously investigated.[6]

In addition to the sharp emission lines at 2.8 and 4.35μ, there are wide hazy bands at 2 and 2.4μ (appears on curve *d*), while from 5 to 6μ there is a wide band which is evidently unresolved, the maximum shifting toward the short wave-lengths with rise in temperature. The extraordinary rapidity which characterizes the growth of the emissivity at 1.5 to 2μ is worthy of notice. In one case where the ratio of emissivities at 4.3μ, for a given increase in energy, is 50, it is almost 200 at 2μ.

2. *Oligoclase*[7] $\left(\begin{matrix} 3\ Na\ Al\ Si_3\ O_8 \\ Ca\ Al_2\ Si_2\ O_8 \end{matrix}\right)$.—[Rod, 9 × 2.8 mm. Energy

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[6] This Bulletin, Vol. 4, p. 533; 1908.

[7] Transmission curves. Carnegie Publication No. 65, p. 64.

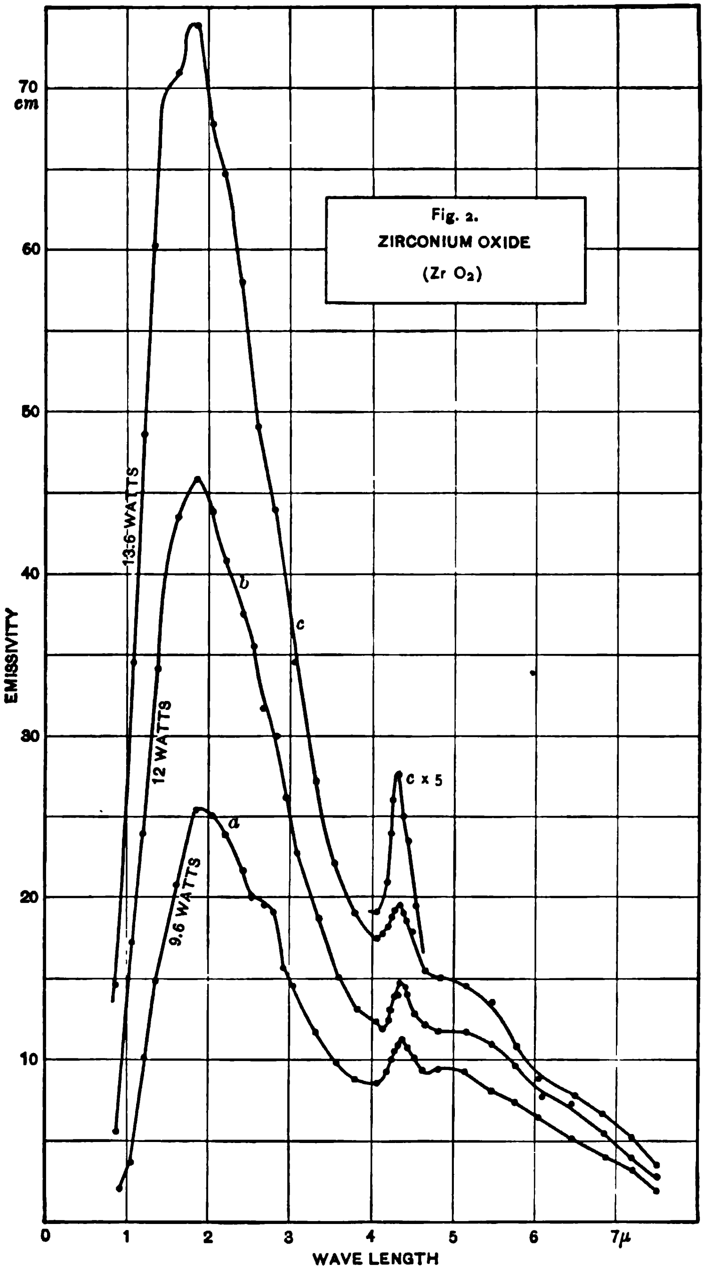

Fig. 2.
ZIRCONIUM OXIDE
(ZrO_2)

supplied was 8.6 watts and 13.3 watts. Curves *a* and *b*, Fig. 3.] The original crystal, as well as the glass rod, were perfectly transparent. The rod showed no color on suddenly throwing off the current. The platinum terminals melted on 16 watts, but the rod showed no color. As in the preceding feldspars there are bands at 2, 2.88, 3.1, 4.1, 4.5, and 7μ. The absorption band at 4.8μ seems to be shifted to 4.5μ in the emission spectrum. Curve *c* gives the emissivity of a rod 2.5 cm long, 2.1 mm diameter, heated (on 29.4 watts) until it became viscous (temperature about 1200° C), but did not sag. The radiation was taken from a length of 5 mm of the central portion of the filament by shielding it. The curve is conspicuous in showing that at 2μ the intensity has increased but little as compared with the band at 2.9μ. The isochromatics of this filament are given in Fig. 14. In curves *a* and *b* the ends of the rod were not shielded, so that some of the radiation at 1 to 2 may be due to internally reflected radiation of the hot platinum terminals.

3. *Albite*[8] ($Na\ Al\ Si_3\ O_8$).—The rod was translucent and seemed to emit no light except that reflected from the platinum terminals. The emission band at 2.88μ is more intense than the one found in orthoclase, as was found in the absorption spectrum. Two other bands are noticeable at 4.1 and 4.5μ, respectively.

4. *Orthoclase*[9] (*Var. Adularia*) ($K\ Al\ Si_3\ O_8$).—[Rod, 8 × 2 mm. Energy supplied, 2.2 and 3.9 watts.] This substance emitted a little more light than albite, although it was more transparent. The emission band at 2μ is prominent. The one at 2.9μ is shifted from its position at 3.2μ in the absorption spectrum, from which it would appear that the group of atoms causing the absorption is different in the two cases. An examination of the transmission spectrum, using polarized light, will be necessary to determine the true position of the absorption band. The bands at 4.1 and 4.5μ are in common with the feldspars.

5. *Beryl* ($Be_3\ Al_2$) ($Si\ O_3$)$_6$.—[Rod, 10 × 3 mm. Energy, 7 to 8 watts. Curve *a*, Fig. 4.] The rod was an opalescent, milky glass, although the original crystal was a transparent green. The

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[8] Transmission curves. Carnegie Publication, No. 65, p. 65.

[9] Transmission curves. Carnegie Publication, No. 65, p. 64.

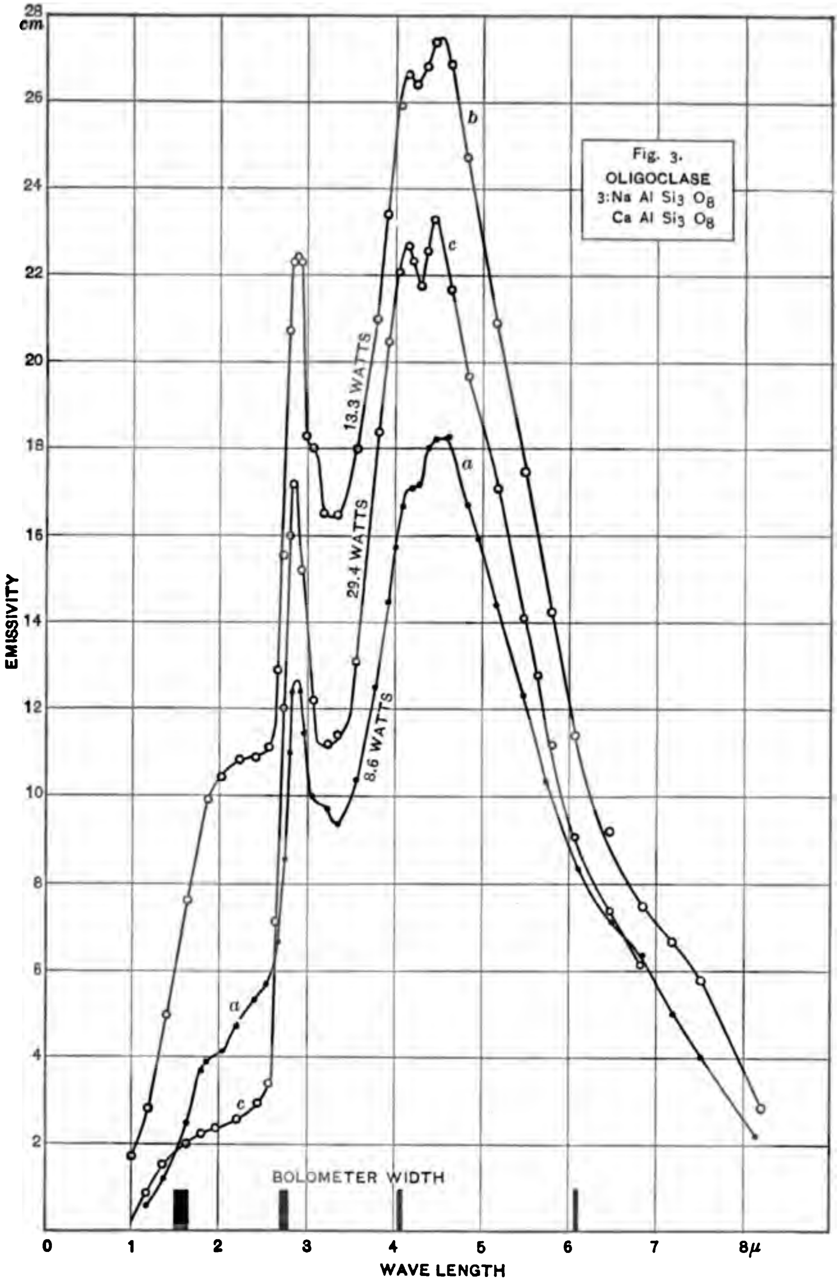
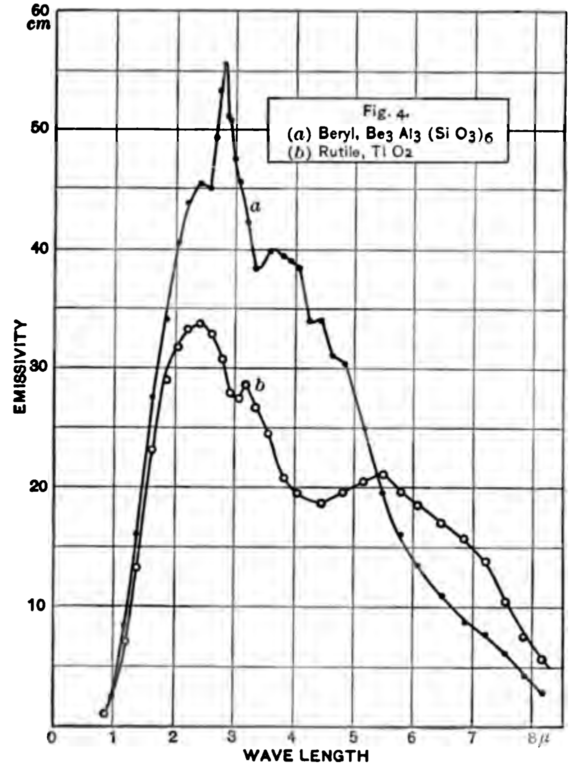


Fig. 3.
OLIGOCLASE
3:Na Al Si_3 O_8
Ca Al Si_3 O_8
28 cm
26
24
22
20
18
16
14
12
10
8
6
4
2
EMISSIVITY
0
1
2
3
4
5
6
7
8μ
WAVE LENGTH
BOLOMETER WIDTH
13.3 WATTS
29.4 WATTS
8.6 WATTS
a
b
c
a
c

apparent temperature was probably about 1100°. The emission spectrum is unusual in appearance, with a sharp maximum superposed upon it at 2.8μ. There appear to be other ill-defined maxima at 1.7, 2.4, 2.9(?), 3.6, 4.4, and 4.8μ, respectively.

6. *Rutile.*[10] ($Ti\ O_2$).—[Flat plate, 8 mm long, tapering from 1.5 to 1.8 mm wide and .25 mm thick. Energy supplied, 6 watts. Curve *b*, Fig. 4.] This mineral was heated to a light red color



corresponding to a temperature of perhaps 1000°. The terminals were of copper melted to the rutile plate. The emission spectrum shows maxima at 2.4, 3.2, 5.5, and 7.0μ. The transmission spectrum is too low to show these as absorption bands; but the band at 3.1μ is visible in the transmission spectrum of brookite, $Ti\ O_2$. This substance is a good conductor of electricity at this temperature, but a very poor radiator of light rays.

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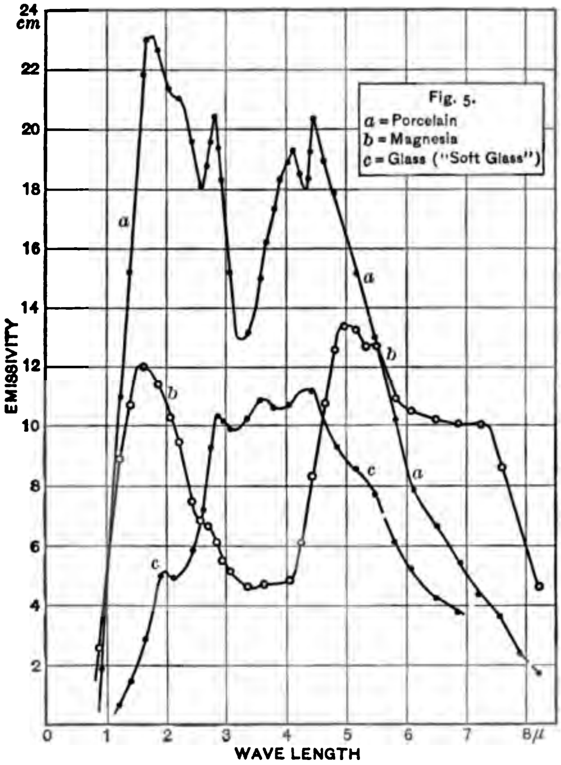
[10] Transmission curve of this plate given in Carnegie Publication, No. 65, p. 67.

7. *Porcelain* (*Pyrometer Tubing*).—[Hollow rod, 15 × 2 mm; hole, 1 mm. Energy supplied, about 8 watts. Curve *a*, Fig. 5.] The sample of pyrometer tubing examined was heated to a light red color (1400°) to keep it conducting. The energy spectrum is marked for its strong emission, with sharp maxima at 1.8, 2.1, 2.83, 3.7, 4.1, and 4.5μ, with indications of bands beyond 6μ.

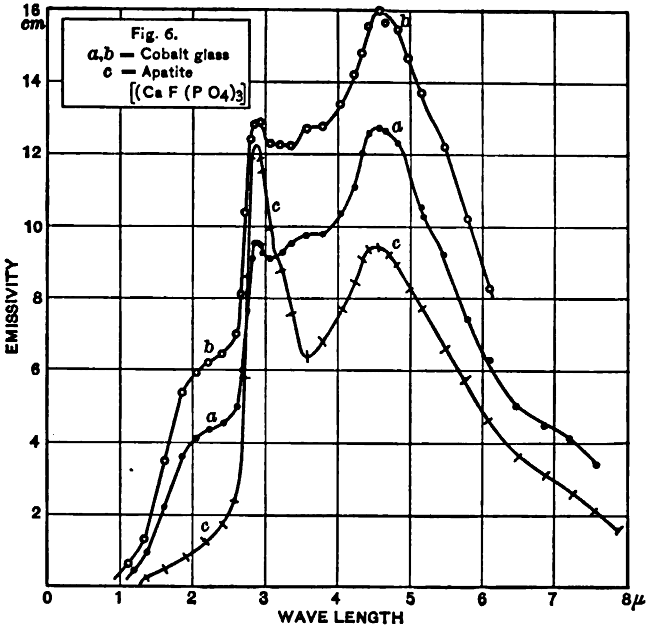
8. *Magnesia* (*Pyrometer Tubing*).—[Hollow rod, 12 × 1.5 mm; hole, 1 mm. Energy supplied, about 8 watts. Curve *b*, Fig. 5.]



This material is used to insulate thermocouples and conducts only at high temperatures. It is probably a compound of magnesium oxide and silica. The temperature was probably 1200° to 1400°. The emission spectrum is conspicuous for two regions of strong emissivity at 1.6 and 5μ, respectively, with a deep depression at 3.5μ. The emission bands at 1.6, 2.7, 5, 5.5, and 7.1μ are not well resolved, but their presence cannot be doubted.

9. *Glass*[11] (*Soft*).—[Rod, 3 × 2 mm, heated to a dull red color. Curve *c*, Fig. 5.] The substance examined was a piece of ordinary "soft" white glass tubing drawn into a solid rod. There are emission bands at 2, 2.9, 3.6, 4.4, and 5.5μ, respectively.

10. *Glass* (*Cobalt Blue*).—[Rod, 12 × 2 mm. Energy supplied, 0.64 and 1.15 watts. Curves *a* and *b*, Fig. 6.] This rod was heated to a dull red. The emission spectrum is quite different from that of soft glass, but there are no such marked bands as one might expect from a knowledge of the prominent absorption bands at 1.6 and 2.2μ, which exist in cobalt glass. The silicate band at 2.9μ is prominent. Other bands occur at 2, 3.6, 4.6, and 5.5μ respectively.



11. *Apatite*[12], $Ca_5\ F\ (PO_4)_3$.—[Rod, 12 × 1.5 mm. Energy supplied, 9.3 watts. Curve *c*, Fig. 6.] The emission spectrum shows bands at 2.9 and 4.5μ. The rod was made from a gray, massive specimen, which may have contained silica, whence the band at 2.9μ in both the emission and absorption spectrum.

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[11] Transmission curve. Carnegie Publication, No. 65, p. 64.

[12] Transmission curve. Carnegie Publication, No. 65, p. 58.

However, many oxides known to be free from silica have a strong emission band in this region, from which it would appear that this band is characteristic of the oxide.

12. *Topaz* [$(Al\ F)_2 Si\ O_4$].—[Rod, 13 mm long, 2 to 2.5 mm diameter. Curves *a* and *b*, Fig. 7]. This rod was made from the massive transparent mineral. The curve is conspicuous for the sharpness of the emission bands which occur at 1.4, 2.85, 4.1, 4.5, and 7.5μ, respectively.

III. EMISSION SPECTRA OF SOLIDS ON NERNST "HEATER-TUBE."

In examining the radiation from solids placed upon a heater it is possible to have some of the radiation from the latter transmitted through the former. If the substance to be examined is in the form of a fine powder and the layer is from 1 to 1.5 mm thick there is but little chance for the radiation to be transmitted from the heater. This fact enables one to study the selective emission of solids which can not easily be formed into solid rods, and is applied in examining the following list of substances:

1. *Zirconium Oxide* ($Zr\ O_2$).
   Magnesium Oxide ($Mg\ O$).

[Layer of oxide 0.8 to 1.5 mm thick, on Nernst heater tube from which the clay covering had been removed. Fig. 8, curve *a* is for zirconia.] The zirconium oxide was the pure material and was heated to a dull red, being at a somewhat lower temperature than the rod heated electrically. The emission spectrum is conspicuous for its two sharp emission bands at 2.83 and 4.3μ, respectively. Smaller maxima appear at 2, 2.4, and 5.4μ.

The magnesium oxide spectrum shows two wide bands, at 3 and 5.3μ, respectively, with a smaller band at 2.1 and a deep depression at 3.5μ.

2. *Cerium Oxide* ($Ce_2\ O_3$).
   Thorium Oxide ($Th\ O_2$).
   Uranium Oxide ($U_2\ O_3$).

The Cerium oxide curve (*c*, Fig. 7) shows a strong emission at 2 to 3μ, as was shown by Rubens, with possible bands at 4.4 and 7.5μ. It was shown by Rubens[4] that beyond 7.5μ the emissivity approaches that of a complete radiator. For the Welsbach gas mantle which is composed of 99% $Th\ O_2 + 1\%\ Ce_2O_3$ Rubens

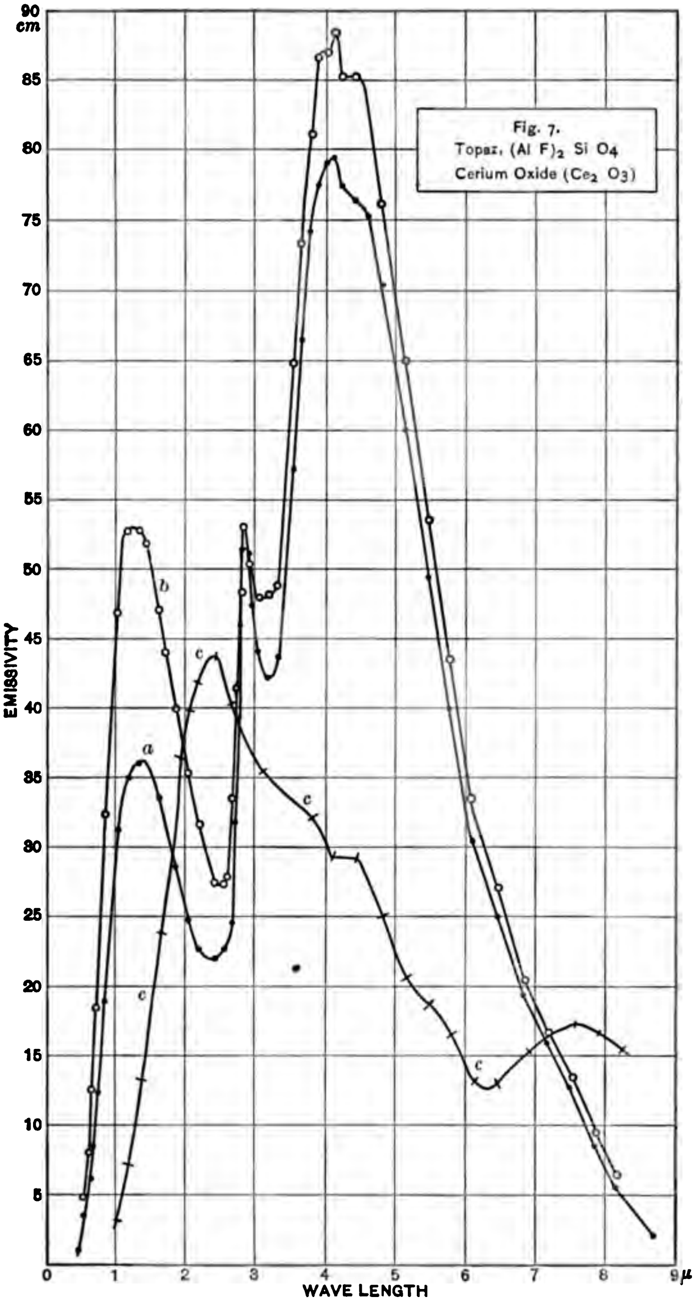
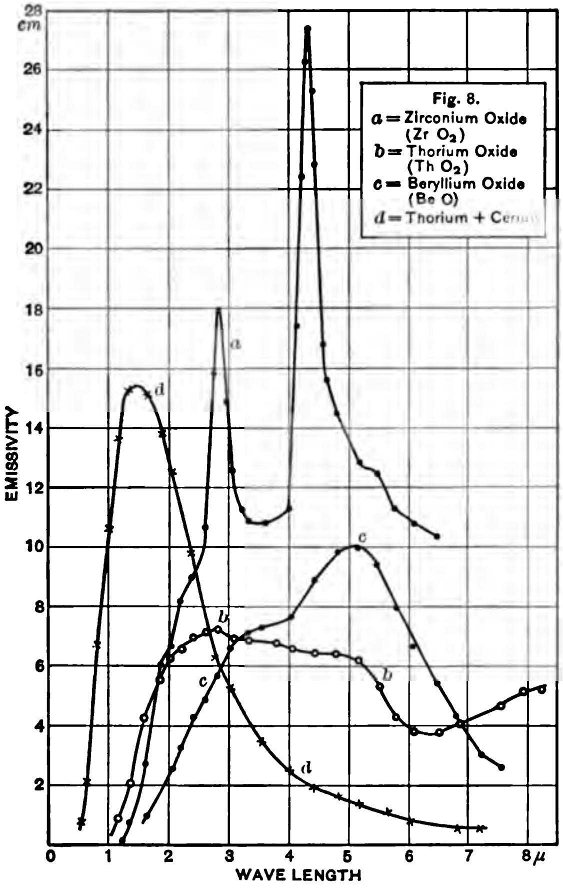


Fig. 7.
Topaz, (Al F)$_2$ Si O$_4$
Cerium Oxide (Ce$_2$ O$_3$)

found the radiation curve to be flat and suppressed, similar to curve *b*, Fig. 8. A rod 1 mm diameter, made of 99% $Th\ O_2$+1% Ce_2O_3 and heated white hot, electrically, gives an entirely different spectrum from the gas mantle, curve *d*, Fig. 8. It is probable that this is due to the thickness of the radiating layer.

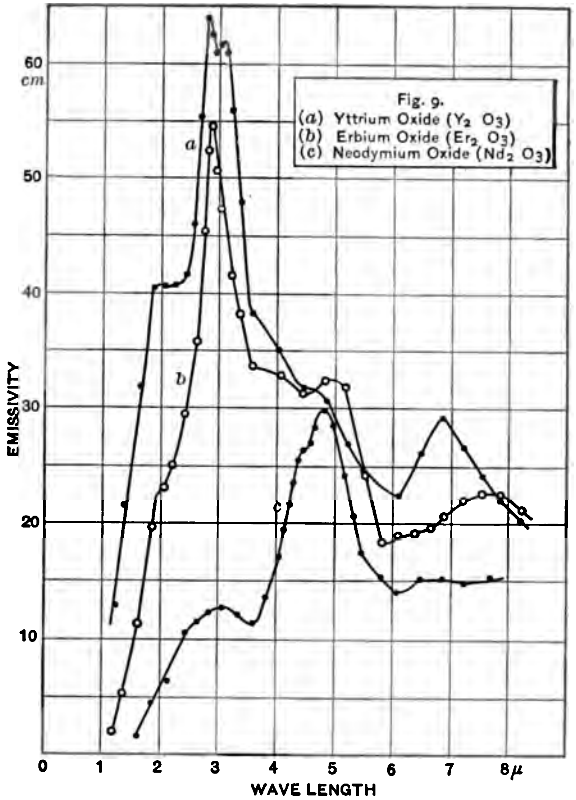
The Thorium oxide curve (*b*, Fig. 8) shows no marked emission bands, and the whole spectrum to 7μ is suppressed. Beyond this point the emissivity is high, corresponding closely to that of a complete radiator, as was shown by Rubens, using a gas mantle of this material.



The Uranium oxide is a greenish-brown powder which gives a smooth, continuous spectrum with hazy maxima at 2.8μ and 3.4μ, respectively.

3. *Beryllium Oxide* (*BeO*).—[Curve *c*, Fig. 8.] The Beryllium oxide emission spectrum is smooth, with two wide maxima at 3.5 and 5μ, respectively. The temperature was such that a faint red showed through the interstices of the layer of white oxide.

4. *Yttrium Oxide* (Y_2O_3).—[Curve *a*, Fig. 9.] The surface color was a deep red, corresponding to a temperature of 900° to 1000°. The curve shows emission maxima at 2, 2.76, 3, 3.6, 4.6, and 6.9μ, respectively, the latter band being unusually sharp.



5. *Erbium Oxide* (Er_2O_3).—Curve *b*, Fig. 9, shows the emission of a layer of the oxide, formed by decomposing a solution of the nitrate on a strip of platinum, heated electrically. There is a sharp emission band at 2.85μ. Other maxima appear at 2, 3.2, 4.1, 5, and 7.5μ. The general outline of the spectrum is similar to that of Yttrium.

6. *Neodymium Oxide (NdO).*
Manganous Oxide (MnO).

[Curve *c*, NdO, Fig. 9.] The neodymium oxide was deposited in a thick layer upon a strip of platinum by decomposing a solution of the nitrate. The radiation curve shows maxima at 3, 4.4, and 4.8μ. Beyond 6μ the emissivity is strong and not unlike that of cerium and thorium.

The manganous oxide was of a grayish-brown color. The thickness of the layer upon the "heater tube" was about 1.2 mm.

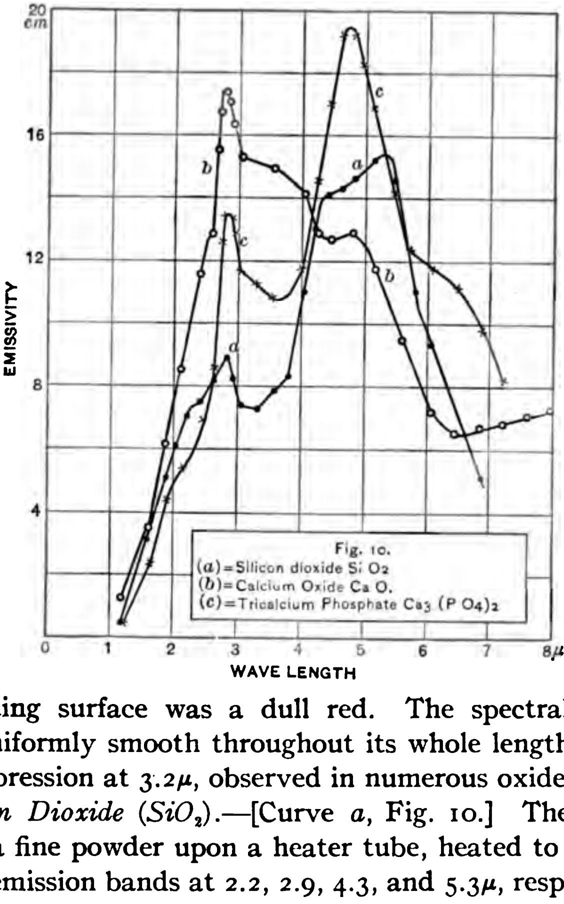


Fig. 10.
(*a*) = Silicon dioxide SiO_2
(*b*) = Calcium Oxide CaO.
(*c*) = Tricalcium Phosphate $Ca_3(PO_4)_2$

The radiating surface was a dull red. The spectral radiation curve is uniformly smooth throughout its whole length, with but a slight depression at 3.2μ, observed in numerous oxides.

7. *Silicon Dioxide* (SiO_2).—[Curve *a*, Fig. 10.] The radiating layer was a fine powder upon a heater tube, heated to a dull red. There are emission bands at 2.2, 2.9, 4.3, and 5.3μ, respectively.

8. *Calcium Oxide* (CaO).—[Curve *b*, Fig. 10.] The radiating surface was a dull red. The emission curve is conspicuous for its

two sharp maxima at 2.8 and 4.75μ, respectively, and a high emissivity at 8μ, which is similar to cerium and thorium. Smaller bands appear at 2.4, 3.3, and 4μ. The calcium oxide was heated to a bright red before mounting it upon the heater and was apparently free from the carbonate (see Fig. 11). Since there are no emission bands belonging to that substance, it appears that the water used in making the CaO into a paste was entirely expelled. A chemical analysis of the calcium oxide by Dr. H. C. P. Weber gave no weighable amount of silica, which shows that the band at 2.8μ is not due to that oxide.

9. *Tricalcium Phosphate, $Ca_3(PO_4)_2$.*—[Curve *c*, Fig. 10.] The tricalcium phosphate has two marked bands, at 2.85 and 4.75μ, and smaller bands at 2μ and at 6.2μ, respectively.

10. *Cobalt Oxide (Co_2O_3).*
Chromium Oxide (Cr_2O_3).
Stannic Acid (SnO_2).

The cobalt oxide curve is smooth throughout, except the depression at 3μ.

Chromium oxide is green in color, and emits a fairly smooth spectrum with a possible maximum at 5μ. The depression at 3.2μ is prominent.

Stannic acid is grayish-white in color, but, unlike many of the white oxides, it emits a continuous spectrum. The depression at 3.2μ is small. The transmission bands in cassiterite, SnO_2, were found to be small.

11. *Zinc Oxide (ZnO).*
Lead Oxide (PbO).

The zinc oxide became a yellowish-green on heating, resuming its former white on cooling. In spite of this selective emission in the visible spectrum, the distribution of energy in the infra-red is uniform with the usual depression at 3.2μ. Lead oxide, "plumbago," melts at a low temperature. On heating the color changes from orange to deep red. The emission curve is smooth except for a depression at 3.3μ, which is marked, and at 5.5μ there is a possible emission band.

12. *Calcium Sulphate*[13] *($CaSO_4$ + 2 H_2O).*—[Curve *b*, Fig. 11. Temperature about 900°.] The calcium sulphate used was a

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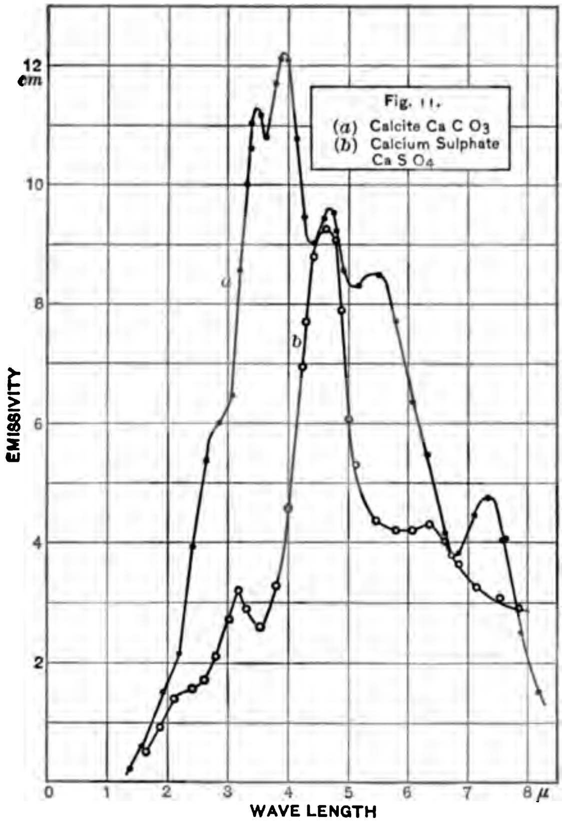
[13] Transmission curve. Carnegie Publication, No. 65, p. 18.

thick smooth layer of "plaster of Paris," which dehydrated, in part at least, at the red heat used. The emission bands at 2, 3.2, 4.65, and 6.3μ coincide in intensity and position with the absorption bands found in previous work. The band at 4.65μ is shifted from its position at 4.55μ in anhydrite, $CaSO_4$, but coincides with the partially dehydrated selenite, $CaSO_4 + 2\ H_2O$, given in Vol. 2, p. 461, Fig. 3 of this Bulletin.



13. *Calcite*[14] (*$CaCO_3$*).—[Curve *a*, Fig. 11.] The sample examined was a layer of finely ground white marble. The color corresponded to a temperature of about 900° C. The emission curve is of interest on account of the two types of emission it contains. In the region of 6.7μ calcite has a band of strong selective reflection. In this region of the spectrum the emissivity is pro-

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[14] Transmission curve. Carnegie Publication, No. 65, p. 70.

portional to the reflecting power, thus placing it in the class with metals. Here the emission is actually suppressed, and we have an emission minimum instead of a maximum. In the remaining part of the spectrum the emission is proportional to the absorption. The emission bands at 2, 2.7, 3.5, 3.9, 4.7, and 5.5μ coincide with the absorption bands previously observed. The apparent maximum at 7.3μ is due to the suppression of the radiation at 6.8μ, as previously described. This fact can, of course, only be determined from a knowledge of the reflective power of the substance under examination. The maximum reflection, and the minimum emission do not coincide on account of the high reflecting power on the side toward the long wave-lengths, which suppresses the emission curve, thus shifting the maximum farther into the infra-red. The only other example is that found by Rosenthal for quartz.[15]

IV. RELATION BETWEEN EMISSIVITY AND ENERGY CONSUMPTION.

The object in examining the isochromatic radiation curves of the solids considered in this paper is to determine whether or not the sharp emission bands, e. g., those in oligoclase, behave like spectral lines or like wide bands which include a considerable portion of the spectrum.

Other phenomena such as the interpretation given to the intersection of the isochromatics are secondary. If the observed bands behave like emission lines of a gas, then, as previously shown,[16] the emission must be proportional to the energy supplied. The fact that the maximum does not shift with rise in temperature is strong evidence of the spectral purity of the emission bands, e. g., at 2.9 μ, hence a direct proportionability of emission and energy consumption. If they are impure spectral lines, then the emissivity can not be proportional to the energy consumption, because the maximum of intensity must shift, due to the difference in the rate of increase in intensity of the different parts of the band, which is caused by the damping action of the different frequencies of the vibrating particles. Heretofore the spectral partition of radiant energy in terms of energy consumption has not been discussed, and it is therefore necessary to examine the isochromatic radiation of a substance of which the radiation laws are known. The criterion for judging whether the sharp emission bands are similar to the emission lines of a gas, or similar to the complex and highly damped emission of a solid, e. g., platinum, is based upon the behavior of the isochromatic radiation curves, which are straight lines for a gas, but which are unknown for platinum and hence must first be determined in the present investigation.

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[15] Rosenthal. Ann. der Phys. (3) 68, p. 791; 1899.

[16] Investigations of Infra-red Spectra, Part II. Publication No. 35 of Carnegie Institution of Washington, 1905.

First of all it is well to consider some of the conceptions of the nature of the mechanism by which the thermal energy within the radiator is transformed into radiant energy.

According to Stark,[17] and to others, the continuous spectrum is due to collision of the free electrons with the molecules or with the positive ions, while the emission bands are due to the internal vibration of the positive ion ("atomion"). The transformation of the thermal energy in the positive ion into radiant energy produces a discontinuous spectrum which follows Kirchhoff's law. The change of the thermal energy of the free electrons into radiant energy produces a continuous spectrum.[18]

A complete radiator emits energy, however small the amount, of all wave lengths, whatever its temperature may be above the absolute zero.[19] The permanent conductivity in metals indicates a permanent ionization, and it must therefore behave like the complete radiator in its emission. The isochromatic energy curves must therefore all begin at the origin of the energy axis; and, as will be shown presently, they may have a double curvature. On the other hand, it is a well-known property of spectral lines that the energy required for excitation is different for different lines, being independent of the wave length. It is beyond the scope of the present paper to consider the question whether or not, for an infinitesimal rise in temperature above the absolute zero, any spectral lines will be emitted, for if, as is generally supposed, there is no damping, the emissivity must be proportional to the energy supplied throughout the whole range of temperatures. It seems quite well established that spectral lines are not emitted before the temperature of the gas has been raised a finite amount above the absolute zero.[20] Therefore, the smaller the radiation coefficient the higher must the temperature be raised in order to cause emission.[21]

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[17] Stark. Ann. der Phys., (4) **14**, p. 506, 1904. A summary of these theories may be found in Carnegie Publication No. 35, p. 325.

[18] Stark. Phys. Zs., 8, p. 918, 1907.

[19] Drude's Lehrbuch der Optik., p. 325.

From these considerations it appears that the emission of a spectral line is proportional to the energy consumption, and since a finite amount of energy must be supplied to excite them, that the isochromatic energy curve must intersect the energy axis at some distance from its origin.

The substances just described must have one or both of two kinds of spectral energy distribution, due (1) to the general absorption which is present to some extent, however small, and which gives rise to a continuous spectrum, and (2) to bands of selective absorption which give rise to emission bands. It is therefore of interest to examine their isochromatic radiation curves by plotting emissivity against energy consumption instead of against temperatures, as has been done heretofore. In plotting isochromatics of emissivity against temperature it is unnecessary to correct for slit width, but if we wish to compare different isochromatics of emissivity plotted against energy consumption, then it is necessary to reduce all the observations to the same bolometer sensibility, and also to correct for slit width in order to reduce the emissivities to a common origin. If the wave lengths of two isochromatics are close together, as for example $\lambda = 1.804\mu$ and $\lambda = 1.968\mu$ in Fig. 12, the error introduced in the determination of the energy consumption (without correcting for slit-width) at the point of intersection of two isochromatics is less than 3 per cent and could be made still less by selecting two wave-lengths which are closer together.

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[20] B. Davis. Phys. Rev., **20**, p 145, 1905, shows that the energy necessary to ionize a molecule is of the order of 5×10^{-12} erg.

See also papers by Townsend, Phil. Mag., (6) **1**, 209, 1901; and by Nutting. Astrophys. Jour., **21**, 404, 1905.

[21] Stark. Phys. Zs., **8**, p 919, 1907, computes that in order to bring the D-lines to emission as the result of collision of moving sodium ions they must have a kinetic energy larger than 3.2×10^{-10} erg., while the collision of the negative electron would cause the D-lines to be emitted with a kinetic energy of less than 10^{-11} erg.

It is of course impossible to determine the true energy consumption in short filaments, on account of the losses by conduction from the ends and by convection, so that the true energy supplied, especially at high temperatures, is probably higher than here recorded. But this can not affect the conclusions to be drawn from the isochromatics, for the error in the observed energy consumption must affect *all* the isochromatics, in the same manner, while in the results to be mentioned presently the isochromatics for the *different wave lengths are affected differently*, which seems to be conclusive evidence that the cause is not due to lack of knowledge of the energy supplied (which would produce a still greater curvature of the isochromatics toward the axis of energy), but is due to a variation in emissivity with change in energy consumption.

The present results show that, for solids having a fairly smooth, continuous emission spectrum, as the maximum emission, E_{max}, shifts toward the short wave-lengths, the isochromatics close to, and of shorter wave-length than, the λ_{max}, E_{max}, suddenly bend upwards (away from the energy axis), and after the E_{max} passes over these wave-lengths, so that the isochromatics lie on the long wave-length side of the λ_{max}, E_{max}, then their curvature changes in the opposite direction. In other words, there is a point of inflection when the isochromatic wave-length is identical with the λ_{max}. The interpretation of this is, of course, the well-known property of spectral emission of a complete radiator and of metals in which the emissivity in the short wave lengths increases more rapidly with rise in temperature than in the long wave lengths, whence the shift of the maximum of emission.

1. *Isochromatics of platinum.*—The aforesaid double curvature is well illustrated in Fig. 12, in the isochromatic radiation curve of platinum at 2.75 μ. The isochromatic at 3.578 μ has but a single (downward) curvature, since the temperatures were all higher than that required to cause the λ_{max} to be equal to or greater than 3.578 μ.

The platinum strip was 50 × 1.5 × 0.02 mm in an exhausted glass bulb with a fluorite window; hence convection losses are small. In this figure the graphs of wave lengths $\lambda = 1.804\ \mu$ and $\lambda = 1.968\ \mu$, intersect at about 3.0 watts, showing that the maximum of the energy curve lies between (but not halfway) these two wave-lengths, the temperature being about 1100° C. Using these two wave lengths the λ_{max}, for this energy input, can be calculated from the equation given by Paschen.

In Fig. 12 the slit-width correction has not been applied to the isochromatics, except for $\lambda = 1.804\ \mu$ and $\lambda = 1.968\ \mu$ (short uppermost graphs, Fig. 12), which shows a small correction to the energy input. The other isochromatics are not comparable on account of different bolometer sensibilities.

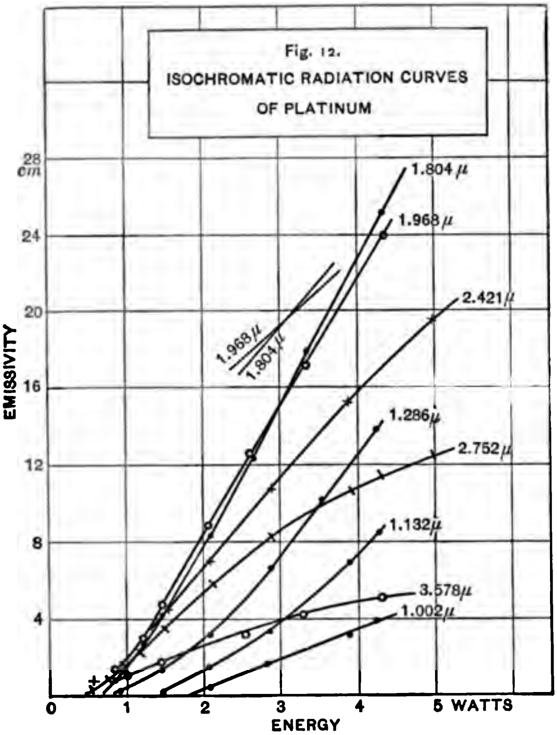
Fig. 12. ISOCHROMATIC RADIATION CURVES OF PLATINUM

2. *Isochromatics of Nernst glower.*—In a preceding paper [22] on the selective emission of the Nernst glower the writer concluded that because the spectrum was discontinuous at low temperatures it must behave likewise at high temperatures (provided a large enough dispersion could be obtained) and hence, since the emission of spectral lines is proportional to the energy, that the appli-

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[22] See results on Nernst glower; this Bulletin, **4**, p. 546, 1908.

cation of isochromatics of emissivity plotted against temperature (assuming Wien's equation), as made by other observers, must give too high values for the temperature of the glower. The isochromatics obtained by the present method indicate that the Nernst radiation behaves somewhat like that of a metal and of a complete radiator (emissivity not quite proportional) to the energy input, but more nearly so than for platinum, so that the former conclusion must be somewhat modified to admit this data. However, the position of the λ_{max} and E_{max} does not admit the highest temperatures given by others.

Furthermore, if we follow the usual assumptions (see Kayser, Spectroscopic, Vol. II, pp. 59, 245, and 331) and consider the separate lines as a part of an energy curve, obtained by drawing the envelope through the highest points of the separate emission bands, then the maximum of the envelope must shift toward the short wave lengths with increase in energy consumption, and the slant of the isochromatics must be similar to those of platinum, Fig. 12. It is difficult to conceive how this is possible with discrete spectral lines which require the application of a certain amount of energy to excite them.

The change of the emission curves of the Nernst glower from a discontinuous into a continuous spectrum has been noticed in a previous paper. They illustrate this envelope type of energy curve just mentioned. But it seems more probable that this is due to the rapid growth of the general emission of the intervening frequencies which, with a doubtful broadening of the emission band (see figs. 1 and 2), obliterates the selective emission at high temperatures. In Fig. 13 are given a series of isochromatics for a 110-volt Nernst glower 1.4 cm long and 1.4 mm in diameter. The current was supplied from a 2000-volt 600-watt transformer on a 110-volt circuit. The supply of energy was regulated by means of resistances in the primary. The voltage was obtained with a multiple-cell electrostatic voltmeter. The curves (not corrected for slit-width and bolometer sensibility) for the most intense part of the spectrum pass through a slight double curvature, thus giving them the general outline of that of platinum. The normal burning is 80 watts and above that point the isochromatics appear to show a slight increase in curvature. The inter-

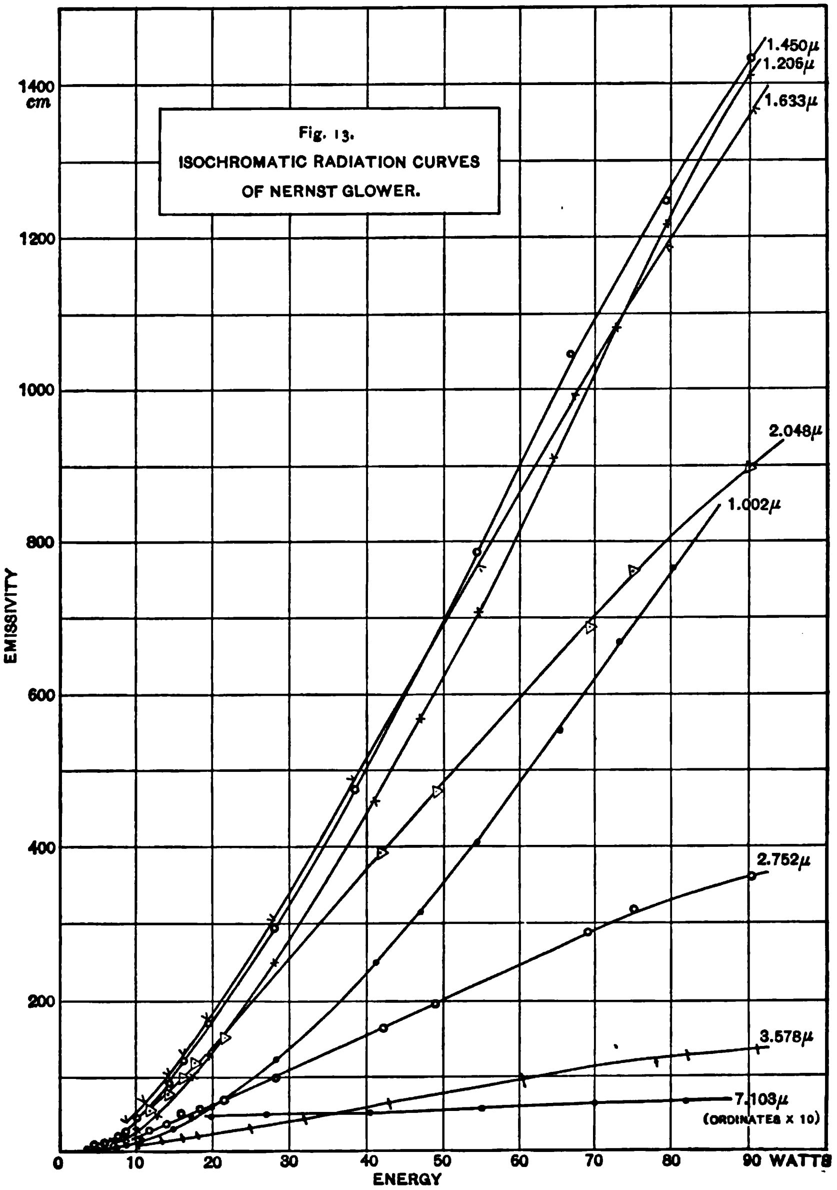

Fig. 13.
ISOCHROMATIC RADIATION CURVES OF NERNST GLOWER.

section of the graphs for wave lengths $\lambda = 1.206\ \mu$ and $\lambda = 1.633\ \mu$ at 73 watts (somewhat greater if slit-width correction is applied) shows that the maximum of the spectral energy curve, for this power consumption, lies between these two points, as previously observed.[22] This method of locating the maximum eliminates the correction for slit width, and is an independent proof of the previous observations that the maximum of the energy curve for normal burning (80 watts) does not lie at such short wave lengths as was observed by previous investigators. This of course is on the assumption that the glowers were of the same material (see curve *b*, Fig. 8, for other substances), the base of which is zirconium oxide with a small percentage of cerium, thorium, or yttrium oxide.

3. *Isochromatics of oligoclase.*—In No. 35, p. 318, of Carnegie Publications, it was shown that in the case of vacuum tube radiation the intensity of the emission lines is proportional to the energy consumption. The graphs there obtained, showing the relation between current and the emissivity of a spectral emission line, are curved due to the fact that Ohm's law does not hold for the vacuum tube discharge. For the present examination, instruments were available to measure the energy consumption when the graph ought to be a straight line, provided the partition of the energy emitted is in discrete lines. As a typical selectively radiating solid, oligoclase was chosen on account of its sharp emission bands, and also on account of its homogeneity. A rod 2.5 cm long and 2.1 mm diameter was prepared in an oxy-hydrogen flame. The spectrometer slit was reduced to 5 mm in length, which permitted the entrance of radiation from only about 5 mm of the central part of the rod, which was a perfectly clear glass, free from air bubbles. At the highest temperatures this central part showed a peculiar faint white "luminescence" similar to the intense white seen in quartz when heated in the oxy-hydrogen flame. The rod was thickened at the ends, which seemed to prevent internal reflection of the radiation from the platinum terminals. The ends of the 0.3 mm platinum terminals within the glass rod were red hot. The rod was viscous at the highest energy input, 29.4 watts, indicating a temperature of at least 1100° to

1200° (melting point about 1300° C); but no light was emitted other than the hazy white glow already mentioned, which is in marked contrast with the radiation from the platinum electrodes. A similar example is given in Wood's Optics, p. 457, where it is stated that sodium sulphate, in a loop of platinum wire, heated in a blast lamp, emits but little light, although the wire glows vividly.

In Fig. 14 are given the isochromatic emission curves of oligoclase at wave-lengths 2.048, 2.905, 4.445, and 6.082 μ respectively, for different values of power consumption. The graphs are for the same rod, under the same conditions of galvanometer sensibility and distance of the radiator from the slit and are corrected for slit width. The complete isothermal radiation curve on 29.4 watts is given in Fig. 3, curve *c*, the scale of ordinates being of course different from those of curves *a* and *b*. Two additional series of observations on different days, and using different adjustments, were made at wave length 2.905 μ. Only one of these graphs was parallel with the one given in Fig. 14, showing that there is a variation in the slant of the isochromatic curve under different con-

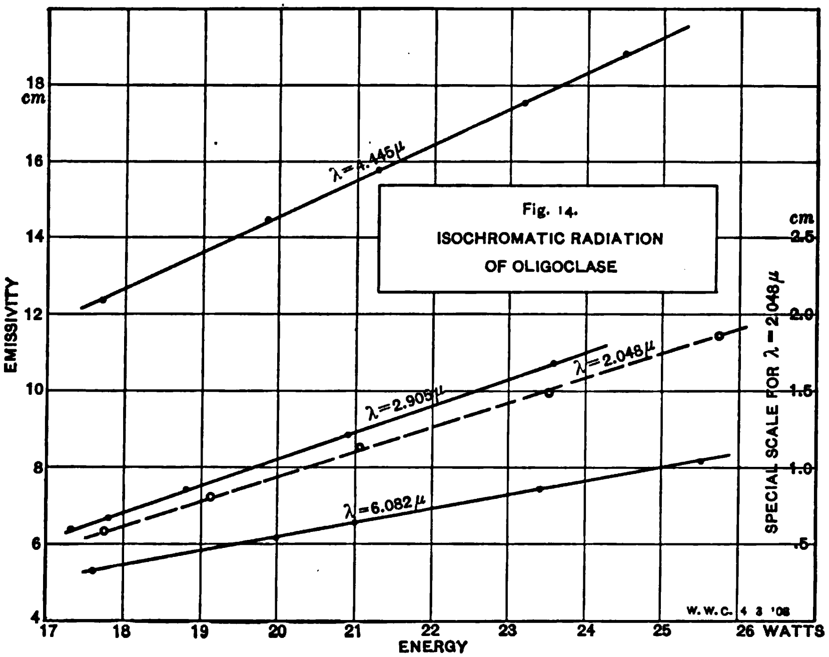
Fig. 14.
ISOCHROMATIC RADIATION OF OLIGOCLASE

ditions due, no doubt, to a variation in the spectrometer setting, to a variation in the homogeneity of the rod, to internal reflection of radiation from the platinum terminals, etc. It will be noticed that, throughout the range investigated, the change in emissivity of all the lines is proportional to the energy supplied, just as is true of gases. It is possible that for some wave-lengths the thickness of the radiator was not sufficient to emit a saturated radiation, and this may explain why the emissivity at 2.048 μ apparently does not follow the same law (its slant should be steeper) as do the other emission bands. In order to have displacement of the maximum of emission, just as is known for solids emitting continuous spectra, it is necessary that the intensity of the emission at the short wave lengths increase more rapidly than it does in the long wave lengths. The $\lambda = 2.048\ \mu$ isochromatic slants only a little less from the normal than does $\lambda = 6.082\ \mu$. In this region of the spectrum there is a weak general absorption while the other wave-lengths are the maxima of selective emission (absorption) bands, and it is possible that what corresponds to the emissivity constant of a complete radiator is different for the two kinds of radiation found in this substance corresponding to what is commonly called "general" and "selective" absorption. It is possible that the isochromatic at 2.048 μ undergoes a sudden change, curving sharply upward, at a higher temperature. The same is true of the platinum isochromatic at 1 μ. In fact, it appears that the emissivity at 2.048 μ must suddenly change in intensity, unless oligoclase is entirely different from the other substances examined; for, like the others, in the oxy-hydrogen flame it emits an intense white light, although, as shown in curve *c*, Fig. 3, the emissivity at 2 μ is still weak when the temperature is close to the melting point.

In oligoclase it is not possible to locate so definitely, if at all, the position of the maximum of the envelope by the intersection of the isochromatics; for the lines show no tendency to curve, as in platinum, although at closely the same temperature. By the extrapolation of the 2.9 μ and the 4.4 μ isochromatics, the intersection is found to be at about 8.5 watts. The maximum emission would then lie at about 3.5 μ, which is the position of the maximum for a complete radiator at 850° Abs. (580° C). On 29.4 watts, when the oligoclase was already viscous, indicating a temperature of 1100° to 1200° (melting point 1300°), the maximum emission should be found at a much smaller wave length than 3.5 μ; the emission curve is, however, but little different in outline from those obtained at lower temperatures, Fig. 3, with no indication of a shifting of the energy distribution. It would therefore appear that it is not permissible to consider the envelope, drawn through the emission maxima, as a criterion for judging the temperature of a substance like oligoclase. On the whole, it appears that the general emission, as distinguished from the bands of selective emission, is less intense in oligoclase than in most of the other silicates studied.

V. SUMMARY.

In general, the results on the oxides furnish an excellent illustration of the shifting of the maximum of intensity of emission toward the short wave lengths with rise in temperature, just as is known for metals, which emit continuous spectra. In this respect the various emission curves of zirconium oxide are particularly conspicuous. In addition to what may be termed general emission, in which the maximum shifts with rise in temperature, the curves of zirconium oxide are unique in having a sharp band of selective emission which does not shift nor broaden with rise in temperature.

The results are not unlike those obtained by Anderson[23] for erbium oxide, in the visible spectrum. He found that the emission spectrum was not continuous, but consisted of bright bands superposed upon a continuous faint background. With rise in temperature the bands became more hazy in outline, and at very high temperatures the spectrum became continuous. If, according to Stark's theory, the continuous spectrum is due to the presence of a large number of free electrons, this is to be expected. At low temperatures the electrical conductivity is small, and the emissivity is confined to particular bands caused by certain groups of electrons. With rise in temperature more electrons, extending over a wider range of wave-lengths, are excited to activity and the separate emission bands become merged into a continuous spectrum. This is not always true, however, in the present work. For example, the sharp emission band of zirconium oxide at 4.3μ seems to retain the same intensity, superposed upon a continuous background of increasing intensity, irrespective of the temperature. In fact many of the emission bands are as sharp as those found in gases.

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[23] Anderson. Astrophys. Jour., **26**, p. 73; 1907.

Many of these oxides have an emission band in common in the region of 2.85μ, from which it would appear that this band may be due to the oxygen atom which is in common with all of them. Furthermore, the emission spectra, when smooth, generally show a depression at 3.2μ, for which no satisfactory explanation has been found. There are no atmospheric absorption bands in this region, and fluorite is not known to have absorption bands at this point. The emission spectrum of platinum, carefully examined at the same time, showed no depression. The emission curve of the bare "heater tube" which consists of a porcelain tube wound with fine platinum wire, likewise gave a smooth emission curve similar to that of the platinum strip. The present data indicate that the depression is a characteristic of the oxides. In other words, the oxides have a large (sometimes small) characteristic emission band at 2.8 to 3μ and a second group of bands at 4.5 to 5μ. If this be true, then it will be necessary to assign the cause of the selective emission to the element common to all the oxides, viz., to oxygen. It is well known that groups of atoms have characteristic bands, but, heretofore, no data has been at hand which in any way indicated the possibility of the characteristic bands being due to a particular atom in the group. The tentative conclusion that these emission bands in the oxide are due to oxygen atoms should perhaps be expected. The oxides are the most important and the most stable of all the important groups of chemical compounds; and the spectra belonging to a particular group are similar. But few substances are inert to the action of oxygen. The spectrum of oxygen[24] shows absorption bands at 3.2 and 4.75μ. The emission spectra of CO and CO_2 show bands in the region of 2.7 and 4.75μ. Oxygen in a vacuum tube showed an emission band at 4.75μ, which at that time was ascribed to CO or CO_2 formed by the electrical discharge. In view of the fact that, with rise in temperature, the CO_2 emission band shifts towards that of CO at 4.6μ, and that eventually all three gases, CO_2, CO, O, when radiating by electrical excitation in a vacuum tube, have their important emission band in common at 4.75μ, it does not seem unreasonable to assume that the latter maximum is due to the oxygen atom, set free during the time intervening between dissociation and recombination brought about by the electrical discharge.

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[24] Carnegie Publication, No. 35, pp. 49 and 313.

In a broad sense the intensity and sharpness of the emission bands are a function of the electrical conductivity. The best insulators, e. g., the refractory silicates, the oxides of zirconium, erbium, etc., have the sharpest emission bands, while the better electrical conductors such as the oxides of cerium, iron, zinc, etc., have a high emissivity but no sharp emission bands throughout the infra-red. The molecular weight of the base seems to affect the sharpness of the bands to as great an extent as does the electrical conductivity. These results, if true, are to be expected from our knowledge of the emission, absorption, and reflection of electrical conductors and insulators.[25]

If the oxides behave like metals, in which the reflecting power is proportional to the electrical conductivity then from one line of theoretical consideration one would expect to find an increase in the reflective power,[26] with rise in temperature, and the accom-

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[25] Kirchhoff's Law says that for a given temperature, T, and wave length, λ, the emissivity E, the absorptivity, A, and the reflectivity, R, of a substance is related to the emissivity, e, of a black body, by the equation $E = Ae = (1-R)\,e$, since $A = 1-R$.

For "transparent media," "electrical insulators" the reflectivity, R, the index of refraction *n*, and the extinction coefficient, k, are related as follows:

$$R = \frac{(n-1)^2 + k^2}{(n+1)^2 + k^2}$$

For electrical conductors, metals, the reflectivity is related to the electrical conductivity by the relation, $100 - R = \frac{36.5\sqrt{w}}{\sqrt{\lambda}}$ where *w* is the specific resistance.

This subject has been thoroughly treated by Aschkinass, Ann. der Phys., (4) **17**, p. 960; 1905.

[26] Rubens. Ann. der Physik, (4) **20**, p. 593, found that the Welsbach mantle may or may not *decrease* in reflective power with rise in temperature, and chemical com position. The reflective power of the oxides is low (5 to 6 per cent) which may account for this disagreement.

panying increase in the electrical conductivity of the oxides. If this be true, then it would seem that the spectral emission ought to become more continuous, as is found in the Nernst glower.

It may be added that the positions of the well-defined maxima of emission are not affected by change in temperature. Whether the emission maxima at 2.85 and 4.75μ are due to the presence of water remains undetermined. If they are due to water, then one would expect to find them in calcium sulphate, $CaSO_4 + 2H_2O$, as well as in calcium oxide (probably some CaOH present); in the sulphate no band was found at 2.9μ where the emission should be the most intense, if due to water. The sulphates[27] have a characteristic band at 4.75μ; but from the present data it is no more satisfactory to assume that the bands at 2.85μ and 4.75μ found in the oxides, are due to the presence of water than to ascribe them to the common constituent, viz., oxygen.

The isochromatics of oligoclase show that the emissivity is proportional to the energy input, thus differing from the other solids investigated, but, in view of the fact that in the oxy-hydroden flame it emits an intense white light, it appears that the emissivity must suddenly undergo a change in the visible spectrum, and perhaps form a more continuous spectrum throughout the infra-red.

WASHINGTON, May 20, 1908.

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[27] This Bulletin, Vol. 2, p. 469.

REMARKS ON THE QUARTZ COMPENSATING POLARISCOPE WITH ADJUSTABLE SENSIBILITY.

By Frederick Bates.

In an article on a "Quartz Compensating Polariscope with Adjustable Sensibility," which appeared in the Bulletin of the Bureau of Standards (Vol. IV pp. 461–466), the writer discussed the effect on the zero reading of the scale when the Polarization angle a was varied. In general, the intensities of the light emerging from the large and small nicols of a Lippich polarizing system are unequal. Hence, when the analyzing nicol is set for equal intensity of the halves of the field its position is different from what it would have been had the beams been of equal intensity. The angular difference, δ, between the two positions of the analyzing nicol was obtained from the equation

$$\tan \delta = \pm \frac{\sqrt{K}-1}{\sqrt{K}+1} \tan \frac{a}{2} \tag{3}$$

$$\text{where } K = \frac{A_2}{A_1}$$

and is the ratio of the intensity of the light emerging from the small nicol of the polarizing system to that emerging from the large nicol.

We have for a Lippich system

$$A_2 = f(A_1 \operatorname{Cos}^2 a, M) \tag{5}$$

where $A_1 \cos^2 a$ is the intensity of the light from the small nicol, if there has been no other loss than that due to the partial crossing of the large and small nicols. M is the algebraic sum of the effects of all the remaining factors upon which the relative intensities of A_1 and A_2 depend. These factors are quite numerous. Among them may be mentioned absorption, depolarization, the inclination of the small nicol to the axis of the system, the relative inclination of the faces of the small nicol, the positions of diaphragms, and reflection as the light enters the small nicol. It is evident that M is a constant for a particular instrument, but varies with different polarizing systems. For this reason, and because it is a simple matter for the instrument makers to correct for it, the quantity M was neglected in discussing δ in the paper referred to above.

In "Bemerkungen zu der vorstehenden Batesschen Arbeit,"[1] Dr. Otto Schönrock calls attention to the fact that the writer has not taken the reflection and absorption in the small nicol into consideration in calculating δ from (3), inasmuch as the equation

$$A_1 = A_2 \cos^2 \alpha \tag{6}$$

was used to obtain K. Schönrock modifies (3) in the following manner:

From the well-known Fresnel equations, when light at perpendicular incidence passes from one refracting medium into a differently refracting medium, the fraction

$$B = \left[\frac{n-1}{n+1}\right]^2 \tag{7}$$

is reflected. Applying (7) to the case of air and Iceland spar, where $n = 1.486$, and neglecting (6), Schönrock finds that

$$A_2 = A_1\left[1 - \left(\frac{n-1}{n+1}\right)^2\right]^2 \tag{8}$$

$$= A_1 \; 0.925 \tag{9}$$

There is therefore a loss of 7.5% by reflection at the faces of the small nicol. A further allowance of 0.005 was arbitrarily made for the loss due to absorption in the small nicol. Thus, he finds, by considering the losses due to the crossing of the large and small nicols and to reflection and absorption, that

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[1] Zs. Ver. Zuckerind, Feb., 1908, p. 111.

$$A_2 = 0.92\ A_1 \cos^2 a \tag{10}$$

and

$$\tan\delta = \frac{1 - \sqrt{0.92}\cos a}{1 + \sqrt{0.92}\cos a}\tan\frac{a}{2} \tag{11}$$

In Table I is shown a comparison of the values of δ given by (3) and (11).

TABLE I.

a	δ IN SUGAR DEGREES.	
	From Equation (3).	**From Equation (11).**
5°	0.014	0.164
10°	0.111	0.412
15°	0.378	0.830

The values from (11) are markedly larger than those from (3). It is to be remembered that (11) takes into consideration the effect on δ of the loss in intensity due to reflection and absorption in the small nicol, but neglects all the other factors of $f(M)$. It will be observed for any value of δ that the value from (11) equals the value from (3) increased by $0.03a$.

Hence,

$$\delta = \tan^{-1}\left[\pm\frac{\sqrt{K}-1}{\sqrt{K}+1}\tan\frac{a}{2}\right] + 0.0103a \tag{12}$$

$$= \tan^{-1}\left(\tan^3\frac{a}{2}\right) + 0.0103a \tag{13}$$

Equation (13) holds for all working values of a and is significantly simpler than (11). A consideration of the remaining factors involved in $f(M)$ would result in decreasing the constant, 0.0103 (=0.03 for δ in sugar degrees). It is thus evident that if $j(M)$ be taken into account the only effect on the value of δ given by (3) is to change it by $\pm Ca$ where C is a constant for each individual Lippich system.

In Fig. 1 the curve A shows the value of δ (change in the zero reading of the instrument) in sugar degrees, given by (3), for any value of a. Curve B shows the change of δ with a if it were necessary to consider only those factors of $f(M)$ which represent

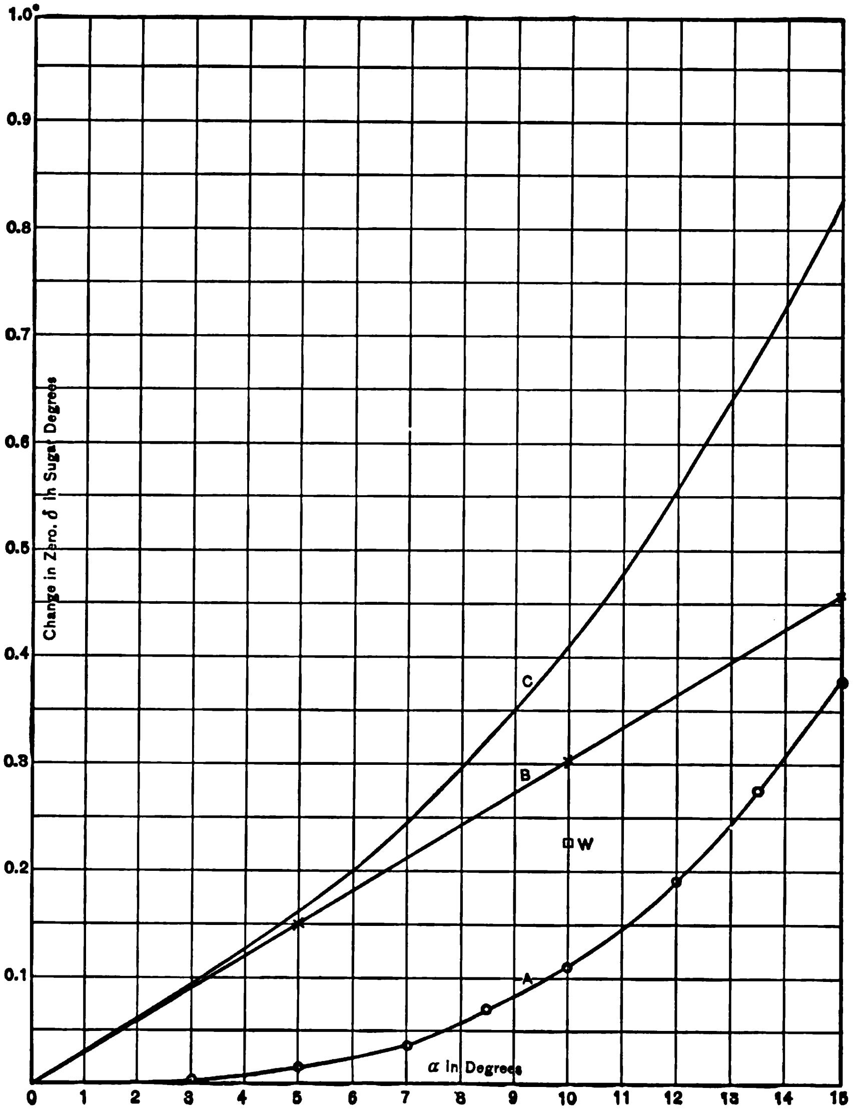

Fig. 1.

losses by reflection and absorption in the small nicol. Curve C shows the values of δ given by (13). Its ordinates are the sums of the ordinates of curves A and B. In building a quartz compensating polariscope with adjustable sensibility, the writer suggested that the analyzing nicol should move through one-half the angular displacement of the large nicol of the polarizing system. The zero point displacements of such an instrument, as a was varied, would approximate curve A with sufficient exactness to give a considerable practical working range of a. The difficulty in displacing the analyzing nicol by the proper amount to eliminate δ as given by curve A is due to the fact that δ is not a linear function of a. It is a simple matter to cause the analyzing nicol to gain or lose on the large nicol, so long as the change is linear. The effect of superimposing on curve A the curve B, which is due to part of $f(M)$, is to make a resulting curve C, which is as nearly linear as A. It will be observed that a straight line can be drawn through curve C, such that for values of a, from 3° to 15°, it will not be distant from the curve at any point by more than 0.05° S.

However, since the value of δ given by (13) does not include any of the factors of $f(M)$, save the reflection and the absorption, it is not the true zero displacement. By considering only the reflection and absorption in the small nicol, Schönrock introduces from $f(M)$ the only factors which could be used to make the apparent zero displacement larger. He has allowed 0.005 for the loss in intensity by absorption in the small nicol, whereas a considerably larger portion of the light lost by first reflection at the surfaces of the small nicol is returned by subsequent reflections to the analyzer. The exact value of δ can best be determined by experiment. An experimental determination with $a = 10°$ for a Lippich system selected at random gave δ at the point W, Fig. I, whereas according to Schönrock it should lie upon the curve C. The curve giving the true values of δ lies between curves A and C.

The change of δ with a is fortunately very nearly linear. As shown above, the only effect of $f(M)$ is to make the change more nearly linear. Hence, in the writer's previous paper $f(M)$ was ignored and only the difficult case was considered when curve A represents approximately the displacement curve of the system.

It is thus entirely practicable to build a quartz compensating polariscope with adjustable sensibility, having a working range of a from 3° to 15° with a negligible zero correction for all ordinary polarizations. It is immaterial whether the true zero point displacement curve approximates curve A or C.

WASHINGTON, June 2, 1908.

ON METHODS OF OBTAINING COOLING CURVES.

By G. K. Burgess, Associate Physicist.

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The rôle of thermal analysis in many metallurgical and chemical problems, involving in many instances the constitution and behavior of very complex substances, is of increasing importance; and great advances are being made in our knowledge of the properties of many alloys, minerals, and chemical compounds at high temperatures by means of the pyrometer.

Any internal change in the physical or chemical nature of a substance usually alters many of its physical properties, as, for example, its magnetic and thermoelectric behavior, electrical resistance, specific heat, dimensions, density, and microscopic structure. A large internal change at a definite temperature or within a small range of temperature—in other words, "a transformation"—will cause sudden or very rapid changes in some or all of these physical properties, and several of them may be used with advantage in the detection and study of such transformations. In this paper, however, we shall confine ourselves to the consideration of such changes as may be detected, measured, and recorded by thermometric means.

METHODS OF THERMAL ANALYSIS.

All methods of thermal analysis are based upon the principle that chemical and physical transformations within a substance are, in general, accompanied by an evolution or absorption of heat. The detection of the temperature and the measurement of the extent of these transformations, and in many cases their interpretation, may be carried out by taking the cooling curve of the substance, which in its simplest form consists in plotting the temperature of the cooling substance against the time. It is evident that the heating curve of a substance may also be taken to find its characteristics, and this is sometimes done, but, in general uniform results are obtained more easily by taking the cooling curve, mainly on account of the greater experimental difficulties in maintaining a uniform rate of heating in the containing furnace. In some problems it is desirable to have both curves, while occasionally, as in transformations involving loss of water of crystallization or of constitution, the heating curves alone are of significance. The same apparatus will evidently serve for both.

The cooling curve, however exactly taken, will of course give no indication of those transformations for which there is no evolution or absorption of heat. If the cooling is at constant pressure, as we shall assume in all of what follows, the absence of a transformation—physical or chemical—may be assumed only when both the energy and the volume changes are zero. The detection of changes in volume, unaccompanied by changes in internal energy, may be effected by the use of an apparatus measuring linear expansion, such as the recording differential dilatometer of Sahmen and Tammann.[1] These cases are exceptional, however, and we shall not consider them further.

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[1] *Sahmen* und *Tammann:* Über das Auffinden von Umwandlungspunkten mit einem selbst-registrierenden Dilatographen. Ann. d. Phys. **10**, p. 879–896; 1903.

There have been developed a considerable number of methods for obtaining cooling curves which are adapted to the study of recalescence points in steels as well as to the investigation of the composition and properties of alloys and chemical compounds. There has as yet been no general discussion[2] of the different methods nor of their availability for special problems and it may therefore be of some interest to have at hand an outline of the principles of the methods that may be used in obtaining cooling curves as well as a brief mention of the various types of apparatus available, with a discussion of their advantages and limitations.

The methods may be classified in various ways; thus we have to distinguish between those adapted for slow cooling, which is the case most commonly met with, and for very rapid cooling as in quenching steels; those methods which require an auxiliary body possessing no thermal transformations on cooling as compared with those requiring only the substance studied; and finally we may have on the one hand methods involving the time, and on the other hand those in which the time may be eliminated. In this paper we have considered in detail only such methods as are adapted for slow cooling, and have classified them in terms of the forms of the curves representing the experimental data.

Many operations which can be carried out in the laboratory can not be applied conveniently in industrial works, so that it will be necessary also to distinguish the various types of apparatus as regards their adaptability either for purely scientific researches or for industrial needs. For the latter, especially, it is very desirable to make all operations as automatic as possible, so that the different methods of autographic and photographic recording should be considered. Again, we shall have occasion to point out those methods which are the most advantageous to use when very minute quantities of heat or differences in temperature are to be detected, as, for example, the secondary breaks in the cooling curves of many alloys and of numerous compounds and mixtures. In such cases it becomes necessary to use methods of the highest possible sensitiveness, which usually necessitates the discarding of autographic and photographic recording.

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[2] A paper by *W. Rosenhain*, on "Observations on Recalescence Curves" was read before the London Physical Society, January 24, 1908, an abstract of which indicates he has compared the merits of the "Inverse Rate" and "Differential Methods."

The Differential Method has also been studied in detail by *Portevin*, Notes sur l'Emploi du Galvanomètre Différentiel, Rev. de Métallurgie, **5**, p. 295; 1908.

We shall mention those methods which are suitable for taking cooling curves in the range of temperatures up to 1500° C, but much of what is said will apply also to higher temperatures if proper precautions be taken to eliminate the effects upon the measuring apparatus of the electrical conductivity of the materials and contents of the furnace. Although several methods of measuring temperature may be used over most of the range indicated, such as the change of electrical resistance of platinum with temperature and the various optical and radiation pyrometers, we shall confine ourselves to the thermoelectric pyrometer made of the platinum metals as being on the whole the most generally suitable over this range for this kind of work, although undoubtedly particular problems may occur in which the use of some other type of pyrometer is preferable.

Use of the Thermocouple.—It may be recalled that the thermocouple possesses most of the desirable attributes of a temperature indicator. With its insignificant volume it may be introduced into a very small space, and so be used with small samples, and it takes up the temperature of the sample with great promptness. When made of the platinum metals, it is very durable and retains the constancy of its indications in a most satisfactory way, even when subjected to contaminating atmospheres, and after deterioration it may usually be restored to its former condition by glowing. Temperatures may be obtained by means of a simple form of galvanometer without any accessories. Such a galvanometer, it is true, indicates electromotive forces, while in general the temperature of a thermojunction is not strictly proportional to the electromotive force generated by it, although such a linear relation, which it is desirable to realize in order to simplify the interpretation of the indications of some types of recording instruments, holds very nearly in the case of the platinum-iridium couple of the composition Pt, 90 Pt—10 Ir.

The following table shows the E.M.F.—Temperature relation, and the E.M.F. in microvolts per degree centigrade (dE/dT) for couples composed of a wire of pure platinum joined to one of the approximate composition: *90 Pt—10 Rh, 90 Pt—10 Ir,* and pure *Ni,* respectively.

Temp. Cent.	Pt, 90 Pt—10 Rh		Pt, 90 Pt—10 Ir		Pt, Ni	
	E.M.F. Microvolts	dE/dT	E.M.F. Microvolts	dE/dT	E.M.F. Microvolts	dE/dT
300	2290	9.0	4080	15.9	7940	11.8
500	4160	9.7	7300	16.7	10510	14.4
700	6170	10.4_5	10720	17.5	13670	17.1
900	8340	11.2	14300	18.3	17400	19.7
1100	10630	11.9	18030	19.1	21640	22.4
1300	13070	12.6	21940	19.9	26300	25.0
1500	15600	13.3_5	26010	20.7	....	....

It will be seen that the platinum-iridium couple is nearly twice as sensitive as the platinum-rhodium couple beside having a more nearly linear E.M.F.—Temperature curve. These advantages are in part offset, however, by the fact that the iridium couple deteriorates more rapidly and is less constant in its indications. The platinum-nickel couple, although very sensitive, is less reliable than the others and the *Ni* wire soon becomes brittle and breaks. Moreover, the effect of the nickel recalescence point (about 375° C) has some influence on readings taken in its neighborhood. There are other thermojunctions, made of alloys of the more refractory baser metals, which are much more sensitive than the above and which may be suitable over particular ranges. These couples are usually constructed of wire of considerable diameter and are, therefore, not adapted for work with small samples. For exact work one should make sure of their constancy of indication over the temperature range to be studied. A more robust and less sensitive indicating instrument may evidently be used with this type of thermocouple, although recently pivot instruments, suitable for use with the platinum couples, have been constructed by Paul of London, the Cambridge Scientific Instrument Company, and by Siemens and Halske.

Methods of Recording.—Before describing the various methods that have been suggested for the taking of cooling curves, it may be well to consider the ways in which the observations may be recorded. Either the observer may himself read and record the indications of the instruments and discuss the data so obtained, analytically or graphically; or he may use, if the method and desired precision permit, an autographic or a photographic self-registering instrument, when it may or may not be necessary to make further reductions, depending upon the method used and the interpretation sought.

It is evidently of great advantage to use self-recording apparatus when possible, and it then becomes necessary to choose between the photographic type and the autographic.[3] The latter possesses the advantage that the experimenter may watch any part of the record, and can therefore control the operation and at any moment vary the conditions affecting the experiment; whereas with a photographic recording apparatus, as usually constructed, the observer does not know whether or not the experiment is progressing properly until it is finished and he has developed the sensitive plate. The manipulation by the photographic method is usually also more delicate and time consuming and the adjustment less sure, and the record often requires further graphical interpretation. The autographic method is in general not adapted for interpreting phenomena taking place within an interval of a few seconds, so that for very rapid cooling it is necessary to employ the photographic method. It is possible to construct the photographic recorder so as to obtain a very considerable range of speeds with the same apparatus, while it is difficult and costly to construct an autographic recorder having more than two speeds.

TEMPERATURE—TIME CURVES.

I. θ *vs* t.—The simplest method of obtaining a cooling curve is to take simultaneous measurements of the temperature of the cooling substance and of the time, from which a plot may be made showing the variation of temperature with time. (See Plate I, Fig. I.)

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[3] The term *autographic* is here used to designate an instrument which is self-recording by any other than photographic means.

The most obvious defect of this method is that it will not distinguish between phenomena proper to the substance studied and those due to outside conditions, such as accidental variations in the rate of cooling of the furnace due to air drafts and like causes. Again, where measurements over a considerable range of temperature are to be taken continuously, it becomes impracticable without elaborate experimental arrangements to combine great sensibility with this great range, so that the experimenter is in general forced to choose between great sensibility over a small temperature range or a relatively small sensibility over a large range; and this is especially true if it is desired to record the data automatically.

This method was naturally the first used[4] and it is to-day perhaps the most common one for taking cooling curves in both metallurgical and chemical laboratories. In its most elementary form it requires a minimum of apparatus—a thermocouple and an indicating galvanometer. Any desired sensibility and range may be had by substituting for the direct reading galvanometer a potentiometer and sensitive galvanometer. With this arrangement it is advantageous, when rapidity of observation is an object, to measure the last increment of temperature in terms of the galvanometer deflection rather than try to balance the potentiometer exactly while the temperature is changing. It is possible in this way to take readings as often as every 5 seconds with a properly devised set-up and quick period galvanometer.[5] A precision of 0°.1 C at 1000° C may be attained. Independently of the method of measurement used, the certainty of the detection of slight transformations may usually be increased by increasing the size of the sample under observation, thus making available larger quantities of heat.

The constant attention of the observer is of course required for either of the above systems of measurement. There have been devised, however, many kinds of self-recording apparatus for using this method, the earlier forms being photographic, while many of the later ones are autographic.

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[4] Frankenheim, Pogg. Ann. der Physik. **37**, **38** (1836–1837).

[5] See W. P. White, Potentiometer Installation, Phys. Rev. **25**, 334; 1907.

Photographic Recorders.—Among the earliest photographic recorders we may mention the apparatus of Roberts-Austen[6] (Fig. 1) in which the photographic plate P was given a vertical motion, either by clockwork, or, in order to secure maximum sensibility for several rates of cooling, by means of buoying the plate on water whose rate of flow could be regulated. The vertical motion of the plate then gave the time while the deflection of the galvanometer gave the corresponding temperature, and a beam of light from the source L reflected from the galvanometer mirror M and incident on the plate, after passing through a fine slit *S* placed horizontally before the moving plate, gave directly on the latter the time-temperature curve. Light reflected from a fixed mirror F and interrupted at equal intervals by a pendulum E gave a fixed zero line as well as a measure of the regularity of the motion of the plate. It was the practice later,[7] when taking measure-

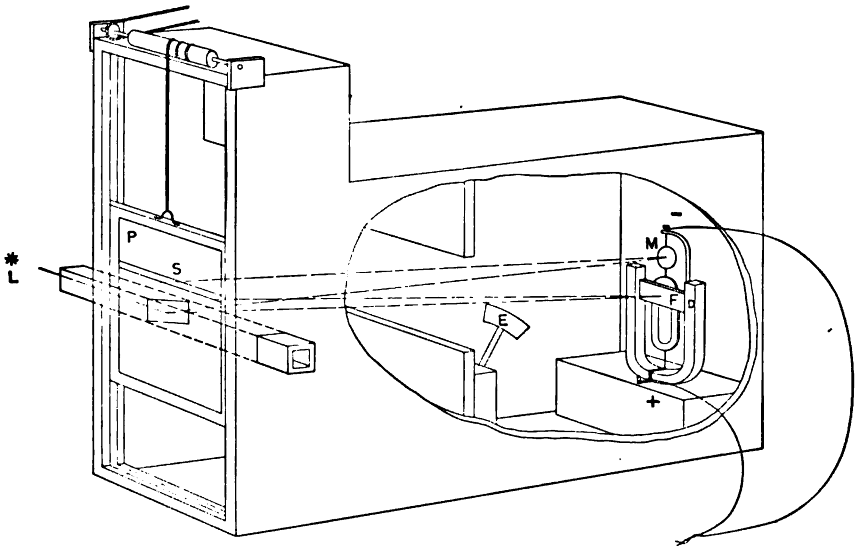

Fig. 1.

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[6] Proc Royal Society, **49**, p. 347; 1891. Nature, **45**, p. 534; 1892. First Report of the Alloys Research Committee in Proc. Inst. Mech. Engs. 1891.

[7] Fourth Report of the Alloys Research Committee, Proc. Inst. Mech. Engs., 1897. A. Stansfield, Phil. Mag. **46**, p 59; 1898.

ments over short temperature ranges, to increase the sensibility by balancing the greater part of the E.M.F. of the thermocouple with an auxiliary measured E.M.F., and giving the galvanometer the maximum sensibility that would keep its deflections on the plate.

In the apparatus used by Charpy,[8] or in its very elaborate form as constructed by Toepfer, of Potsdam, for Kurnakow,[9] the vertically moving plate is replaced by a rotating cylinder wound with the sensitized paper on which the deflections of the galvanometer are registered. This form of recorder had also been used and discarded by Roberts-Austen. Kurñakow's apparatus, which must be placed in a dark room, is furnished with an auxiliary telescope and scale system using red light, so that the experiment may be controlled during the taking of a record. As constructed, five speeds may be given to the cylinder; and there is provided an E.M.F. compensating system for maintaining the maximum sensibility over a series of temperature ranges.

There is another device, used by C. L. A. Schmidt,[10] by which the experiment may be watched while a photographic record of a cooling curve is being taken. It consists in shunting the sensitive photo-recording galvanometer G (Fig. 2), in series with a high resistance R, across a direct reading millivoltmeter V. If the resistance of R+G is great compared with that of V, the readings of the millivoltmeter will not be altered appreciably by this operation. Schmidt moves the photographic plate, mounted as in the apparatus of Roberts-Austen, by means of a screw driven by a small motor. In this way any desired speed may be given to the plate.

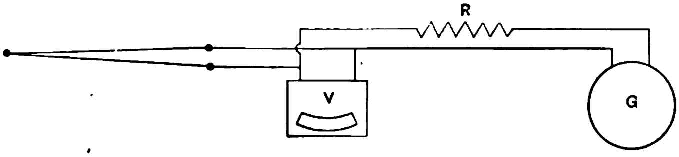

Fig. 2.

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[8] G. Charpy, Bull. de la Soc. pour l'Encouragement, **10**, p. 666; 1895.

[9] N. S. Kurnakow, Zs. f. Anorg. Chemie, **42**, p. 184; 1904.

[10] C. L. A. Schmidt, Chem. Eng., **6**, p. 80; 1907.

In practice it has been found difficult to realize conveniently a sufficiently steady motion of the plate in the Roberts-Austen system of recording, and attempts have been made to devise methods in which the photographic plate remains fixed in position. This has been successfully accomplished by Saladin, whose apparatus (Fig. 8, p. 215) has been modified by Wologdine[11] to give the temperature-time curve by removing the prism M and substituting for the second galvanometer G_2 a plane mirror turning about an horizontal axis. This mirror may be controlled by an hydraulic system as in Roberts-Austen's apparatus, or by clockwork as in the model constructed by Pellin, of Paris. The deflection of the galvanometer G_1 gives to the beam of light an horizontal motion over the plate proportional to the temperature, while the vertical motion of the beam of light is given by the mirror turning at a uniform rate, and is therefore approximately proportional to the time as registered on a flat plate.

Autographic Recorders.—To obtain a satisfactory autographic or pen record without sacrifice of sensibility of the galvanometer it is necessary to eliminate the friction of the pen or stylus upon the paper. This has been accomplished by the use of mechanisms which cause the pen or stylus at the end of the galvanometer boom to make only momentary contact with the moving paper.[12]

In the Siemens and Halske [13] form of instrument, Fig. 3, the paper P is driven forward by the same clockwork that controls the pressing down, by means of the arm B, of the stylus N, which imprints dots periodically on the paper by means of a typewriter ribbon running across and beneath the record sheet. This system permits of taking a record continuously over very long periods of time. In most of the other recorders the paper is wound upon a drum, and various devices are used to obtain the record; thus in the Hartmann and Braun type a silver stylus makes sulphide dots on a prepared paper, and in the Cambridge thread recorder rectangular coordinates are obtained by having the galvanometer boom strike an inked thread which runs parallel to the drum.

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[11] S. Wologdine, Rev. de Métallurgie, **4**, p. 552; 1907.

[12] There are a considerable number of thermoelectric recorders. Among the manufacturers of these instruments are: Siemens and Halske, Berlin; Hartmann and Braun Frankfort a. M ; Pellin, Chauvin and Arnoux, Carpentier, and Richard, Paris; Leeds and Northrup, and Queen, of Philadelphia; The Scientific Instrument Company, of Cambridge, England and Rochester, N. Y.; The Bristol Company, Waterbury, Conn.

[13] Zs. f. Instrk., **24**, p. 350; 1904.

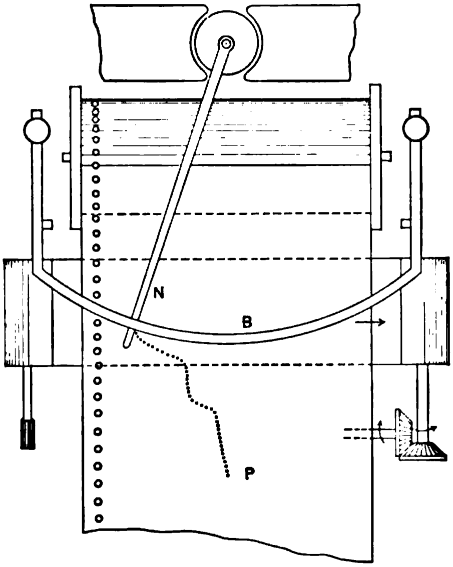

Fig. 3.

As previously stated, these autographic instruments all give intermittent records and are limited to one or two speeds, and although they may be made very sensitive they are not adapted for the detection of transformations which take place very rapidly, since the recording interval can not readily be shortened much below 10 seconds, and in most instruments this interval is greater than 15 seconds. In other words, they can be used advantageously only for slow cooling.

II. *θ vs t, θ vs t' (or θ vs θ').*—In order to eliminate the effect of irregularity of outside conditions which influence the rate of cooling, a method commonly used when endeavoring to detect small transformations consists in placing a second thermocouple in the furnace, but sufficiently removed from the substance studied to be uninfluenced by its behavior. Alternate readings on the temperature of the test piece (θ) and of the furnace (θ') are then taken, preferably at definite time intervals. The data are most readily discussed by plotting the two temperature-time curves side by side as shown in Fig. 4, or by plotting the difference in temperature $\theta-\theta'$ against the temperature θ of the test piece.

This method may be made recording either by using two instruments or by modifying one of the above mentioned autographic recorders so as to trace the curves of two thermocouples on the same sheet.[14] In practice, however, this method is usually resorted to only when great sensibility is desired, as in detecting minute internal energy changes, when the potentiometer combined with the deflection galvanometer is the most sensitive and quick-working arrangement for taking the measurements. (See p. 205.) It is convenient to use thermocouples of the same composition so as to have readings of both the temperature of the sample and of the furnace given by the same potentiometer setting, and so depend upon the galvanometer deflections for measuring the residual parts of θ and θ'.

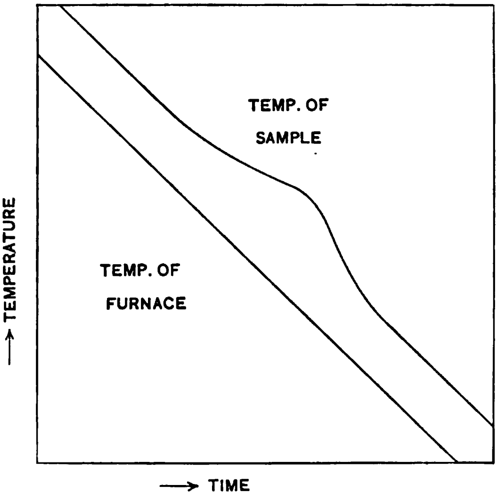

Fig. 4.

Regarding the precision of this method, it is to be noted that the quantity it is really desired to measure is $\theta-\theta'$ in terms of θ, and this is accomplished by measuring θ and θ', hence the sensibility of $\theta-\theta'$ is no greater than that of θ or θ'. In other words the method requires the maximum refinement of measurement to obtain the quantity sought, as well as the maximum of computation or plotting to reduce the observations.

DIFFERENTIAL CURVES.

III. *θ vs t, $\theta-\theta'$ vs t.*—The preceding method, II, may readily be modified so as to give $\theta-\theta'$, the difference in temperature between the test piece and furnace, by direct measurement instead of by computation, with the added advantage that the precision of $\theta - \theta'$ may be made very great as compared with that of θ, the temperature of the sample. This may be accomplished, for example, by placing a commutator in the thermocouple circuit at A, Fig. 5, so that alternate measurements on θ and $\theta - \theta'$ may be taken in terms of the time. Evidently the connections may be so made that either the galvanometer G_2 of the same direct reading or potentiometer system that measures θ, or a separate instrument G_1, as shown in the figure, may be used to measure $\theta - \theta'$.

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[14] The Siemens and Halske instrument has been so arranged. See Zs. f. Instrk., **24**, p. 350; 1904.

Use of a Neutral Body.—Accidental variations in the indications of the auxiliary thermocouple giving θ', the furnace temperature may largely be eliminated by placing this couple within a blank or neutral substance. The material of the neutral body should be such that it undergoes no transformations involving an absorption or evolution of heat within the temperature range studied, such as a piece of platinum, porcelain, or even in some cases nickel or nickel steel. It is also desirable that the sample and neutral have as near as may be the same heat capacities and emisivities. The sample to be studied and the neutral piece are placed near together and arranged symmetrically with respect to the temperature distribution within the furnace.

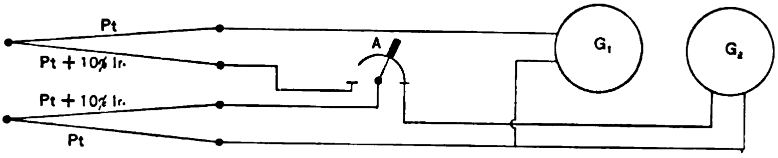

Fig. 5.

To Roberts-Austen [15] again was due the credit of first devising a sensitive differential method using the neutral body. He also modified his photographic recorder (Fig. 1) so as to give, by means of a second galvanometer, the $\theta - \theta'$ *vs* t curve on the same plate with the θ *vs* t curve, from which a curve giving $\theta - \theta'$ in terms of θ could be constructed. His arrangement of the direct reading and differential thermocouple and galvanometer circuits is shown in Fig. 6 in which S is the sample or test piece and N the neutral body possessing no transformations; the galvanometer G_2 measures the temperature θ of the sample, and G_1 measures the difference in temperature $\theta - \theta'$ between the sample and the neutral. Curves for steels and alloys were usually taken with the samples in vacuo.

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[15] Fifth Report of the Alloys Research Committee, Proc. Inst. Mech. Engs., p. 35; 1899. Metallographist, 2, p. 186; 1899.

It is evident that Roberts-Austen's final photographic apparatus, although very sensitive, was also complicated and very delicate of adjustment, and in practice it took great skill in its use, requiring for instance some three or four successive exposures adjusted to the proper adjacent temperature ranges, to take the cooling curve of a steel from 1100° C to 200° C.

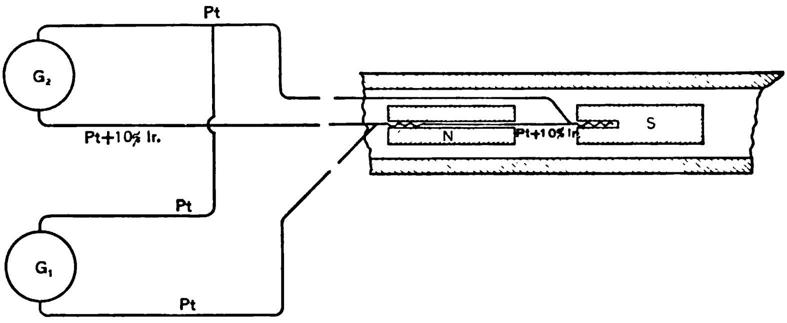

Fig. 6.

Most of the recent exact work [16] employing the principle of this method has been done by taking the observations of θ directly on a potentiometer and $\theta - \theta'$ on the same or an auxiliary galvanometer. In this case of direct reading, the simpler arrangement of thermocouples indicated in Fig. 5 may advantageously replace Roberts-Austen's (Fig. 6), or the modification shown in Fig. 7, such as used by Carpenter and others, thus dispensing with one thermocouple and the drilling of a second hole in the sample.

This method is evidently capable of attaining maximum sensitiveness since the galvanometer connected to the differential thermocouple, giving $\theta - \theta'$ *vs* t, may be made as sensitive as desired independently of the θ *vs* t system. There is the further advantage that no limits are set to the range of temperatures over which a given precision in $\theta-\theta'$ may be had. There is, however, a limitation on the certainty of interpretation of results by this method, especially when the rate of cooling is rapid, due to the fact that it is practically impossible to realize the ideal condition of having $\theta-\theta' =$ a constant, or keeping the cooling curves of the test piece and neutral parallel for temperature intervals within which there are no transformations of the test piece. The rate of cooling, and hence the value of $\theta-\theta'$, is influenced by several factors, among the most important of which are the mass of each substance—the unknown and the neutral—its specific heat, conductivity, and emissivity, as well as the relative heat capacities of the furnace and inclosed samples. The $\theta-\theta'$ *vs* t line is, however, always a smooth curve, except for the regions in which there are transformations in the substance under study.

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[16] See for example: Carpenter and Keeling, Collected Researches, Natl. Phys. Lab., 2, 1907.

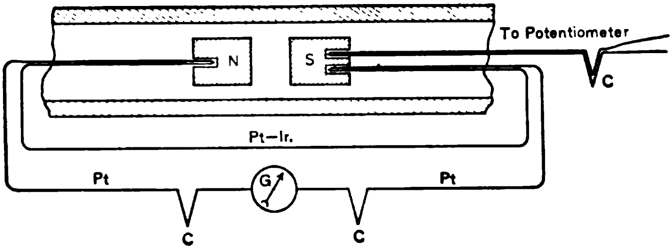

Fig. 7.

The autographic system of recording may also be used, and it is possible to construct an apparatus by means of which both the θ *vs* t and $\theta-\theta'$ *vs* t curves shall be recorded simultaneously on the same sheet by the same galvanometer boom. In order to accomplish this we have made use of a Siemens and Halske recording millivoltmeter having a total range of 1.5 millivolts and a resistance of 10.6 ohms. The E.M.F. generated by the differential thermocouple, proportional to $\theta-\theta'$, is recorded directly by this instrument. 1° C corresponds to from 16 to 19 microvolts between 300° and 1100° C for a platinum-iridum couple, or to about 1.8 mm on the record paper. In series with the Pt-Ir thermocouple giving temperatures is a suitable resistance, about 200 ohms in this case, so that the galvanometer boom may be kept within the limits of the paper when recording values of θ. The circuit is made alternately through the direct and the differential thermocouple circuits in series with the recorder by means of a polarized relay actuated by the same battery that depresses the galvanometer boom when the mark is made on the paper. The thermocouple circuits may be those of either Figs. 5, 6, or 7, but with the galvanometer G_2 indicating temperatures suppressed. It is evident that by recording the two curves, $\theta-\theta'$ *vs* t and θ *vs* t, on the same sheet there is some sacrifice in the ability to detect small and rapid transformations, since the spacing is doubled. Usually also, with such an arrangement, the galvanometer will not be completely aperiodic for one or the other system. On the other hand, it is of great advantage to have the curves together and obtained independently of inequalities in clock rates, which are a serious source of error in locating transformation points exactly when two separate instruments are used.

When it is desired merely to detect the existence of a transformation without measuring its temperature exactly, the sensitive form of recording millivoltmeter may be connected directly to the differential thermocouple without other accessories.[17]

IV. θ *vs* $\theta-\theta'$.—It is sometimes of advantage to be able to record and discuss the data independently of the time, and so express $\theta-\theta'$, the difference in temperature between sample and neutral, directly in terms of θ, the temperature of the sample. This may evidently be accomplished by replotting the results obtained from the curves of the previous differential methods which involve the time. It was reserved, however, to Saladin, engineer of the Creusot Works, to invent, in 1903, a method[18] that would record photographically the θ *vs* $\theta-\theta'$ curve directly, thus obviating any replotting. His method possesses also the advantage of having the photographic plate fixed in place. The forms of curve obtained in this way are illustrated in Plate I, Fig. IV.

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[17] Hoffmann und Rothe, Zs. Instrk., **25**, p. 273; 1905.

[18] Saladin: New Autographic Method to Ascertain the Critical Points of Steel and Steel Alloys, Iron and Steel Metallurgy and Metallography, **7**, p. 237; 1904. First presented at Reunion des Membres français et belges de l'association Internationale des Methodes d'Essais, 28 Fev., 1903.

The arrangement of the apparatus in its simplest form, due to Le Chatelier,[19] is shown in Fig. 8. Light from the source S strikes the mirror of the sensitive galvanometer G_1 whose deflections measure the differences in temperature (θ–θ') between the sample under study and the neutral body. The horizontal deflections of the beam of light are now turned into a vertical plane by passing through the totally reflecting prism M placed at an angle of 45°. A second galvanometer G_2, whose deflections are a measure of the temperature of the sample and whose mirror in its zero position is at right angles to that of G_1, reflects the beam horizontally upon the plate at P. The spot of light has thus impressed upon it two motions at right angles to each other, giving, therefore, on the plate a curve whose abscissæ are approximately proportional to the temperature θ of the sample and whose ordinates are proportional to θ–θ'. The sensitiveness of the method depends upon that of the galvanometer G_1, which may readily be made to give five or six millimeters for each degree centigrade. The arrangement of the thermocouple circuits is the same as in Figs. 6 or 7. If so desired, the time may also be recorded by means of a toothed wheel driven by a clock and placed in the path of the beam of light. Compact forms of

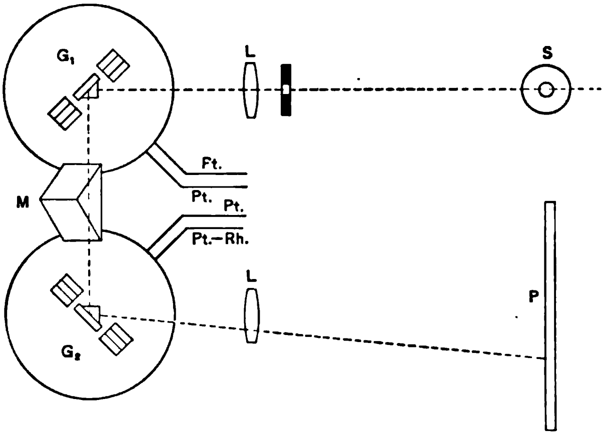

Fig. 8.

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[19] H. Le Chatelier, Rev. de Métallurgie, 1, p. 134; 1904.

this apparatus, which is used considerably in metallurgical laboratories, are made by Pellin, Paris, and by Siemens and Halske, Berlin.

When steels and metallic alloys in the solid state are being investigated, advantage may be taken of the thermoelectric behavior of the sample itself to record the critical regions with Saladin's apparatus. Thus Boudouard[20] measures $\theta-\theta'$ by means of platinum wires set into crevices at each end of the sample, taking advantage of the fact that the transformation will usually be progressive along the sample. This modification eliminates the neutral piece and one platinum or alloy wire, but, as Le Chatelier has shown,[21] is less accurate than the usual form of Saladin's apparatus; and its indications may even be indeterminate or ambiguous, as the reaction may start midway between the embedded wires or at either end.

Saladin's method, it should be noted, is a perfectly general one for recording the relations between any two phenomena which may be measured in terms of E.M.F. or as the deflections of two galvanometers. This method can not readily be made autographic, as this would require the simultaneous action of two galvanometer systems on a single pen. When used as a direct reading method it reduces to III.

IVa. θ *vs* $\theta-\theta'/\Delta\theta$.—For even moderately rapid cooling the differential method gives distorted curves which are often difficult of interpretation. This distortion is due largely to the different heat capacities and emissivities of the sample and neutral piece, thus causing differences in their rates of cooling in the furnace. Rosenhain has suggested (l. c.) that these irregularities may be eliminated by taking what he calls the "Derived Differential Curve," or expressing the temperature θ of the sample in terms of the difference in temperature $\theta-\theta'$ between the sample and neutral for equal temperature decrements $\Delta\theta$ in the sample. The experimental method is the same as in IV, but in addition it is necessary to replot the data obtained from the θ *vs* $\theta-\theta'$ measurements so as to give the value of $\theta-\theta'$ per degree change in temperature of the sample in terms of its temperature; but this appears to be worth while when the cooling curves of the sample and of the neutral differ considerably.

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[20] O. Boudouard, Rev. d. Metallurgie, **1**, p. 80; 1904.

[21] Rev. de Metallurgie, **1**, p. 134; 1904.

DIRECT AND INVERSE RATE CURVES.

V. *θ vs $d\theta/dt$.*—In many problems it is of interest to measure the speed of the transformations under observation. This may be done by determining directly the rate of change of temperature of the sample in terms of its temperature. Le Chatelier [22] used this method in 1887 in his study of the properties of clays. He was also the first to employ a photographic apparatus for the recording of cooling or heating curve data, using an arrangement in which the plate remained stationary. The sparks from an induction coil were made to pass at intervals of two seconds before a slit and gave upon the plate, after reflection from the galvanometer mirror, images of the slit whose spacing was a measure of the speed of heating, which in his experiments was about 2° C per second.

Another method of recording the rate of heating or cooling in terms of the temperature has been devised by Dejean.[23] The new feature of this method is the use of an induction galvanometer or relay which may be inserted in the circuit of the more sensitive galvanometer G_1 of the Saladin system (Fig. 8). The principle of the apparatus is shown in Fig. 9. The induction relay is a modified d'Arsonval galvanometer having an electromagnet and a movable coil, the latter consisting of two distinct

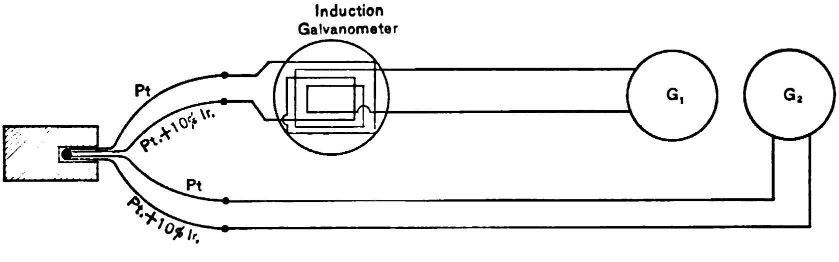

Fig. 9.

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[22] H. Le Chatelier, Comptes Rendus (Paris), **104**, p. 1443; 1887.

[23] P. Dejean, Rev. de Metallurgie, **2**, p. 701; 1905. **3**, p. 149; 1906.

insulated windings, one of which is connected to a thermocouple. Heating or cooling one junction of this couple causes the coil to be deflected and its motion in the field of the electromagnet induces an E. M. F. in the second winding of the coil which is proportional to its angular speed and hence to the rate of change of E. M. F. of the couple, or approximately to the rate of cooling or heating, i. e., to $d\theta/dt$. The induced E. M. F. is measured by joining this winding to the sensitive galvanometer G_1. The galvanometer deflection passes through a minimum when the heating or cooling passes through a minimum, that is for a region in which there is an absorption or evolution of heat. A second thermocouple in series with the other galvanometer G_2 of the Saladin system gives the temperature of the sample. We have, therefore, on the plate P (Fig. 8) when the record is taken photographically, the temperatures as abscissæ and the rate of cooling $d\theta/dt$ as ordinates as shown in Plate I, Fig V. Dejean has used this method in the study of steels and has also investigated with it the copper-cuprous oxide system. The transition temperatures are very sharply marked. If desired, direct reading may be substituted for the photographic recording, with an increase in precision, as discussed on p. 205. This method is evidently a perfectly general one for recording the rate of change of E.M.F. (dE/dt).

For neither Le Chatelier's nor Dejean's arrangement can differences in the rate of heating or cooling due to the substance itself be distinguished from those due to external causes, since no neutral piece is used.

VI. θ *vs* $dt/d\theta$.—Among the methods adapted for slow cooling, we shall mention last one of the earliest which was used to throw into prominence the abnormalities of a cooling curve, namely, the *inverse rate method*, which was employed as early as 1886 by Osmond[24] in his classic researches in metallurgy. It consists in noting the intervals of time required for a substance to cool by equal decrements of temperature and plotting this quantity ($dt/d\theta$) in terms of the temperature. (See Plate I, Fig. VI.) Osmond thus describes his method: "The time taken by the thermometer during the heating or cooling of the sample to rise or fall one division of the scale (1 mm) was registered by means of a Morse telegraph ribbon or on a rotating cylinder turned by a small electric motor. * * * A halt of the thermometer is transcribed as a cusp and a slowing down by a swell of the curve, whose area is proportional to the quantity of heat set free "

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[24] F. Osmond, Comptes Rendus (Paris), **103**, p 743, p. 1122; 1886. **104**, p. 985; 1887. Annales des Mines, **14**, p. 1; July, 1888.

But one thermocouple is needed and no neutral piece is used, so that the apparatus is the same as required for a θ *vs* t curve, I, although it is necessary, if work of precision is to be undertaken, to record very exactly by means of a chronograph and key the intervals of time (Δt) required to pass over a given number of degrees ($\Delta \theta$), say 5° C. or 10° C intervals.[25] The method, however, can not readily be made automatically recording for the variables θ and $dt/d\theta$ in terms of each other, and therefore requires the active presence of the observer. It has the same disadvantage as method II in that the precision of a difference in temperature ($\theta - \theta'$ or $\Delta\theta$) can be made no greater than that of the temperature θ. The inverse rate method is perhaps best considered as one for interpreting and plotting the θ *vs* t data in such a way as to emphasize its irregularities and so the more readily permit the detection of any critical regions.

RAPID COOLING.

None of the experimental arrangements so far described is adapted for measuring the very rapid cooling, i. e., several hundred degrees in a few seconds, met with in quenching or chilling.

Le Chatelier,[26] in an investigation of the quenching of small samples of steel and the effect of various baths, made use of a galvanometer having a period of 0.2 second and a resistance of 7 ohms, whose deflections were recorded on a photographic plate moving vertically at a speed of 3 mm per second. A half second's pendulum vibrating across the path of the beam of light, from a Nernst glower as source, gave a measure of the time. He succeeded in recording satisfactorily temperature intervals of 700° C in 6 seconds, using as samples cylinders 18 mm on a side. It would be desirable to increase the precision and sensitiveness of this method, which might be done, as Le Chatelier himself suggests, by using an oscillograph arrangement, or a string galvanometer such as Einthoven's, in which the displacements of a silvered quartz fiber of high resistance in an intense magnetic field are measured photographically.

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[25] This method is well illustrated by F. Wüst, Metallurgie (German), 8, p. 1; 1906.

[26] H. Le Chatelier, Rev. de Metallurgie, 1, p. 473; 1904.

CHARACTERISTICS OF COOLING CURVES.

In conclusion, let us consider briefly the forms that the different cooling curves may take and their approximate interpretation, for three typical kinds of transformation:

(*a*) The substance remains at a constant temperature throughout the transformation.

(*b*) The substance cools at a reduced rate, which may or may not be constant over a portion of the transformation.

(*c*) The substance undergoes an increase in temperature during the first part of the transformation.

An approximation to the first case, that of a strictly isothermal transformation, is met with in the freezing of chemically pure substances of sufficient thermal conductivity which do not undercool appreciably; in the formation of eutectics; and occasionally in other transformations. The second case is perhaps the most common; and the third is represented by the phenomena of recalescence and of undercooling preceding crystallization.

In the case of an isothermal transformation (*a*) heat is generated at the same rate that it is lost by radiation, convection, and conduction, or, considering the phenomena as confined to the sample alone:

$$\frac{dQ}{dt}=Ms\ \frac{d\theta}{dt}\ \cdots\cdots\cdots\cdots\cdots\cdots (a)$$

where M is the mass, and s the specific heat of the body supposed constant during the transformation, $\frac{dQ}{dt}$ the rate of generation of heat, and $\frac{d\theta}{dt}$ the rate of cooling the body would have, when passing through the temperature of the transformation, if there were no transformation. The heat Q generated in such a transformation lasting a finite time Δt, is therefore:

B

$$Q = Ms \frac{d\theta}{dt} \varDelta t \text{..................................} (\beta)$$

in which $\frac{d\theta}{dt} \varDelta t$ is a measure of the fall of temperature, $\varDelta\theta$ the substance would undergo if there were no evolution of heat during the time $\varDelta t$, that is, during the transformation, whence:

$$Q = Ms.\ \varDelta\theta \text{..................................} (\gamma)$$

In the case of the transformation (*b*) we have

$$\frac{dQ}{dt} < Ms \frac{d\theta}{dt}$$

and for the transformation (*c*):

$$\frac{dQ}{dt} > Ms \frac{d\theta}{dt}$$

In Plate I are illustrated these three types of transformation as given by the following cooling curves:

I.	θ *vs* t	Temperature-Time.
IV.	θ *vs* $\theta - \theta'$	Differential.
V.	θ *vs* $d\theta/dt$	Temperature-Rate.
VI.	θ *vs* $dt/d\theta$	Inverse rate.

The first horizontal line of figures refers to case (*a*), the second line to (*b*), and the third line to (*c*). For all of the curves the ordinates are temperatures; and the corresponding parts of the several curves are indicated by the same letters. The vertical lines O, O, represent the zero of abscissæ for each group of curves.

In the case of the temperature-time curves, Figs. I, I′, I″, they are drawn for an accompanying neutral piece (p. 211), A′B′C′, as well as for the sample under study, ABCD. In I the sample and neutral are cooling at the same rate, while in I′ and I″ they are cooling at different rates. In each figure the point C_1 indicates the temperature that would have been reached by the sample if there had been no transformation. C_1 may be considered as practically coinciding with C_2, the temperature that would have been reached if the sample had continued to cool at the same rate as at B, the beginning of the transformation.

For each of these transformations (*a*), (*b*), and (*c*), as represented by the curves I, I′, I″, the heat evolved is approximately proportional to $\Delta\ \theta \doteq CC_1 \doteq CC_2$, or to the maximum difference in temperature produced by the phenomenon. It is to be noticed that in general the rate of cooling just to the right of C will be greater than that just to the left of B, on account of the presence of the furnace, for the furnace walls, to which the heat is being lost, have continued to cool during the time BC. Even if the furnace were at a constant temperature, the rates at B and C might still be different due to the difference in specific heats and in emissivities of the substance before and after the transformation. The end of a transformation is always marked by a point of inflexion as shown at E (Figs. I′, I″).

From the temperature-time curve, therefore, we may determine the temperatures of beginning and ending of a transformation, its duration, an approximate measure of the heat evolved by the transformation, as well as the rate of cooling and therefore of transformation, at any instant. The interpretation of the variations in these factors forms the basis of thermal analysis, which has already been so productive in the determination of the constitution of alloys and chemical compounds.[27]

Although the forms of the curves representing the various methods are those corresponding to what is actually obtained from a sample cooling inside a furnace, yet it should be noted that the equations representing the evolution of heat are written in a form that practically neglects the influence of the furnace.

A complete discussion of the cooling within a furnace of a substance possessing transformation points would be more complicated than contemplated by this paper; requiring a knowledge of the law of cooling of the furnace and that of the interchange of heat between the sample and the furnace, the latter depending on their relative heat capacities and conductivities, the emissivity of the sample, and the distribution of temperature within the furnace.

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[27] The principles of thermal analysis as based on the θ *vs* t curve are quite fully described in the following papers by *Tammann:* Zs. Anorg. Chem., **37**, p. 303; 1903. **45**, p. 24; 1905. **47**, p. 289; 1905, a résumé of which is given by *Portevin* in Rev. de Metallurgie, **4**, p. 979; 1907. See also *Bancroft*, J. Phys. Chem., **6**, p. 178; 1902. *Shepherd*, ibid., **8**, p. 92; 1904, Iron and Steel Mag., **8**, p. 222; 1904, and *Bakhuis Roozeboom*, Die Heterogenen Gleichgewichte, 1901.

If small quantities of heat are involved in a transformation, the simplest assumptions that can be made are that at each instant the sample is losing heat to the furnace at a rate proportional to their difference in temperature, and that the parts of the furnace receiving this heat are all at the same temperature throughout at any instant. On this basis, the quantity of heat Q, lost by the sample during a transformation, as interpreted by the θ *vs* t curve, would be more nearly proportional to $\overline{CC}_1^2$ than to CC_1.

The practical conditions of cooling, however, are in general somewhere between the two extremes of the sample cooling independently of the furnace as required by our equations, and within the simplified furnace above described; so that CC_1 may be considered a minimum measure of the heat evolved, the actual measure of this quantity varying with every different experimental arrangement.

In what follows, the limitations just discussed, concerning the measurement of the heat evolved during a transformation, are assumed to apply to each of the types of cooling curve.

Figs. IV, IV′, and IV″ give the forms of the curves traced when the difference in temperature ($\theta - \theta'$) between the sample and the neutral piece (p. 211) of approximately the same thermal capacity is plotted in terms of the temperature (θ) of the sample. In Fig. IV, the line *bc*, since it is the approximate equivalent of $CC_1 \doteq \Delta\theta$ of Fig. I, may be taken as a measure of the heat evolved during the transformation. If the cooling curves of the sample and of the neutral are not parallel, as shown in Figs. I′ and I″, it is necessary, in estimating the heat evolved from the difference curve *ab′cd* (Figs. IV′ and IV″) to take into account the variations in $\theta - \theta'$ during the transformation. This change in $\theta - \theta'$ is given by $BB' - C_1C'$ (Figs. I′, I″). Furthermore, successive differential curves for the same sample are comparable only when the sample and neutral are cooled in the different experiments so as to always maintain the same temperature-time relation over a given range. When this is the case, *bc* (IV′ and IV″) is still an approximate measure of the heat generated.

It can be shown that, except for horizontal tangents, the points of inflexion E of the θ *vs* t curves do not correspond to points of inflexion at the same temperatures on the differential curves, hence the end of a transformation can not be determined from the latter. If equal time intervals are marked on the differential curve, the rate of cooling or of transformation over an interval Δt may be obtained by finding the value of $\Delta\theta/\Delta t$.

Although, as we have seen (p. 210), the differential method may be made more sensitive and certain than the direct method, yet, from the above, it is evident that it furnishes a less complete basis for the interpretation of the physico-chemical phenomena involved in the indicated transformations.

Figs. V, V′, V″ represent the *temperature-rate* curves (θ *vs* $d\theta/dt$) and Figs. VI, VI′, VI″ the *inverse rate* curves (θ *vs* $dt/d\theta$). The two types may be considered as reciprocals of each other. Comparing them with the θ *vs* t curves (Figs. I, I′, II′) it is seen that a sharp break in the latter corresponds to a perpendicular to the temperature axis for both the former; and that for an isothermal transformation, $d\theta/dt$ becomes zero at C and $dt/d\theta$ infinite at *c*. When the θ *vs* t curve is convex to the θ axis, the $d\theta/dt$ curve is concave and the $dt/d\theta$ curve convex. Both curves give negative values (OE′C, Fig. V″, and *g′gg″* Fig. VI″) for a region of recalescence.

The end of a transformation, corresponding to a point of inflexion E on the θ *vs* t curve, is sharply indicated (E, *e*) on both these rate curves by the tangent becoming parallel to the θ axis and the curve having a maximum or minimum abscissa. For any region of cooling at a constant rate, the tangent remains parallel to the θ axis during this interval.

From neither of the rate curves can the relative amounts of heat evolved during a transformation be readily computed for the different kinds of transformation. The area of the inverse rate curve, however (Fig. VI′), taken between the limits *b′*, *e′* is proportional to Δt or to $Q/d\theta/dt$ (see equation β); that is, to the quantity of heat generated divided by the rate of cooling.[28] If the rates of cooling at the beginning of two transformations are equal, then the areas, taken as above for the inverse rate curves (Fig. VI'), become an approximate measure of the heat generated. This condition, however, is rarely realized in practice. An examination of Fig. VI'' shows the practical impossibility of constructing any instrument other than one using an optical system, which would record the complete inverse rate curve.

We see, therefore, that both these rate curves mark the limits of a transformation more sharply than either the temperature-time or differential curves, and in general show greater changes in form for slight heat changes, but the rate curves do not in general give a simple measure of the heat evolved, nor is a neutral piece used to eliminate extraneous heat fluctuations.

A comparison of the properties of these various cooling curves indicates that the one from which the most comprehensive view of a transformation can be obtained is the simple temperature-time curve (I) when this method is made sufficiently sensitive; the one giving the least information is the temperature-rate curve (V); and those which cannot readily be recorded directly by any form of instrument yet devised are the inverse rate curve (VI) and the derived differential curve (IV*a*); while the one that can be made the most sensitive and certain is the differential curve (IV). The analytical discussion, therefore, leads to the same result as the examination of the experimental methods, namely, that for great range combined with greatest sensitiveness, the most certain and complete data may be obtained by combining the temperature-time observations with those obtained by the differential method.

WASHINGTON, August 3, 1908.

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[28] Area $b'bcee' = \int_{\theta_1}^{\theta_2} \frac{dt}{d\theta} d\theta = t_2 - t_1 = \Delta t$, or time occupied in falling a temperature $\Delta \theta$.

NOTE ON THE APPROXIMATE VALUES OF BESSEL'S FUNCTIONS FOR LARGE ARGUMENTS.

By Louis Cohen.

In many physical problems we frequently meet with Bessel's functions of a complex argument where the real part of the argument is a large quantity. Under these conditions the Bessel functions can be expressed in terms of exponential functions. Thus:

$$\left.\begin{aligned} J_0(ix)&=\frac{e^x}{\sqrt{2\pi x}}, \quad J_1(ix)=\frac{ie^x}{\sqrt{2\pi x}} \\ G_0(ix)&=\frac{\sqrt{\pi}e^{-x}}{\sqrt{2x}}, \quad G_1(ix)=\frac{-i\sqrt{\pi}e^{-x}}{\sqrt{2x}} \end{aligned}\right\} \qquad (1)$$

J_0 and J_1 represent the Bessel functions of the first kind and of the zero and first order, respectively. G_0 and G_1 represent functions which are combinations of the Bessel functions of the first and second kinds, and have the property that the functions approach zero when the arguments become very large. The values of $G_0(ix)$ and $G_1(ix)$ in terms of the Bessel functions of the first and second kinds are given by the following equations:[1]

$$\left.\begin{aligned} G_0(ix)&=(\log 2-\gamma)J_0(ix)-Y_0(ix) \\ G_1(ix)&=(\log 2-\gamma)J_1(ix)-Y_1(ix) \end{aligned}\right\} \qquad (2)$$

Y_0 and Y_1 are the Bessel functions of the second kind and γ is Euler's constant ($\gamma=0.57721$).

The values of the Bessel functions as given by equation (1) are usually obtained by the aid of the theory of functions of a complex variable; the proof is very laborious and difficult.

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[1] See Gray and Mathews: Treatise on Bessel Functions, pp. 68, 91.

I have worked out a simple demonstration, which, though it is not rigorous, will, I trust, be of interest to physicists.

When the real part of the argument is positive we can express the Bessel function of the first kind in the following form:[2]

$$J_n(y)=\sqrt{\frac{2}{\pi y}}\left[P\cos\left\{\frac{(2n+1)}{4}\pi-y\right\}+Q\sin\left\{\left(\frac{2n+1}{4}\right)\pi-\right\}\right] \quad (3)$$

where

$$P=1-\frac{(4n^2-1)(4n^2-9)}{2!\,(8y)^2}+\ldots\ldots$$

$$Q=\frac{4n^2-1}{8x}-\frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!\,(8y)^3}+\ldots\ldots$$

and hence

$$J_0(y)=\sqrt{\frac{2}{\pi y}}\left[P\cos\left(\frac{\pi}{4}-y\right)+Q\sin\left(\frac{\pi}{4}-y\right)\right]$$

$$J_1(y)=\sqrt{\frac{2}{\pi y}}\left[P\cos\left(\frac{3\pi}{4}-y\right)+Q\sin\left(\frac{3\pi}{4}-y\right)\right]$$

now

$$\cos\left(\frac{\pi}{4}-y\right)=\frac{1}{\sqrt{2}}(\cos y+\sin y)$$

$$\sin\left(\frac{\pi}{4}-y\right)=\frac{1}{\sqrt{2}}(\cos y-\sin y)$$

$$\cos\left(\frac{3\pi}{4}-y\right)=\frac{1}{\sqrt{2}}(-\cos y+\sin y)$$

$$\sin\left(\frac{3\pi}{4}-y\right)=\frac{1}{\sqrt{2}}(\cos y+\sin y)$$

$$\left.\begin{aligned}\therefore\ J_0(y)&=\frac{1}{\sqrt{\pi y}}[(P+Q)\cos y+(P-Q)\sin y]\\ J_1(y)&=\frac{1}{\sqrt{\pi y}}[(Q-P)\cos y+(P+Q)\sin y]\end{aligned}\right\} \quad (4)$$

If y is very large we can put $P=1$ and neglect Q as compared with P, and we shall have

$$J_0(y)=\frac{1}{\sqrt{\pi y}}\{\cos y+\sin y\}$$

$$J_1(y)=\frac{1}{\sqrt{\pi y}}\{-\cos y+\sin y\}$$

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[2] Gray and Mathews: Bessel Functions, p. 40.

We also have

$$\cos y = \frac{1}{2}(e^{iy} + e^{-iy})$$

$$\sin y = \frac{-i}{2}(e^{iy} - e^{-iy})$$

put now $y = ix$, and we get

$$J_0(ix) = \frac{1}{2\sqrt{\pi i x}}\left\{(1-i)e^{-x} + (1+i)e^{x}\right\}$$

$$J_1(ix) = \frac{1}{2\sqrt{\pi i x}}\left\{-(1+i)e^{-x} + (i-1)e^{x}\right\}$$

Neglecting e^{-x} as compared with e^{x}, we get

$$\left.\begin{aligned} J_0(ix) &= \frac{1}{2\sqrt{\pi i x}}(1+i)e^{x} = \frac{1}{\sqrt{2\pi x}}e^{x} \\ J_1(ix) &= \frac{1}{2\sqrt{\pi i x}}(i-1)e^{x} = \frac{i}{\sqrt{2\pi x}}e^{x} \end{aligned}\right\} \qquad (5)$$

The values of the G functions can be obtained in the following way: There exists a well-known relation between the Bessel functions of the first and second kinds, which is as follows:

$$J_1(ix)\,Y_0(ix) - J_0(ix)\,Y_1(ix) = \frac{1}{ix} \qquad (6)$$

and if we substitute in equation (6) the values J_0 and J_1 as given by equation (5) we get

$$\begin{aligned} \frac{e^{x}}{\sqrt{2\pi x}}\left\{i\,Y_0(ix) - Y_1(ix)\right\} &= \frac{1}{ix} \\ \text{or } i\,Y_0(ix) - Y_1(ix) &= \frac{1}{i}\sqrt{\frac{2\pi e^{-x}}{x}} \end{aligned} \qquad (7)$$

From equation (2) we can get the following relation:

$$iG_0(ix) - G_1(ix) = (\log 2 - \gamma)\big(iJ_0(ix) - J_1(ix)\big) - i\,Y_0(ix) + Y_1(ix)$$

Since, however, $J_1(ix) = iJ_0(ix)$ (see equation 5) we get

$$iG_0(ix) - G_1(ix) = Y_1(ix) - iY_0(ix) = -\frac{1}{i}\sqrt{\frac{2\pi}{x}}\,e^{-x}$$

or

$$i\,G_0(ix) - G_1(ix) = i\sqrt{\frac{2\pi}{x}}e^{-x} \tag{8}$$

The values of G_0 and G_1 as given by equation (1) will satisfy the above equation; introducing the values we get

$$\left(i\sqrt{\frac{\pi}{2x}} + i\sqrt{\frac{\pi}{2x}}\right)e^{-x} = i\sqrt{\frac{2\pi}{x}}\,e^{-x}$$

which is identical with (8), and hence it is reasonable to suppose that the values of G_0 and G_1 as given by equation (1) are the proper solutions.

WASHINGTON, August 24, 1908.

THE INFLUENCE OF TERMINAL APPARATUS ON TELEPHONIC TRANSMISSION.

By Louis Cohen.

The problem of telephonic transmission has been very ably treated by several investigators. One has only to examine the collected papers of Dr. Oliver Heaviside and the two papers on the Propagation of Electrical Waves by Prof. M. I. Pupin to find a discussion of almost every phase of the problem. But while there is very little that one could add to the mathematical treatment of the problem, yet there is one point in connection with this problem, "The Influence of Terminal Apparatus on Telephonic Transmission," which has not been given sufficient attention, and I trust a brief discussion of this phase of the problem will be of some interest.

The only treatment of the problem that I am familiar with, that takes into account the effect of the terminal apparatus, is that by Dr. O. Heaviside: "Effect of a periodic impressed force acting at one end of a telegraphic circuit with any terminal condition."[1] The derivation, however, of the formula as given in this paper is not very easy to follow, and the expressions he obtains for the current and potential along the line are of a very complicated form, and therefore do not make apparent the detrimental effect of the terminal apparatus.

In general whatever influence the line may have in distorting a telephonic wave, when the wave reaches the receiving instrument part of it will be reflected, and will merely dissipate itself on the line, and part will be absorbed and furnish the energy necessary to operate the instrument. Either portion of the wave, however, be it the part reflected or absorbed, is a function of the frequency, and therefore every harmonic will be affected differently, which will necessarily produce a certain amount of distortion. The extent of the distortion that the wave will suffer in passing through the receiving instrument may under certain circumstances be considerable.

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[1] Collected papers, vol. 2, p. 245.

The final expressions obtained in this paper for the current and potential at the end of the line are of very simple form and convenient for numerical calculations, enabling us to calculate the extent of the distortion produced by the terminal apparatus.

An examination of the final expressions (20) will, I think, make apparent at once the effect of the receiver on the distortion. Certain improvements which suggest themselves will be considered in the latter part of this paper.

Let L, R, C denote the inductance, resistance, and capacity per unit length of line; let also L_0, R_0, C_0 and L_1, R_1, C_1 denote the inductance, resistance, and capacity of the sending and receiving instruments, respectively. Assume also that the leakage conductance of the line is negligible. Denoting by X the current at any point on the line and s the distance from the sending end, which we shall consider as the origin, then we shall have for any point on the line

$$L\frac{dx}{dt}+Rx=-\frac{dv}{ds} \qquad (1)$$

which expresses the fact that the algebraic sum of the electromotive forces acting at any point is zero. We also have the relation

$$C\frac{dv}{dt}=-\frac{dx}{ds} \qquad (2)$$

From equations (1) and (2) we get the differential equation of propagation, now commonly known as the "telegraphist equation," which is as follows:

$$L\frac{d^2x}{dt^2}+R\frac{dx}{dt}=\frac{1}{C}\frac{d^2x}{ds^2} \qquad (3)$$

If the current is simple harmonic with respect to time, we may put

$$X=x'e^{ipt}$$

and equation (3) becomes

$$(-CLp^2+RipC)x'=\frac{d^2x'}{ds^2} \qquad (4)$$

The complete solution of equation (4) is

$$x' = A \sin \mu s + B \cos \mu s \tag{5}$$

and therefore

$$X = \{A \sin \mu s + B \cos \mu s\} e^{ipt} \tag{6}$$

Introducing the value of X as given by (5) into equation (4), we find that

$$-\mu^2 = -C\,Lp^2 + RipC$$

Putting $\mu = \alpha + i\beta$ and then equating the reals and imaginaries separately, we shall get

$$\left.\begin{aligned} \alpha &= \sqrt{\frac{1}{2}}\sqrt{Cp\{\sqrt{p^2L^2+R^2}+pL\}} \\ \beta &= \sqrt{\frac{1}{2}}\sqrt{Cp\{\sqrt{p^2L^2+R^2}-pL\}} \end{aligned}\right\} \tag{7}$$

The value of V, the potential at any point on the line, can be obtained from (6) by the aid of equation (2).

We thus find

$$V = \frac{\mu}{Cip}\left\{B \sin \mu s - A \cos \mu s\right\} e^{ipt} \tag{8}$$

The constants A and B in equations (6) and (8) must be determined from the boundary conditions.

When

$$s = l$$

$$V_l = X_l\left\{R_1 + i\left(PL_1 - \frac{1}{C_1 p}\right)\right\}$$

Introducing the values of V_l and X_l, we get

$$\frac{\mu}{Cip}\left\{B \sin \mu\, l - A \cos \mu l\right\} = \left\{R_1 + i\left(L_1 p - \frac{1}{C_1 p}\right)\right\}\left\{A \sin \mu l + B \cos \mu l\right\}$$

or

$$\left\{B \sin \mu l - A \cos \mu l\right\} = (H_1 + iF_1)\left\{A \sin \mu l + B \cos \mu l\right\} \tag{9}$$

where

$$H_1+iF_1=\frac{Cip}{\mu}\left\{R_1+i\left(L_1p-\frac{1}{C_1p}\right)\right\}$$

Equating reals and imaginaries separately, we get

$$\left.\begin{aligned}H_1&=\frac{Cp}{\alpha^2+\beta^2}\left\{-\alpha\left(L_1p-\frac{1}{C_1p}\right)+\beta R_1\right\}\\ F_1&=\frac{Cp}{\alpha^2+\beta^2}\left\{\beta\left(L_1p-\frac{1}{C_1p}\right)+\alpha R_1\right\}\end{aligned}\right\}\tag{10}$$

When $s=0$ we have

$$V_0=Ee^{ipt}-X_0\left\{R_0+i\left(L_0p-\frac{1}{C_0p}\right)\right\}$$

Ee^{ipt} is the impressed electromotive force.

Introducing the values of V_0 and X_0, we get

$$\frac{-\mu A}{Cip}=E-B\left\{R_0+i\left(L_0p-\frac{1}{C_0p}\right)\right\}$$

$$\frac{E\,Cip}{\mu}=\frac{B\,Cip}{\mu}\left\{R_0+i\left(L_0p-\frac{1}{C_0p}\right)\right\}-A$$

or

$$E\frac{Cip}{\mu}=\left(T_1+iT_2\right)B-A\tag{11}$$

where

$$\left.\begin{aligned}T_1&=\frac{Cp}{\alpha^2+\beta^2}\left\{-\alpha\left(L_0p-\frac{1}{C_0p}\right)+\beta R_0\right\}\\ T_2&=\frac{Cp}{\alpha^2+\beta^2}\left\{\beta\left(L_0p-\frac{1}{C_0p}\right)+\alpha R_0\right\}\end{aligned}\right\}\tag{12}$$

From (9) and (11) we can determine the value of the constants A and B, which are as follows:

$$\begin{aligned}A&=\frac{E\,Cip\,\{\sin\mu l-(H_1+iF_1)\cos\mu l\}}{\mu(T_1+iT_2)\,\{(H_1+iF_1)\sin\mu l+\cos\mu l\}-\mu\{\sin\mu l-(H_1+iF_1)\cos\mu l\}}\\ B&=\frac{E\,Cip\,\{\cos\mu l+(H_1+iF_1)\sin\mu l\}}{\mu(T_1+iT_2)\,\{(H_1+iF_1)\sin\mu l+\cos\mu l\}-\mu\{\sin\mu l-(H_1+iF_1)\cos\mu l\}}\end{aligned}\tag{13}$$

Introducing these values of A and B in equations (6) and (8) we get

$$X=\frac{E\,Cip\{\cos\mu(l-s)+(H_1+iF_1)\sin\mu(l-s)\}e^{ipt}}{\mu(T_1+iT_2)\{(H_1+iF_1)\sin\mu l+\cos\mu l\}-\mu\{\sin\mu l-(H_1+iF_1)\cos\mu l\}} \tag{14}$$

$$V=\frac{E\mu\{(H_1+iF_1)\cos\mu(l-s)-\sin\mu(l-s)\}e^{ipt}}{\mu(T_1+iT_2)\{(H_1+iF_1)\sin\mu l+\cos\mu l\}-\mu\{\sin\mu l-(H_1+iF_1)\cos\mu l\}} \tag{15}$$

Remembering now that $\mu=a+i\beta$, we can write

$$\left.\begin{aligned}\sin\mu l&=\frac{1}{2}\sin al\,(e^{\beta l}+e^{-\beta l})+\frac{i}{2}\cos al\,(e^{\beta l}-e^{-\beta l})\\ \cos\mu l&=\frac{1}{2}\cos al\,(e^{\beta l}+e^{-\beta l})-\frac{i}{2}\sin al\,(e^{\beta l}-e^{-\beta l})\end{aligned}\right\}\quad(16)$$

Introducing the above values of $\sin\mu l$ and $\cos\mu l$ in the denominators of equations (14) and (15) and then separating the reals and imaginaries, we shall have

$$\mu(T_1+iT_2)\{(H_1+iF_1)\sin\mu l+\cos\mu l\}-\mu\{\sin\mu l-(H_1+iF_1)\cos\mu l\}=K_1+iK_2$$

where

$$\left.\begin{aligned}K_1=&\left(\frac{e^{\beta l}+e^{-\beta l}}{2}\right)\left(T_1a\cos al-T_2\beta\cos al-a\sin al\right)\\ &+\left(\frac{e^{\beta l}+e^{-\beta l}}{2}\right)\left\{-Cp\cos al\left(L_1p-\frac{1}{C_1p}\right)-T_2CpR_1\sin al\right\}\\ &\qquad-T_1Cp\left(L_1p-\frac{1}{C_1p}\right)\sin al\\ &+\left(\frac{e^{\beta l}-e^{-\beta l}}{2}\right)\left(CpR_1\sin al-T_1CpR_1\cos al\right.\\ &\qquad\left.+T_2Cp\left(L_1p-\frac{1}{C_1p}\right)\cos al\right)\\ &+\left(\frac{e^{\beta l}-e^{-\beta l}}{2}\right)\left(T_1\beta\sin al+T_2a\sin al+\beta\cos al\right)\end{aligned}\right\}\quad(17)$$

and

$$\left.\begin{aligned}
K_2=&\left(\frac{e^{\beta l}+e^{-\beta l}}{2}\right)\left(-T_1\beta\cos al+T_2a\cos al-\beta\sin al\right)\\
&+\left(\frac{e^{\beta l}-e^{-\beta l}}{2}\right)\Bigg(CpR_1\cos al+CpR_1T_1\sin al\\
&\qquad-T_2Cp\left(L_1p-\frac{1}{C_1p}\right)\sin al\Bigg)\\
&+\left(\frac{e^{\beta l}-e^{-\beta l}}{2}\right)\left\{Cp\left(L_1p-\frac{1}{C_1p}\right)\sin al-T_2CpR_1\cos al\right\}\\
&\qquad-T_1Cp\left(L_1p-\frac{1}{C_1p}\right)\cos al\\
&+\left(\frac{e^{\beta l}-e^{-\beta l}}{2}\right)\left(-T_1a\sin al+T_2\beta\sin al-a\cos al\right)
\end{aligned}\right\}\quad(17)$$

If we neglect the constants of the transmitting apparatus, then $T_1=0$ and $T_2=0$, and consequently the expressions for K_1 and K_2 will be greatly simplified, thus:

$$\begin{aligned}
K_1=&\left(\frac{e^{\beta l}+e^{-\beta l}}{2}\right)\left(-a\sin al-Cp\left(L_1p-\frac{1}{C_1p}\right)\cos al\right)+\left(\frac{e^{\beta l}+e^{-\beta l}}{2}\right)\\
&\left(\beta\cos al+CpR_1\sin al\right)\\
K_2=&\left(\frac{e^{\beta l}+e^{-\beta l}}{2}\right)\left(-\beta\sin al+CpR_1\cos al\right)+\left(\frac{e^{\beta l}-e^{-\beta l}}{2}\right)\Bigg(-a\cos al\\
&+Cp\left(L_1p-\frac{1}{C_1p}\right)\sin al\Bigg)
\end{aligned}\qquad(18)$$

If we put for convenience $l-s=\xi$ and then substitute for $\cos\mu\xi$ and $\sin\mu\xi$ their values as given by equation (16), we shall have

$$\begin{aligned}
X=\frac{ECip(iK_1+K_2)}{2(K_1^2+K_2^2)}\Big\{&\cos a\xi(e^{\beta\xi}+e^{-\beta\xi})-i\sin a\xi(e^{\beta\xi}-e^{-\beta\xi})+\\
&H_1\sin a\xi(e^{\beta\xi}+e^{-\beta\xi})+iH_1\cos a\xi(e^{\beta\xi}-e^{-\beta\xi})+iF_1\sin a\xi(e^{\beta\xi}+e^{-\beta\xi})\\
&-F_1\cos a\xi(e^{\beta\xi}-e^{-\beta\xi})\Big\}e^{ipt}
\end{aligned}$$

$$V=\frac{E(a+i\beta)(K_1-iK_2)}{2(K_1^2+K_2^2)}\Big\{H_1\cos a\xi(e^{\beta\xi}+e^{-\beta\xi})-iH_1\sin a\xi(e^{\beta\xi}-e^{-\beta\xi})$$
$$+iF_1\cos a\xi(e^{\beta\xi}+e^{-\beta\xi})+F_1\sin a\xi(e^{\beta\xi}-e^{-\beta\xi})-\sin a\xi(e^{\beta\xi}+e^{-\beta\xi})$$
$$-i\cos a\xi(e^{\beta\xi}-e^{-\beta\xi})\Big\}e^{ipt}$$

After some simplification and a few reductions the real part of the above two expressions will be as follows:

$$\left.\begin{array}{c} X=\dfrac{ECp}{2\sqrt{K_1^2+K_2^2}}\Big\{[e^{\beta\xi}\cos(pt-a\xi+\theta)+ \\ e^{-\beta\xi}\cos(pt+a\xi+\theta)]+\sqrt{H_1^2+F_1^2}[-e^{\beta\xi}\cos(pt-a_{\xi}-\varphi+\theta) \\ +e^{-\beta\xi}\cos(pt+a_{\xi}-\varphi+\theta)]\Big\} \\ V=\dfrac{E\sqrt{a^2+\beta^2}}{2\sqrt{K_1^2+K_2^2}}\Big\{[-e^{\beta\xi}\cos(pt-a\xi+\theta+\omega)+ \\ e^{-\beta\xi}\cos(pt+a\xi+\theta+\omega)]+\sqrt{H_1^2+F_1^2}[l^{\beta\xi}\cos(pt-a\xi-\phi+\theta+\omega) \\ +e^{-\beta\xi}\cos(pt+a\xi-\phi+\theta+\omega)]\Big\} \end{array}\right\}(20)$$

where

$$\tan\theta=\frac{K_1}{K_2},\ \tan\omega=\frac{H_1}{F_1},\ \tan a=\frac{\beta}{a}$$

K_1 and K_2 are given by equations (17) or (18).
H_1 and F_1 are given by equations (10).
a and β are given by equations (7).

$$H_1^2+F_1^2=\frac{C^2p^2}{a^2+\beta^2}\Big\{(L_1p-\frac{1}{C_1p})^2+R^2\Big\} \qquad (21)$$

If we neglect the sending apparatus, then $K_1^2+K_2^2$ will reduce to the following:

$$K_1^2+K_2^2=\frac{e^{2\beta l}+e^{-2\beta l}}{4}\Big\{(a^2+\beta^2)+C^2p^2\Big(R_1^2+(L_1p-\frac{1}{C_1p})^2\Big)\Big\}$$
$$-\frac{1}{2}\cos 2al\Big\{(a^2+\beta^2)-C^2p^2\Big(R_1^2+(L_1p-\frac{1}{C_1p})^2\Big)\Big\}+Cp\sin al$$
$$\Big\{a(L_1p-\frac{1}{C_1p})-\beta R_1\Big\}-\frac{e^{2\beta l}-e^{-2\beta l}}{2}\Big\{\beta Cp(L_1p-\frac{1}{C_1p})+aCpR_1\Big\} \quad (22)$$

Equations (19) and (20) are the most complete and general expressions for the current and potential at any point on the line, and take into consideration the electrical constants (resistance, inductance, capacity) of the sending and receiving apparatus. An examination of equation (19) will show that we have four distinct waves traveling on the line. The first part represents two waves oppositely directed, sent out by transmitting apparatus, and the second part represents the reflected waves. It is quite evident that the reflected wave will not affect the receiver, and this appears from equation (19); when $s=l$, $\xi=0$, and equation (19) reduces to

$$X_l=\frac{E\,Cp\cos(pt+\theta)}{\sqrt{K_1^2+K_2^2}} \qquad (23)$$

A similar condition obtains for the potential, except that in this case it is the second part of equation (20) that represents the part of the wave absorbed by the receiving instrument, for it is evident that the drop on the receiver must be proportional to the impedance of the instrument, which is the factor $\sqrt{H_1^2+F_1^2}$, and thus when $s=l$ we have

$$V_1=E\frac{\sqrt{a^2+\beta^2}\,\sqrt{H_1^2+F_1^2}}{\sqrt{K_1^2+K_2^2}}\cos(pt-\varphi+\theta+\omega)$$

Introducing the value of $\sqrt{H_1^2+F_1^2}$ from equation (21) we get:

$$V_1=\frac{E\,Cp\sqrt{\left(L_1p-\frac{1}{C_1p}\right)^2+R_1^2}}{\sqrt{K_1^2+K_2^2}}\cos(pt-\varphi+\theta+\omega) \qquad (24)$$

For the purpose of this paper it will be sufficient to consider equation (23). Suppose we calculate the maximum value of X, that is $\frac{Cp}{\sqrt{K_1^2+K_2^2}}$ for several different frequencies, first by neglecting the terminal apparatus, and second by taking into account the terminal apparatus; then evidently the ratio of the values as calculated for the different frequencies will be a measure of the distortion. Now comparing these ratios for the different frequencies with and without the terminal apparatus will evidently give us a measure of the effect produced by the terminal apparatus.

In my calculations I have neglected the transmitting apparatus and considered only the effect of the receiving apparatus on the distortion. This reduces the labor of computation very materially, and at the same time shows just as clearly the effect of any terminal apparatus on distortion.

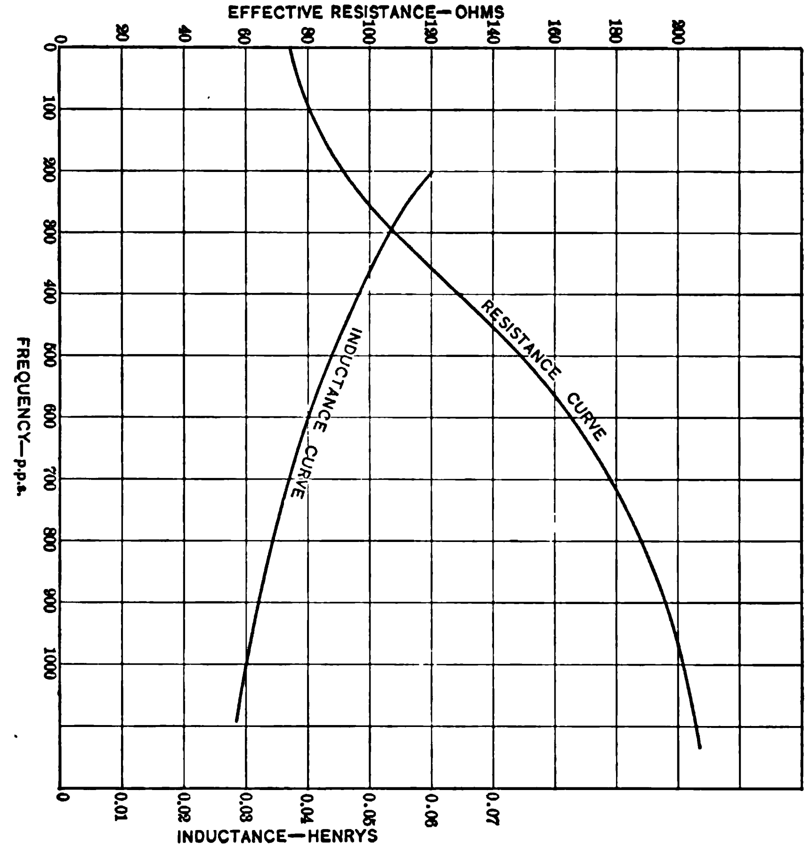

Fig. 1.

In computing the value of X_t as given by equation (23) it must be remembered that the inductance and resistance of the receiver are functions of the frequencies, and I am indebted to Mr. C. E. Scribner, of the Western Electric Company, for furnishing me curves showing the relation of the resistance and inductance with the frequency of the receiver manufactured by the Western Electric Company. The curves are given in Fig. (1).

In calculating the value of X as given by the above equation I have assumed three distinct cases:

(1) The terminal apparatus neglected entirely;

(2) The receiver included in circuit;

(3) The receiver in series with a condenser in circuit.

I have also assumed three distinct frequencies, 500, 1000, and 1500, and two different lines, one 30 km long and one 300 km long.

The constants of 30-km line (cable, 19 B. S.) are approximately as follows:

L=0.00001 henry per km.
C=0.045 microfarad per km.
R=27.5 ohms per km.

Using these constants and assuming that the impressed voltage corresponding to the frequencies 500, 1000, 1500 are E_1, E_3, E_5, respectively, we shall have the following table:

TABLE I.

	Frequency=500	Frequency=1000	Frequency=1500
Case 1—			
No terminal apparatus	$X_l=0.00114E_1$	$X_l=0.00099E_3$	$X_l=0.00089E_5$
Case 2—			
Receiver in circuit	$X_l=0.00175E_1$	$X_l=0.005E_3$	$X_l=0.0046E_5$
Case 3—			
Receiver and a condenser of 1.5 mf in circuit	$X_l=0.0018E_1$	$X_l=0.0019E_3$	$X_l=0.0045E_5$

An examination of the above table shows that the receiver in the circuit increases the distortion, and it is further seen that the introduction of a condenser will improve the transmission.

Now while telephoning at such short distance is possible with almost any apparatus, yet it will be admitted that there is considerable room for improvement, and it would seem that the introduction of a condenser of the proper magnitude in series with the receiver ought to produce good results.

To show the effect on the distortion in long distance transmission I have made similar calculations for a cable (13 B. S.) 300 km long.

The electrical constants of such a cable are approximately as follows:

$L = 0.0001$ henry per km
$C = 0.045$ microfarad per km
$R = 7$ ohms per km.

Using these values and the constants of the receiver the same as in the previous case, we get the following table:

TABLE II.

	Frequency=500	Frequency=1000	Frequency=1500
Case 1—			
No terminal apparatus	$X_l = 0.003E_1$	$X_l = 0.00011E_3$	$X_l = 0.000053E_5$
Case 2—			
Receiver in circuit	$X_l = 0.00126E_1$	$X_l = 0.00017E_3$	$X_l = 0.00004E_5$
Case 3—			
Receiver and a condenser of 1.5 mf in circuit	$X_l = 0.0014E_1$	$X_l = 0.0002E_3$	$X_l = 0.00005E_3$

From the table it can be seen that the introduction of the telephone receiver does not materially influence telephonic transmission.

WASHINGTON, August 3, 1908.

THE PRINCIPLES INVOLVED IN THE SELECTION AND DEFINITION OF THE FUNDAMENTAL ELECTRICAL UNITS TO BE PROPOSED FOR INTERNATIONAL ADOPTION.*

By F. A. Wolff.

INTRODUCTION.

In a paper presented to the International Electrical Congress at St. Louis in 1904,[1] the writer reviewed the efforts which had previously been made to secure international uniformity of the electrical units, and pointed out their failure as shown by the discrepancies in the laws thus far enacted.

These efforts practically begin with the Siemens unit and the C. G. S. system of the British Association, of which the practical unit of resistance, intended to represent 10^9 electro-magnetic units, took a concrete form.

As pointed out in the paper referred to, the error found in the B. A. unit led to the convocation of the first International Electrical Congress in 1881, which recommended that the practical electrical units be defined in terms of the C. G. S. (electro-magnetic) units, and that the unit of resistance be represented by a column of mercury 1 sq mm in cross section, at the temperature of 0° C and of a length to be determined by an international commission on the basis of new absolute measurements.

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* Paper read before the American Physical Society April 25, 1908, and before the American Electrochemical Society April 30, 1908.

[1] Transactions St. Louis Int. Elec. Cong., **1**, 148; 1904. This Bulletin, **1**, 39; 1904.

In accordance with the above, the International Commission, which met in Paris in 1882 and 1884, recommended the following definitions:

The legal ohm is the resistance of a column of mercury 1 square millimeter in cross section and 106 centimeters in length at the temperature of melting ice.

The ampere is equal to one-tenth of a C. G. S. unit of the electro-magnetic system.

The volt is the electromotive force which will maintain a current of 1 ampere in a conductor of which the resistance is a legal ohm.

It will thus be seen that a distinction was made between the concrete and the absolute ohm. The ampere was defined in terms of absolute units, while the unit of electro-motive force was derived from a concrete unit of resistance and an absolute unit of current.

The last formal international action was taken by the Chicago Congress of 1893, which adopted the following definitions:

"*Resolved*, That the several governments represented by the delegates of this International Congress of Electricians be, and they are hereby, recommended to formally adopt as legal units of electrical measure the following:

"As a unit of resistance, the *international ohm*, which is based upon the ohm equal to 10^9 units of resistance of the C. G. S. system of electro-magnetic units, and is represented by the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14.4521 grammes in mass, of a constant cross-sectional area and of the length of 106.3 centimeters.

"As a unit of current, the *international ampere*, which is one-tenth of the unit of current of the C. G. S. system of electro-magnetic units, and which is represented sufficiently well for practical use by the unvarying current which, when passed through a solution of nitrate of silver, and in accordance with the accompanying specifications, deposits silver at the rate of 0.001118 of a gramme per second.

"As a unit of electromotive force, the *international volt*, which is the electromotive force that, steadily applied to a conductor whose resistance is 1 international ohm, will produce a current of 1 international ampere, and which is represented sufficiently well for practical use by 1000/1434 of the electromotive force between the poles or electrodes of the voltaic cell known as the Clark cell, at a temperature of 15° C, and prepared in the manner described in the accompanying specifications."

The disagreement of the laws defining the units subsequently enacted by the various countries may be traced to the following causes:

First. The three fundamental units, although connected by Ohm's law, were either directly or indirectly defined in terms of concrete standards, naturally resulting in inconsistencies in the definitions, which by new measurements of the value of the Clark cell were shown to be almost one-tenth per cent.

Second. The specifications were not sufficiently definite, thus limiting the accuracy of reproduction.

Third. The *international* units were not recognized as separate and distinct from the *absolute* units on which they were based.

As some of the laws were enacted before and some after the inconsistencies in the definitions were recognized, considerable differences resulted, a subject fully discussed in the paper above referred to. For our present purpose it need only be recalled that in some cases the same unit was defined effectively in as many as three or four different ways in the same law. Official international recognition of the discrepancies between the laws was taken by the St. Louis Congress, which adopted resolutions to the effect that the questions could best be dealt with by an international commission representing the governments concerned, that such a commission might in the first instance be appointed by those countries in which legislation on electric units had been adopted, and that provision should be made for securing the adhesion of other countries prepared to adopt the conclusions of the commission. The hope was expressed that the commission referred to might eventually become a permanent one.

In accordance with the provisions of the above resolution, invitations were extended to England, France, Austria, Belgium, and the United States, by Professor Warburg, president of the

Reichsanstalt, on behalf of Germany, for a conference which was held in Berlin in October, 1905. The decisions reached by the Berlin Conference[2] were as follows:

(1) That only two electrical units shall be chosen as fundamental units.

(2) The international ohm, defined by the resistance of a column of mercury, and the international ampere, defined by the deposition of silver, are to be taken as the fundamental electrical units.

(3) The international volt is the electromotive force which produces an electric current of 1 international ampere in a conductor whose resistance is 1 international ohm.

Recommendations were also made that more detailed specifications be adopted for the mercury ohm, and that the Weston cadmium cell with solid hydrated cadmium sulphate and a 12 to 13 per cent cadmium amalgam be adopted as the standard cell.

The opinions were further expressed by the conference:

"(1) That the information before it is not sufficient to enable it to propose any alteration in the formerly accepted value for the ampere.

"(2) That the information before it is not sufficient to enable it to lay down exact directions in respect to the silver voltameter and the standard cell.

"(3) That if a proposal for a change in the accepted value of the ampere is to be brought from any source before a formal conference to be held later, an agreement in writing on the point should be come to previously between the parties interested. If differences of opinion in the matter can not be removed, a new preliminary conference should be held. The same procedure should be observed in regard to the specifications for the silver voltameter and the standard cell, in the event of such specifications being submitted to a formal conference from any quarter."

The recommendations made at the conference by Professor Carhart and by the Bureau of Standards, that the volt be selected as the second fundamental unit were rejected, as were also the recommendations of the Bureau advocating a systematic program of absolute measurements before the next International Electrical Congress was called.

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[2] Verh. Int. Konf. Elektr. Masseinheiten, p. 42.

The conclusions reached by the Berlin Conference should be regarded as preliminary in view of the fact that considerable work has since been done on both the standard cell and coulometer, particularly as the conference expressed the opinion[3] that the information before it was not then sufficient to enable it to lay down exact directions with respect to the silver voltameter and the standard cell.

The questions likely to be considered by the next International Congress, for which it is proposed to issue a call in the near future, are as follows:

(1) The selection of the two units to be taken as fundamental, and in terms of which all the others will be derived.

(2) The adoption of specifications for the units selected.

(3) The adoption of numerical values in the definitions of the fundamental units.

(4) The definition of the remaining international electrical units in terms of the two taken as fundamental.

(5) The definition and naming of the international magnetic units.

(6) The discussion of international photometric standards.

Of these, the last two topics will not be discussed in this paper.

THE CHOICE OF FUNDAMENTAL UNITS.

As it is universally agreed that the ohm be taken as one of the two fundamental units, the question is reduced to the choice between the ampere and the volt, defined respectively in terms of the coulometer and standard cell, as the second unit.

In the opinion of the writer[4] it would be well, before a final decision is made, to consider fully the principles involved in the choice, which should, of course, be made on the basis of merit and without prejudice or undue deference to previous practice.

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[3] Verh. Int. Konf. Elek. Masseinh., p. 50.

[4] Wolff, loc. cit., pp. 50–59.

Arranged in the order of their importance, these are:

(1) Accuracy of Reproduction from Specifications.
(2) Concreteness.
(3) Ease of Reproduction.

1. ACCURACY OF REPRODUCTION.

The factors determining the accuracy of reproduction in the case of the coulometer and the standard cell may be stated to be: (*a*) The nature of the fundamental principle underlying the definition; (*b*) the purity of the materials employed; (*c*) the number of measurements involved in the realization of the definition.

(*a*) The preference heretofore generally given to the coulometer can be attributed in part to the fact that its indications are assumed to be exact, being based on Faraday's law, generally regarded as a fundamental law of nature. The more recent work[5] on the coulometer seems, however, to establish the conclusion that while Faraday's law may be rigorously exact, account must certainly be taken of the possible effect of secondary reactions accompanying the electrolysis, thus complicating the results as shown by the differences obtainable in the use of coulometers of different types. On the other hand Smith[6] obtained "identical values within one or two parts in 100,000 for the electrochemical equivalent of silver for six different types" and "no influence of pressure, temperature or current density." Dushak and Hulett[7] have, however, recently found for the porous cup type a difference of almost 0.01 per cent between the deposits formed in the presence and absence of air. The difference was further established by an analysis of the deposited silver, which on heating gave up water and gas. Dushak and Hulett state that "the idea of the included matter being other than trapped electrolyte finds support, first in the constancy of the quantity of impurity in deposits formed under the same conditions, and second, in its apparent uniform distribution throughout the silver crystals." The writer, while disclaiming any intention of discrediting the work of any of the investigators, simply desires to call especial attention to the discrepancies in their results, which indicate that either the same procedure has not been followed by all, or that unrecognized sources of variation exist. It can not, therefore, be regarded as established that the *increase of weight* in the coulometer is in strict accordance with Faraday's law.

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[5] Richards, Collins and Heimrod, Proc. Amer. Acad., **35**, p. 123; 1900. **37**, p. 415, 1902. Zs. Phys. Chem., **32**, p. 321; 1900. **41**, p. 302; 1902.
Guthe, Phys. Rev., **19**, p. 138; 1904. This Bulletin, **1**, p. 21 and p. 349; 1904–5.
Carhart, Willard and Henderson, Trans. Amer. Electrochem. Soc., **9**, p. 375; 1906.
Van Dijk, Ann. Phys., **19**, p. 249; 1906.

[6] Phil. Trans. Roy. Soc., **207**, 581; 1908.

[7] Dushak and Hulett, Trans. Amer. Electrochem. Soc., **12**, p. 257; 1908.

The principle underlying the definition of the volt is equally fundamental, though not capable of expression in as simple a form as Faraday's law, since electrode potentials are determined solely by the nature, concentration, and temperature of the materials involved. In reversible cells these are not changed by the passage of small currents in either direction.

The formula which expresses the relation and which was first deduced by Helmholtz from thermodynamical principles, may be put in the form

$$Q=Ct\left(E-T\frac{dE}{dT}\right)$$

in which Q represents the *total* energy of the reaction, including that set free or absorbed in solution, hydration, change of state, etc., E the electromotive force, T the absolute temperature and Ct the number of coulombs corresponding to the reaction. It follows, therefore, that if, for the particular reaction, there is a constant relation between the number of coulombs and the total energy per gram molecule, as well as a constant relation between the free and total energy, the electromotive force must have a definite value for given external conditions.

Since the first condition does not involve any assumptions as to the simplicity or complexity of the ions, or even as to the nature of electrolysis, and since the second must be granted as being based on the fundamental principles of thermodynamics, it will be seen that the electromotive force of a reversible voltaic combination follows a law quite as rigorous as that of Faraday. In case of the silver coulometer, secondary prdoucts, the electrolysis of which would modify the result, may be formed, but in the case of the standard cell no secondary reactions of this kind need be considered as the electromotive force of the cell is always compensated, so that there is only a negligible amount of electrolysis produced. Reactions may, however, take place between the ingredients in the cell which give rise to secondary products affecting the electromotive force, but as both Clark and Weston cells can be constructed which will maintain their values for considerable periods, such reactions are either absent or take place very slowly, so that they need not be considered in discussing the accuracy of reproduction. The question of constancy will be taken up under the next heading.

(*b*) The question of the purity of the materials employed is most important in both cases, with the advantage resting slightly with the silver coulometer, as the number of materials is less. On the one side it is necessary to start with pure silver and silver nitrate, while on the other hand pure cadmium and cadmium sulphate (or zinc and zinc sulphate), together with mercury and mercurous sulphate, are required. It will be generally admitted that the metals, silver, cadmium, zinc, and mercury can readily be prepared in the highest state of purity, as also the silver nitrate and zinc and cadmium sulphate. The recent work on the standard cell has demonstrated that mercurous sulphate of uniform electromotive properties can also be obtained by a number of different methods, so that it might be considered that there is little choice on the basis of the number of materials entering into the coulometer and cell. The question of the purity can, however, almost be eliminated if arrangements be made by the national standardizing laboratories to furnish the necessary materials, which can be prepared in considerable quantities by the best methods, so that this would not have to be done by each investigator

(*c*) The greater number of materials required in the cell is more than offset on the other hand by the number of measurements, each subject to possible error, which must be made in the case of the coulometer to realize the ampere from its definition. The cell automatically takes a value determined by the materials employed, while the results of a coulometer measurement depend on two washings and subsequent weighings made with the greatest quantitative care (particularly the last on account of the granular and loosely adherent character of the deposited silver), the regulation of the current during the run, as well as an accurate measurement of its duration.

The author wishes to lay particular stress on the fact that while standard cells set up by different individuals in different laboratories have been compared in order to establish the accuracy of reproduction, the corresponding step has not as yet been taken with reference to coulometer measurements, so that the results obtained by different investigators only establish relative reproducibility. The indirect comparison is now possible, owing to the progress made in the construction of standard cells and standard resistances which may be carried from place to place, and through which alone the results of coulometer measurements made in different localities can be accurately compared. Before making a final choice, the relative merits of the cell and coulometer with respect to accuracy of reproduction should, however, be definitely established by a systematic campaign of international cooperation involving the exchange of specifications and materials by the various national laboratories, each of which would undertake to make a careful series of coulometer measurements and to set up a considerable number of cells in accordance with the various specifications proposed. The value of the results obtainable can be further increased by the cooperation of the national bureaus with qualified individuals in the respective countries. All the results could subsequently be reduced to a common basis of comparison through the interchange of resistance standards and standard cells.

2. CONCRETENESS.

Another most important consideration is that the definitions of the units adopted as fundamental should both be capable of realization in the form of concrete standards. This makes possible the construction of standards which will maintain their values for considerable periods at least, and in terms of which measurements can, at all times, be made. It also makes possible the direct intercomparison of the concrete standards of the various laboratories by the exchange of standards designed for portability.

In this respect the standard cell has, perhaps, its greatest advantage over the silver coulometer, which merely gives the average value of a current which has been passed through it, and which has ceased to exist. The result of the measurement is, therefore, a silver deposit, and not a standard in terms of which other current measurements can be made, making it necessary to employ a standard resistance and a standard cell as secondary units to fix the results. Both the secondaries must be assumed to remain constant between coulometer measurements which are ordinarily made at long intervals. The more recent work has shown that both Clark and Weston cells can be set up which will maintain their values, within one part in 100,000, for considerable periods, and that when properly set up they will assume their normal values immediately. It is true, however, that some Weston cells have been under observation which have decreased in voltage with time. Hulett attributes this to an unstable equilibrium in the paste limb, but this can not be regarded as established, as the effect may possibly be due to imperfect washing of the mercurous sulphate prepared in strongly acid solutions, or to traces of impurities in the materials employed. A number of cells with abnormal values have been under observation at the Bureau for some time, and were included with others in a redetermination of the temperature coefficient, in the course of which it was found that they exhibited enormous hysteresis. As the mercurous sulphate was originally present in considerable excess, it seems that the retardation of the establishment of concentration equilibrium may be due to the formation of a difficultly soluble surface film. It is hoped that an analysis of the materials, which will shortly be undertaken, will definitely establish the cause of the abnormal behavior. Even though the Weston cell should be found to represent a system in unstable equilibrium, and should be rejected, the Clark cell, in which no abnormal behavior has thus far been observed, would still be available as the concrete standard of electro-motive force. The cracking at the amalgam limb can be overcome by suitable construction, and the objections on account of the high temperature coefficient can be practically eliminated by the employment of thermostats. With the attributes of *constancy* as well as *reproducibility* established, the standard cell would have greater merit than the mercury ohm, since the mean value of a number of fillings is always taken in order to eliminate errors of filling, and since in the interval between mercury ohm measurements the results are usually carried by manganin copies and not by the primary standards.

3. EASE OF REPRODUCTION.

In respect to ease of reproduction the standard cell again has the advantage. In the first place the facilities required for a coulometer measurement, consisting of a precision balance, calibrated weights, platinum bowls, accurate time service, and means for accurately measuring the duration of a run, together with a supply of purified silver and silver nitrate, are to be compared with the necessary cell blanks, and a supply of specially prepared or purified cell materials. For results of the highest precision the balance must, in addition, be especially protected from temperature changes, and provision must be made for controlling the humidity in the balance case.

In the case of the coulometer the operations consist in washing, drying, and weighing the cathode dish before and after receiving its deposit, in maintaining the current at a constant value during the run, and in the measurement of the duration of the run, all of which operations and measurements must be performed with the highest quantitative care, while the operations involved in setting up a standard cell require only care in washing the mercurous sulphate, since relatively large variations in the composition of the amalgam only slightly affect the electromotive force.

Although the time required for setting up a few cells might be greater than that required for a corresponding number of coulometer measurements, if the time for purifying the materials were included, it must be remembered the latter could be prepared in considerable quantity and preserved under proper conditions for future use. This could also be undertaken by the national laboratories from which investigators could obtain them from time to time. The main conclusion under this head is, therefore, that while in the case of the coulometer special facilities are required and numerous *quantitative* measurements and manipulations must be made in order to make a single current measurement, the manipulations required in setting up a standard cell are all *qualitative* and less in number, so that, without special facilities and in the hands of individuals of equal skill, a higher accuracy or reproduction should be obtained.

The question of ease of reproduction, while not of the highest importance, should nevertheless be considered from the standpoint of all who deal with electrical measurements of precision and who should not be compelled to depend any more than necessary on the national laboratory. It has, however, been held that facility of reproduction is no great advantage, since similar experimental difficulties result from the definition of the ohm in terms of the mercury column. With the ampere taken as the second fundamental unit the construction of mercury ohms and coulometer measurements would infrequently be undertaken, mainly by the national laboratories. In this case it is necessary to resort to wire copies of the mercury ohm on the one hand and standard cells on the other, each regarded as secondary.[8]

The case appears to be, however, very different, as in the definition of unit resistance in terms of the mercury column we are left without any satisfactory alternative, so that we are forced to resort to secondary standards, multiples and submultiples of which can readily be constructed and compared. On the other hand, we might select either the coulometer or the standard cell as the second fundamental unit. In the first case the result can only be fixed by the aid of some current measuring device, the best of which have a limited accuracy, or, as is usually done, by the combination of a standard resistance and a standard cell, the latter being taken as the secondary standard. In the second case the cell is taken directly as a fundamental standard. It will, therefore, be seen that the standard cell, which itself satisfies the requirements of a fundamental standard as well as the mercury ohm, should not be regarded in the same light as wire copies of the latter which can not be reproduced from specifications.

The question of choice should also be regarded from the standpoint of the national standardizing institution. As the selection of the ohm and ampere necessitates the adoption of the standard cell as a secondary unit to fix results in the interval between coulometer measurements, the value to be taken for the cell will involve the errors introduced in the reproduction of the mercury ohm as well as those involved in the coulometer measurements, so that unless these together are less than the differences met with from time to time in setting up standard cells directly from the specifications or from materials drawn from the same source, the question would certainly arise as to whether the value to be adopted for the cell should be changed to correspond with each reference to the mercury ohm and coulometer, or whether weight should be given to previous results in the face of a possibly more accurate agreement between old cells and cells freshly set up. In this connection attention should be called to the practice in Germany, where the ohm and ampere are taken as fundamental units. In 1896 and 1898 two series of coulometer measurements were made to determine the values of the Clark and Weston standard cells. The results,[9] which differed by over 0.02 per cent from the directly determined ratio, were adjusted and the values of the Clark and Weston cells thus derived have been in use ever since in Germany. In effect the standard cell has been employed as a fundamental standard, since, by ignoring the definition of the ampere, the result obtained is equivalent to legalizng the values found for the standard cells.

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[8] Mitth. Reichsanstalt, E. T. Z. **25**, 671; 1904.

Further light is thrown on this phase of the question by the practice at the Reichsanstalt in reference to the unit of resistance; four manganin standards have been referred, from time to time, to the mercury units of that institution,[10] but since 1898 the mean value of the four coils has been assumed constant, as the apparent changes of the mean were found to be within the limits of accuracy with which the mercury ohms could be reproduced. It is therefore confidently predicted that the standard cell, which must be depended on between coulometer measurements, will, even if the ohm and ampere are taken as fundamental units, in effect replace the coulometer, unless the accuracy attainable through the ohm and ampere is found to exceed the accuracy with which the cell can be reproduced.

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[9] Kahle, Zs. Instkn. **18**, p. 229 and p. 267; 1898.
Wied. Ann., **67**, p. 1; 1899.

[10] Ber. Int. Konf., 1905, p. 63.
Zs. Instkn., **26**, p. 15; 1906

This is a point of considerable importance from the standpoint of the standardizing institution, on account of the fact that standard cells and not coulometers are submitted for certification, making it necessary to adopt values for the reference cells in terms of which results are expressed.

Moreover, since in actual practice, instantaneous values of currents and voltages of any magnitude are measured with the greatest ease and accuracy directly in terms of resistance standards and standard cells, the choice of the ohm and volt as fundamental units is the more logical.

SPECIFICATIONS.

In reference to the adoption of specifications for the units chosen, attention has already been called to the opinion expressed at the Charlottenburg Conference to the effect that the information before it was not regarded as sufficient to enable it to lay down exact directions with respect to either the silver voltameter or the standard cell. Considerable work has since been done along both lines, but no systematic comparison has been made of the specifications proposed by the various investigators. As the principal object of the next congress will be to redefine the international units in terms of concrete standards reproducible to the highest attainable accuracy from specifications, the latter must be drawn with the greatest care. The various specifications proposed, which differ in regard to details, should therefore be carefully subjected to comparison by numerous investigators, so that any unrecognized defects might be brought to light. It is felt that this would be one of the important objects of the international cooperation above proposed, since, by the aid of the national laboratories, all results could be reduced to the same basis of comparison, and differences which might be found and which might furnish a clue to defects in the specifications could be further investigated.

VALUES TO BE ADOPTED FOR THE UNITS.

In the first place it is necessary to recognize that the C. G. S. system, while giving us *ideally* perfect definitions of the electrical units, fails in practice owing to the necessarily limited accuracy of the absolute measurements in comparison with the precision attainable in relative measurements. The desirability of referring results to concrete standards was recognized by the British Association in the proposal of the B. A. unit of resistance, intended and for some time supposed to represent 10^9 C. G. S. electro-magnetic units. With the advances made in the construction of the mercurial resistance standard, the first Paris Congress recommended its adoption for defining the unit of resistance, notwithstanding the great improvements made up to that time in absolute measurements, the choice being made on the basis of accuracy of reproduction of the results. This was also the principle guiding the Chicago Congress in substituting for the mean cross section of the mercury column, the mass of mercury corresponding to a stated length, following the already established practice of employing the liter instead of the cubic decimeter for volumetric work. As all resistance measurements are now referred to the mercurial unit without reference to its relation to the absolute ohm, this should also be done with the second fundamental unit. In other words, the absolute and international units should be recognized as separate and distinct, so that in the definition of the latter no reference need be made to the former. The particular values adopted in the definitions of the international units should, however, be selected to secure approximate agreement with the absolute units which they will replace for all practical as well as scientific purposes. The few measurements which it is occasionally necessary to express in absolute units could be reduced to the same by applying a correction factor obtained from the results of the various absolute measurements, all in turn fixed by being expressed in terms of the reproducible concrete standards employed to define the international units.

As the values adopted by the next conference should be allowed to stand for at least twenty-five years, to limit the confusion naturally involved in any change of the basis of reference, it seems desirable before making a decision as to the numerical values to await the results of further absolute measurements, a number of which are already under way, since with the great advances made within the last ten years in absolute measurements it might confidently be expected to keep the errors within 0.01 per cent, thus making the difference between the absolute and international units practically negligible. With this degree of approximation the numerical values employed in the definitions could be allowed to stand indefinitely until some better method of defining the units is recognized.

The writer expressed himself on this subject at the St. Louis Congress, and found opposition to the plan, particularly on the part of some of the foreign representatives, who strongly advocated adherence to the absolute units, though recognizing the necessity of concrete standards the values of which it was proposed to alter from time to time to bring them into better accord with the latest absolute measurements. The objection of confusion as to the basis of reference in the literature has already been pointed out above. In addition, unless the question were regulated by an international commission, each national laboratory would be inclined to give greater weight to its own absolute measurements, thus defeating the aim of international uniformity. The better way would be to consider the international units as fixed for a definite period, and the relation between these and the corresponding absolute units could be reported on from time to time by an international commission in the light of all the data before it.

This should dispose of the argument in favor of the coulometer, to which considerable weight seems to have been attached, that a current can be directly measured in terms of the absolute units, whereas the unit of electromotive force must be derived from the absolute measurement of resistance and current. The definition of the ampere in terms of some specific value taken as the electro-chemical equivalent of silver would, with the unit of resistance fixed, result in fixing the value of the standard cell within the limits of error involved. The numerical value for the cell would, however, be exactly the same if derived from the same absolute current and resistance measurements on which the definitions of the international ohm and ampere are based.

It has also been claimed that the choice of the volt as a second fundamental unit would result in the ohm entering twice into the derivation of unit current. It is, however, evident that this is not the case so long as the unit of resistance, from which the value of the cell is initially determined, remains constant. This makes the question raised an academic one, since in practice all current measurements are directly referred to a standard cell and a standard resistance.

DERIVED UNITS.

It will be generally agreed that the remaining units, capacity, inductance, magnetic flux, power, energy, etc., should all be defined in terms of the definitions adopted for the fundamental units, notwithstanding that they are now defined in terms of the C. G. S. system, as a distinction can be made by designating the units thus defined as international. If the values adopted for the international units agree with the absolute units, upon which they are based, to within even one part in 2000, as will certainly be the case, no serious objections can be made, as the replacement of the absolute units will be fully justified by the greater accuracy of reproduction made possible.

SUMMARY OF CONCLUSIONS.

(1) The decisions of the Berlin Conference of 1905 should not be regarded as binding in view of the large amount of work since published on the silver coulometer and standard cell.

(2) The international electrical units to be adopted should be regarded as separate and distinct from the absolute units on which they were originally based.

(3) The choice between the ampere and the volt should be made exclusively on the basis of merit.

(4) The principles on which the decision should be made are—

- (*a*) Accuracy of reproduction from specifications.
- (*b*) Concreteness.
- (*c*) Ease of reproduction.

(5) The accuracy of reproduction of the coulometer remains to be established, as coulometer measurements made by different

investigators establish only relative accuracy, no accurate comparisons of the results obtained by different investigators having thus far been made.

(6) If a decision must be made at the present time and on the basis of merit, the cell should be selected on account of the advantages it offers from the standpoint of concreteness and ease of reproduction, since no greater accuracy of reproduction can be claimed for the coulometer in the light of the data at hand. In addition this choice is more logical from the standpoint of the standardizing laboratory, as it leads to fundamental units corresponding to the standards submitted for certification as well as corresponding to the practice of measuring currents in terms of standards of resistance and electromotive force.

(7) In view of the preference indicated for the coulometer at the Berlin Conference, the reproducibility of the cell and coulometer should, however, first be definitely established by systematic international cooperation, which would also result in bringing to light unrecognized defects in the specifications proposed.

(8) If the ampere is defined in terms of the coulometer the value to be taken for the standard cell, which must be depended on between coulometer measurements, must be determined at each of the national standardizing institutions, the results obtained reduced to a common basis by exchange of cells and resistance standards, and a mean value taken to insure international uniformity.

Unless the accuracy of the result, involving errors in the realization of both the ohm and ampere, is greater than that attainable in the reproduction of the cell from specifications the definition of the ampere will subsequently be ignored, making the cell in effect a primary standard. The same result can be more directly obtained by selecting the volt as the second fundamental unit, since a practically equivalent value may be assigned to the cell, in which case the corresponding value for the electro-chemical equivalent of silver need not be known to the highest accuracy, as it could only be used for the purpose of loosely defining the ampere.

WASHINGTON, September 3, 1908.

THE LUMINOUS EQUIVALENT OF RADIATION.

By P. G. Nutting.

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I. INTRODUCTION.

1. DISCUSSION OF THE GENERAL PROBLEM.

Light has yet to be accurately defined in precise mathematical terms. The word is applied sometimes to the subjective sensation, sometimes to the objective stimulus causing or capable of causing the sensation. While the precise definition of light, like the definition of heat or of temperature, is more or less arbitrary, the relation of light to radiation involves primarily the precisely definable concept of the luminous equivalent of radiation. The general problem is to determine how much light and of what quality is represented by a given amount of radiation having a given spectral distribution.

But, while the heat equivalent of mechanical energy is a fixed quantity, the luminous equivalent of radiation varies enormously with the wave length of the radiation concerned and to a slight extent with its intensity as well. The principle of the conservation of energy does not apply to light, either objective or subjective. Light is not a simple multiple of radiation, but a complicated function of its quality, quantity, and duration. This article is devoted to the formulation of these complex relations.

The general problem of the luminous equivalent of radiation has been within the range of investigators for decades if not for centuries, but has been as a whole practically untouched, partly because it does not lend itself to treatment by infinitesimal analysis based on the conservation of energy, but largely no doubt because of its variability and extra-physical nature. Within the last few years many special problems in photometry, pyrometry, and spectroscopy have arisen which insistently demand the establishment of some definite relation between light and radiation.

Data are available of sufficient accuracy and diversity to determine the character of the functions involved and to determine their form to within a slight uncertainty, but not to fix the values of their parameters. This paper deals chiefly with the essential interrelations between functions rather than with their precise determination. If very exact data, constant for different individuals at all times, were available, how might that data be used in determining the luminous equivalent of radiation? That is the problem dealt with in this paper.

Visual sensibility of various kinds is subject to considerable variations with the individual, with fatigue, attention, and expectation, so that special problems will often require its redetermination by the individual at the time he is working. Hence the value of a foreknowledge of short cuts and pitfalls in this field. On the other hand, many general problems in illumination and luminous efficiency require the specification of the mean properties of the average normal human eye. Existing data are sufficient for outlining such specifications.

Radiation, as now known and studied, includes a wide diversity of wave lengths, ranging from kilometers to ten-millionths of a meter. It is interpreted by many different classes of instruments—electric wave resonators, radiometers of various types, and by the eye and the photographic plate; some are absorbers, some resonators, and others act by mechanical pressure. None of these instruments is equally sensitive over the whole range of wave lengths; thermal receivers have the widest range, the eye and electric-wave resonators the narrowest. Some give indications very nearly proportional to the intensity of the radiation, while others depart widely from proportionality. Resonating receivers are very sensitive to differences in wave length, while thermal. mechanical. and chemical receivers are not.

Light is the interpretation given to radiation by the eye, just as the electric current, mechanical pressure, heat, and chemical effects are interpretations by other forms or radiometers. In any case, the interpretation of radiation by any kind of radiometer depends upon the properties of that form of radiometer as a physical instrument. The amount of light in a given quantity of radiation of a given quality is determined by the sensibility of the eye, sensibility to the same amounts of radiation of different wave lengths, and sensibility to radiation of constant wave length but varying intensities.

In other words, in order to express light in terms of mechanical units or radiation in terms of light units, three relations must be known in mathematical form: (1) The spectral distribution of the radiation in question, (2) the sensibility of the eye to radiation in different spectral regions, and (3) the sensibility of the eye to variations in the intensity of the radiation; that is, if $E(\lambda)$ is the radiation E expressed as a function of wave length λ, then $V(\lambda)$ is a function that converts this radiation $E(\lambda)$ into light $L(\lambda)$, or, in other words, that weights the radiation according to its visibility. Then, having obtained $L(\lambda)$, the sensation of visual brightness produced by the radiation E or the light L is some function $B(L)$ of the light L. The chief objective points of this investigation are the determination of V and B and the correlation of E, L, and B.

2. OUTLINE OF TREATMENT.

Given the sensibility of a physical instrument at all parts of its scale or for all values of the exciting stimulus, the scale reading corresponding to each value of the stimulus may be determined. Hence, by the aid of the general theory of sensibility of physical instruments, light may be expressed in terms of radiation when the sensibility of the eye is known.

Another method of attacking the problem is that of synthetic function theory. The sensation of visual brightness B is a function $B(E,\lambda,t)$ of the intensity of radiation $E(\lambda)$, the wave length λ and the duration of the impression t. By varying λ alone or t alone or E alone, three separate functions of these single variables are constructed and then these three functions may be combined to form the complete function.

A third method is that of setting up an arbitrary rational light scale and determining its relation to the radiation scale of watts per unit wave length. It is necessary to identify the zero of each scale on the other, to determine the relative magnitudes of scale divisions and ultimate values approached on each scale. Because the luminous sensation is not proportional to illumination (Purkinje phenomenon), over a considerable range, it is necessary to construct an auxiliary intermediate scale. In all, three scales, each in two dimensions, are involved.

Available data may be thus summarized: $E(\lambda)$ for B constant and $P(L)$ for λ constant. In other words, the relative amounts of radiation of different wave lengths necessary to produce the same sensation of brightness and the least perceptible increment P as a function of intensity at constant wave length. Besides these there are $E_0(\lambda)$, the threshold value or least perceptible intensity measured as radiation, and $L_0(\lambda)$, the same quantity measured as light. To determine the time function there are rough direct determinations of $B(t)$ and more precise measurements of the critical frequency for no flicker, each for various wave lengths. From these sets of data the general function $B(E(\lambda), \lambda, t)$ is to be constructed. Of related interest but of little assistance are some careful measurements of $\delta\lambda(\lambda)$, the least perceptible increment of wave length as a function of wave length.

3. SUMMARY OF THE PROPERTIES OF THE EYE AS A PHYSICAL INSTRUMENT.

The more important characteristics of the visual response to radiation may be thus summarized:

a. **Sensibility to Radiation of Various Wave Lengths.**—(*1*) The eye responds to radiation between ill-defined limits at about 300 and 1000 $\mu\mu$. Its sensibility is highest between 500 and 600 $\mu\mu$. Good seeing requires radiation between 410 and 760 $\mu\mu$. Radiation is easily visible to most eyes out as far as 330 $\mu\mu$ in the violet and 770 $\mu\mu$ in the red.

(*2*) The spectral sensibility curve has a single maximum in the green, slopes off very steeply at first and then very gradually toward the extreme wave lengths.

(*3*) At low intensities the maximum of the curve lies between 500 and 520 $\mu\mu$ for perhaps 90 per cent of all persons. It is approximately symmetrical and nearly or quite independent of color blindness, partial or complete. It is coincident with the reciprocal of the threshold value of the radiation if reduced to the same maximum ordinate.

(*4*) At moderate and high intensities the maximum of the visibility curve broadens and shifts slightly toward the yellow, varying considerably with color blindness in the subject.

b. **Sensibility to Radiation of Varying Intensity.**—Sensibility falls off steadily with increasing intensity. It is approximately inversely proportional to the intensity over a wide range. The ratio of optical intensity to intensity of radiation increases more rapidly for red than for blue and green. (Purkinje phenomenon.)

c. **Sensibility to Small Differences in Intensity.**—The least perceptible increment, measured as a fraction of the whole, is approximately—

(*1*) Independent of intensity (Fechner's Law). It is about 0.016 for moderate and high intensities and greater for very low and extremely high intensities.

(*2*) Independent of wave length (König's Law), at constant luminosity—extremes again excepted.

(*3*) Independent of the individual.

d. **Sensibility to Slight Differences in Wave Length** has two pronounced maxima, one in the yellow and one in the green, and two slight maxima in the extreme blue and red. These maxima vary considerably with the individual and probably also with the intensity of the radiation used.

e. **Visual acuity** or resolving power, so far as studied, appears to follow the same laws as does sensibility to small intensity differences, (3) namely, it is approximately proportional to the luminosity and independent of color and of the individual.

f. The **growth and decay** of the visual responses **with time**, so far as studied, appear to follow the ordinary exponential law. The parameters of the time functions vary with wave length and intensity. A steady impression is the resultant of a pure reception and a fatigue.

4. GENERAL THEORY OF SENSIBILITY OF PHYSICAL INSTRUMENTS.

A physical instrument, responsive to stimulus, indicates in general the quality, intensity, duration, or extension of that stimulus, and in some cases two or more of these properties at once. Simple instruments like chronometers and cathetometers indicate but a single property; most psycho-physical receivers indicate all four simultaneously. An ideal physical instrument is entirely free from subjective variations (pressure, temperature, attention, experience, fatigue, etc.) affecting its constants.

Let the quality, intensity, duration, and extension of any stimulus be measured in the variables q, i, t, v, respectively; let S_q, S_i, S_t, and S_v be the sensibilities of the instrument for these four properties of the stimulus and let R be the response or scale reading indicated by the instrument when subject to stimulus of quality q, intensity i, for a time t or over a space v, as the case may be. Then these nine variables are related by the defining equations:

$$S_q = K_q \left[\frac{R\,(q,\,i)}{i\,(q)} \right]_q^{q+dq} \quad \text{hence } R\,(q,\,i) = \frac{S_q}{K_q}\, i\,(q)$$

$$S_i = K_i\, \frac{\partial R}{\partial i} \qquad \text{''} \quad R\,(i) = \frac{1}{K_i} \int S\,(i)\, di + \text{const.}$$

$$S_t = K_t\, \frac{\partial R}{\partial t} \qquad \text{''} \quad R\,(t) = \frac{1}{K_t} \int S\,(t)\, dt + \text{const.}$$

$$S_v = K_v \frac{\partial R}{\partial v} \qquad \text{hence } R(v) = \frac{1}{K_v}\int S(v)\,dv + \text{const.}$$

K is a constant of proportionality dependent upon the unit of sensibility or scale magnification (or both) chosen. The integration constants depend upon the choice of scale zero.

In the special case of a uniform scale, S is constant and hence R is a linear function of the stimulus. When S is a linear function of R, R is a quadratic function of the stimulus. If a scale has been calibrated and is free from corrections, K is unity, S is unity, and the integration constant zero.

In general i is a function of q, otherwise all these relations would be exceedingly simple. Either S or R is determined experimentally as a function of one variable at a time and the other calculated. Sensibility to quality S_q is not to be confused with sensibility to differences in quality, its derivative. S_i is inversely proportional to the least perceptible difference in intensity. An instrument usually does not begin to respond until the stimulus has attained a finite value (i_0) called the threshold value, which is a function of q. The least perceptible increment is also related to the uncertainty of the readings due to subjective errors, but the experimental determination of S_i from the probability curve of errors is too laborious to be of much service except as a last resort.

Evidently S_q and S_i can be correctly determined only by null methods. In finding S_q, for instance, the scale reading R is held constant, while i and q are varied. If i were held constant and R varied, it would be impossible to distinguish between the two variations $R(q)$ and $R(i)$; much data on visual sensibility is worthless for this very reason.

5. GENERAL METHODS OF SYNTHETIC FUNCTION THEORY.

Instead of treating the eye as a physical instrument to find the relation of light to radiation, the necessary functions may be constructed directly from observed data. The problem is simpler in plan but more difficult to carry out by the methods of synthetic function theory. The magnitude (B) of the visual sensation is a function of the three objective variables, wave length (λ), spectral energy $E(\lambda)$, and time of action (t) of the radiation. We are required to find B (λ, E (λ), t) not by approximation to numerical data but from the general properties (as regards maxima, minima, and singular points) of the functions. This function B (λ, E, t), like a potential function, is useful not so much for itself as a whole as for its derivatives, invariants, and subsidiary functions.

The general method of constructing functions is as follows: Given that the function required is finite and single valued (as nearly all physical functions are), for all finite positive values of the argument, we set up the most general function possessing these properties. Given further that the function has no finite maxima or a single maximum or a given number of given maxima, the general function is limited in accordance with this condition. Then if the function be known to approach certain limiting values for very high or very low values of the argument, the general function may be still further restricted in form. Finally, if the properties of the required function be sufficiently well known, there remains but to determine its parameters from numerical data in a limited region.

At any stage of construction the simplest possible instead of the most general function may be chosen as a special working basis, much as a particular solution of a differential equation may be chosen as a working solution. But it must be borne in mind that in passing from the general to the special forms we are passing from necessary to sufficient functions.

Such a process finally gives us a function satisfying certain general conditions and fitting numerical data to within experimental uncertainty. Unlike a mere power series fitted to numerical data, such a function may be used with care even outside the range of the experimental data which it fits and may even be integrated and differentiated without introducing any large additional uncertainty.

Consider two functions of x, $A(x)$ and $B(x)$ which differ by no more than a certain small amount within a given region, from $x=x_1$ to $x=x_2$ say, then the definite and indefinite integrals as well as the derivatives of A and B will differ by no more than a small quantity of the same order within that same region. Hence, even when the function constructed is merely an approximation to the actual physical relation sought, it may be used in all ordinary mathematical operations without introducing any large additional uncertainty. The chief liability to serious error lies in the interpretation of parameters as functions of other arguments and operations on parameters in general.

After separate functions have been constructed for each argument (variable), these may if desired be combined in any manner such that the form of none of the subsidiary functions is affected in form by the combination. Functions of related arguments are usually combined as a function of some combination of these arguments, while functions of independent arguments are multiplied or added together.

II. OBJECTIVE LIGHT.

6. CHROMATIC SENSIBILITY AT LOW INTENSITY.

Chromatic sensibility or visibility is a function of wave length which fixes the relation of radiation to objective light; that is, to light measured by photometric or other zero methods. The spectral light curve is the product (EV) of the spectral energy $E(\lambda)$ and the visibility $V(\lambda)$ at each wave length. Its dimensions are light units per watt, hence its factorial parameter is determinable only by experiment. At low intensities (below 0.1 m-c.) vision appears to be accomplished by the retinal rods alone and to be achromatic, while chromatic vision at higher intensities brings into play the less sensitive but more efficient retinal cones. The two functions overlap widely so that the transition from rod to cone vision is very gradual; it is about half and half at ordinary working intensities, 50 meter-candles.

Visibility is determined by measuring the relative spectral energy necessary to produce a given luminous sensation throughout the visible spectrum. The reciprocal of this energy is then taken as proportional to the relative visibility. The constant luminous sensation used may range from the threshold of vision up.

Sufficient data is available for at least roughing out the visibility curve at low intensity for the individual and for the average eye. The best data is by König,[1] while that by Pflüger[2] and Langley[3] is useful for the determination of constants. Other data is useless; some because the energy of the source used was not recorded, some because of defective methods.

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[1] König and Dieterici, Zs. Psy. Phys. Sinnesorgane, **4**, pp. 241–347; 1893.

[2] A. Pflüger, Ann. Ph. **9**, p. 185; 1902.

[3] S. P. Langley, Am. Jour. Sc., **36**, p. 359; 1888.

König's data on visibility at low intensity is reproduced below. It relates to nine individuals, of whom three were totally color blind (monochromats); two were partially color blind (dichromats), and four of normal (trichromatic) vision. As indicated, some of the curves were obtained by stepping off a luminosity curve at low intensity, while others were obtained as reciprocals of threshold

König's Data on Visibility at Low Intensity.

Subject	Type	Method	λ=430	450	470	490
A. Beyssell	Monoch.	Step	0.015	0.074	0.27	0.57
E. Hering	“	“	0.10	(0.27) 0.30	(0.67) 0.64	0.90
F. C. Donders	“	“	0.038	(0.18) 0.14	0.31	0.58
R. Ritter	Red Bl.	“	0.062	0.25	0.50	0.74
“	“	Threshold	0.059	0.26	(0.50) 0.52	(0.75) 0.85
E. Brodhun	Gr. Bl.	Step	0.064	0.21	0.47	0.80
A. König	Normal	“	0.059	0.21	0.49	0.60
“	“	Threshold	0.047	0.23	0.50	0.86
E. Köttgen	“	Step	0.069	0.23	0.49	0.69
F. Hillebrand	“	“	0.31	0.47	0.74	(0.93) 0.96
Pereles	“	“	0.089	0.31	(0.73) 0.58	(0.77) 0.84

Subject	505	520	535	555	575	590	605	λ max
A. Beyssell	....	1.00	0.78	0.45	0.20	0.085	0.035	518
E. Hering	1.00	(0.98) 0.96	0.77	0.47	0.20	0.076	....	507
F. C. Donders	0.85	1.00	0.85	0.49	0.25	0.15	0.09	521
R. Ritter	1.00	0.85	0.64	0.33	0.12	0.051	0.022	
“	1.00	0.83	0.60	0.33	0.12	0.033	0.019	504
E. Brodhun	(1.00) 0.98	(0.98) 0.97	0.74	0.41	0.17	0.067	0.026	512
A. König	1.00	0.99	0.77	0.42	0.17	0.061	0.022	
“	1.00	0.98	0.75	0.36	0.12	0.045	0.015	511
E. Köttgen	1.00	0.87	0.66	0.38	0.14	0.071	0.024	506
F. Hillebrand	1.00	0.90	0.71	0.44	0.21	0.10	0.048	502
Pereles	0.99	1.00	0.81	0.47	0.13	....	....	514

values. All are reduced to the same maximum ordinate unity. All values are means of several determinations. Values lying off from the smooth mean curve are inclosed in parentheses, these values being largely in the blue and being due apparently to a slight absorption in the eye media and to larger errors of measurement in that region. The values given below the numbers in parentheses as well as values of the maximum were read from carefully plotted curves and are used in subsequent work.

Pflüger measured threshold values directly with a Rubens thermopile. The data given below is from smoothed means of recip rocals of these threshold values. As his investigation was chiefly of variations due to fatigue and ill health and his curves diverge widely from one another, not much weight can be attached to them. Still his values add weight to König's and supply additional data on the position of the maximum.

Pflüger's Data on Visibility.

λ =	420	440	460	480	500	520	540	560	580	λ max
La	0.03	0.13	0.29	0.52	0.79	1.00	0.61	0.19	0.07	517
Ha	0.03	0.12	0.24	0.48	0.81	0.99	0.62	0.26	0.06	516
G	0.03	0.13	0.32	0.63	0.98	0.79	0.34	0.12	0.02	504
Cl	0.04	0.10	0.23	0.51	0.94	0.93	0.58	0.27	0.10	509
M	0.08	0.25	0.50	0.83	0.99	0.80	0.43	0.20	0.07	501
K		0.09	0.21	0.53	0.98	0.80	0.42	0.18		504
A	0.08	0.21	0.45	0.78	1.00	0.68	0.31	0.15	0.06	500
S	0.05	0.17	0.48	0.87	0.99	0.81	0.34	0.12	0.06	499
E	0.04	0.22	0.52	0.80	0.98	0.87	0.54	0.21	0.05	505

Langley's bolometric measurements of the amounts of energy of various wave lengths necessary for making print legible are given below. The values given are means of widely divergent data, and as these are for wave lengths 50 $\mu\mu$ apart they are of no great value in determining the exact form of the visibility curve. Values reduced to unit maximum ordinate are given below Langley's values.

The chief value of Langley's data lies in the fact that it is in absolute measure. The number 5.79 for F. W. V. at 550 represents, he says, the reciprocal of 360 x 10^6 ergs acting for $\frac{1}{2}$ second. Hence the unit of visibility employed in the above data is roughly

2.4×10^{-10} light units per watt. Langley's own eye is seen to possess but about one-tenth the maximum sensibility of the other eyes investigated (due evidently to a deficiency of retinal rods), so that we are not surprised to find this maximum shifted over to 563 $\mu\mu$ in the region where it ordinarily lies at moderate and high intensities.

Langley's Data on Visibility.

$\lambda=$	400	450	500	550	600	650	700	λ max
S.P.L.		0.042	0.194	0.706	0.475	0.073	0.004	
" **max = 1**		0.056	0.26	0.94	0.63	0.097	0.008	563
F.W.V.	0.104	1.50	7.90	5.79	0.551	0.036	0.005	
" **max = 1**	0.011	0.16	0.84	0.62	0.06	0.004	0.0005	516
B.E.L.	0.139	3.75	10.10	6.31	1.17	0.089	0.009	
" **max = 1**	0.013	0.35	0.96	0.60	0.11	0.008	0.0008	510
D.M.	0.140	2.85	4.73	4.04	1.14	0.084	0.023	
" **max = 1**	0.03	0.59	0.97	0.83	0.23	0.02	0.005	513

Position of Maximum at Low Intensity.—The mean position of the maximum from the foregoing data is as follows:

Data		Weight
König's data	510.8	5
Pflüger's data	506.1	2
Langley's ex (S. P. L.)	513.0	1
Weighted mean of all data	509.7	

Pflüger's data has been assigned less weight than König's on account of its wide diversity, Langley's on account of its representing fewer subjects and on account of the uncertainty in the position of the maximum due to the few points of the curve given. Langley's own value (563 $\mu\mu$) has been excluded for reasons stated in the preceding paragraph.

We may accept, then, for the present 510 $\mu\mu$ as the position of the maximum of sensibility of the average normal human eye at low intensities approaching the threshold value, with a probable error of not more than 1 or 2 $\mu\mu$. Maxima for different individuals lie usually within 10 $\mu\mu$ of this mean. At higher intensities, with the establishment of cone vision, the maximum shifts toward the yellow at least as far as 560 $\mu\mu$.

Form of the Visibility Curve.—To determine roughly the form of the visibility curve at low intensities, König's data has been carefully plotted and then replotted to a common maximum ordinate. Of the two sets of data given for Ritter and König, those obtained by the threshold value method have been chosen as not being subject to correction for the spectral intensity of the source.

Reduced Data on Visibility at Low Intensity (König).

$\lambda-\lambda_m=$	−100	−80	−60	−40	−20	0.0	+20	+40	+60	+80	+100
Beyssell	0.005	0.030	0.14	0.38	0.72	1.00	0.72	0.39	0.17	0.045	0.008
Hering		0.09	0.28	0.61	0.89	1.00	0.88	0.57	0.29	0.09	
Donders	0.015	0.09	0.22	0.44	0.77	1.00	0.74	0.41	0.20	0.10	0.055
Ritter		0.025	0.18	0.44	0.76	1.00	0.77	0.48	0.22	0.055	0.015
Brodhun		0.075	0.22	0.49	0.83	1.00	0.81	0.47	0.19	0.055	0.02
König		0.055	0.23	0.52	0.85	1.00	0.82	0.44	0.15	0.035	0.01
Köttgen		0.04	0.19	0.45	0.76	1.00	0.80	0.51	0.23	0.08	0.025
Hillebrand			0.38	0.59	0.88	1.00	0.89	0.64	0.36	0.16	0.05
Pereles		0.07	0.24	0.50	0.87	1.00	0.85	0.48	0.16		
Means	0.02	0.06	0.23	0.49	0.81	1.00	0.81	0.49	0.22	0.077	0.026

7. THE VISIBILITY FUNCTION. LUMINOSITY OF RADIATION.

This data is graphically reproduced in the accompanying figure. Calculated curves of the form

$$V=V_0e^{-\kappa(\lambda-\lambda_m)^2}$$

with $\kappa=4$ and $\kappa=5$ have been plotted in the figure in dotted lines.

Numerical values are given below.

$\lambda-\lambda_m=$	−100	−80	−60	−40	−20	max	+20	+40	+60	+80	−100
Observed data (mean)	0.03	0.06	0.23	0.49	0.81	1.00	0.81	0.49	0.22	0.08	0.03
Probability, $\kappa=4.0$	0.02	0.08	0.24	0.53	0.85	1.00	0.85	0.53	0.24	0.08	0.02
" $\kappa=4.5$	0.01	0.06	0.20	0.49	0.83	1.00	0.83	0.49	0.20	0.06	0.01
" $\kappa=5.0$	0.01	0.04	0.16	0.45	0.82	1.00	0.82	0.45	0.16	0.04	0.01
Goldhammer, $n=235$	0.01	0.03	0.16	0.46	0.83	1.00	0.86	0.54	0.27	0.11	0.03

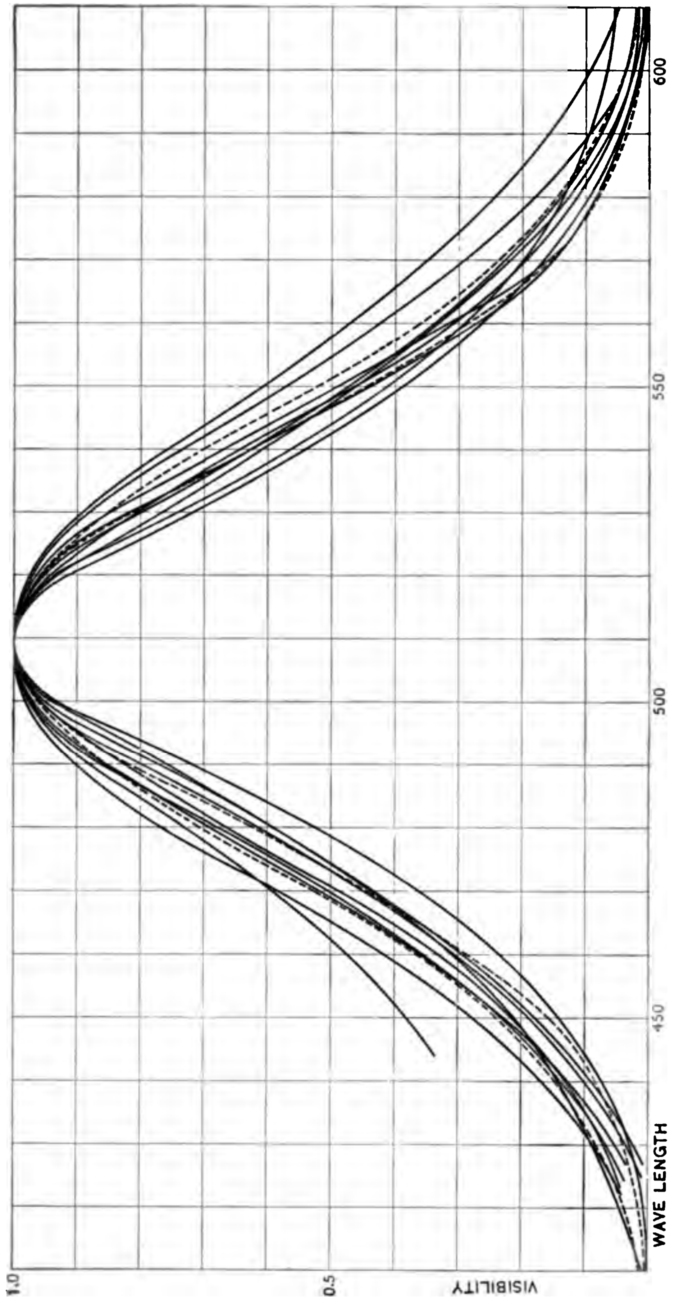

Fig. 1.

The probability function with $\kappa=4.0$ or $\kappa=4.5$ represents the data quite satisfactorily considering the large differences in the individual curves. The value $\kappa=\pi$ would be advantageous mathematically since it gives the curve unit area for unit maximum ordinate, but while it might be adopted arbitrarily, it is more nearly an envelope than a mean value of the experimental data.

Goldhammer has proposed the formula

$$\left(\frac{\lambda_m}{\lambda} e^{1-\frac{\lambda_m}{\lambda}}\right)^n$$

to represent the variation of sensibility with wave length. The wave length for which the sensibility is a maximum is indicated by λ_m while n is a number varying from 150 to 300. This is identical with the well-known Wien emission function for "black" radiation, which is

$$\frac{E}{E_m}=\left(\frac{\lambda_m}{\lambda} e^{1-\frac{\lambda_m}{\lambda}}\right)^n$$

when written in terms of its maxima E_m and λ_m in order to be independent of units of energy and wave length.

Goldhammer's function has the same number of parameters (n and λ_m) as the probability function, so that it is no more flexible. It is always steeper on the short wave length side, while the experimental curves are unsymmetrical on either side in about equal numbers. The discrepancy is not great, however, as may be seen from the data appended to the above table.

Hertzsprung,[4] in discussing the relation between the luminosity and temperature of "black bodies," adopts the following numerical factors to convert radiation into light:

$\lambda=$	400	450	500	550	600	650	700
Factor	0.044	0.083	0.340	1.000	0.631	0.107	0.0065

These values he takes as rough means of observed data at all intensities. He does not attempt to formulate any relation between them.

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[4] E. Hertzsprung, Zs. Wiss. Phot., IV, pp. 43-54; 1906.

8. OTHER POSSIBLE FORMS OF VISIBILITY FUNCTION.

The probability function and the formula for the radiation of black bodies, with altered parameter, were both found to be fairly satisfactory visibility functions, but perhaps rather too simple in form; that is, too inflexible to be made to represent some of the wide individual variations observed. Let us consider more general forms, which include both of these and other possible forms. The ultimate choice of a function is to be determined not so much by its simplicity, flexibility, or precise agreement with data, as by the form of variation of its parameters with intensity.

The more important characteristics of the visibility function are that it has a single maximum, it approaches zero for very long and very short waves, and that it is everywhere smooth, continuous, and single valued. The same may be said of its first derivative except that it has the single root $\lambda=\lambda_m$, and of the second derivative except that it has two roots which are in general not symmetrical with respect to λ_m.

Our data relates not to luminosity at constant spectral energy but to the inverse function, spectral energy at constant luminosity $E(\lambda)$, L constant. This function has a minimum instead of a maximum and is infinite where the visibility function is zero, otherwise the description of the preceding paragraph applies. Since the first derivative has a single root $\lambda=\lambda_m$ and is infinite at $\lambda=0$ and $\lambda=\infty$,

$$\frac{dE}{d\lambda}=a(E)(\lambda-\lambda_m)\left(\lambda,\frac{1}{\lambda}\right).$$

where (E) and $\left(\lambda,\frac{1}{\lambda}\right)$ denote functions of these quantities having no finite roots.

The function of E and indicated by (E) must be exceedingly simple, else $E(\lambda)$ would be so complex as to be quite useless in practice. The form $(E)=E$ is the simplest possible and yet sufficiently general to make $E(\lambda)$ include the probability and black body functions as well as a wide variety of others. The factor $\lambda-\lambda_m$ might of course be replaced by $f(\lambda)-f(\lambda_m)$ where f denotes any function (including $E(\lambda)$, but we wish to consider only the simpler and more specific of the possible forms of $E(\lambda)$. As to the function $\left(\lambda, \frac{1}{\lambda}\right)$ we shall consider in turn various simple possible forms.

$$\left(\lambda, \frac{1}{\lambda}\right) = \text{const. gives } E = \alpha e^{\beta(\lambda - \lambda_m)^2}$$

$$= E_m e^{\kappa\left(\frac{\lambda}{\lambda_m} - 1\right)^2}$$

$$\left(\lambda, \frac{1}{\lambda}\right) = \frac{1}{\lambda} \qquad \text{``} \qquad E = E_m\left(\frac{\lambda_m}{\lambda} e^{\left(\frac{\lambda_m}{\lambda} - 1\right)}\right)^n$$

$$\left(\lambda, \frac{1}{\lambda}\right) = \frac{1}{\lambda^2} \qquad \text{``} \qquad E = E_m\left(\frac{\lambda}{\lambda_m} e^{\left(\frac{\lambda_m}{\lambda} - 1\right)}\right)^n$$

The forms in positive powers of λ are inadmissible.

The first of these three functions is the reciprocal of the probability function proposed by the author. The third is the Wien-Goldhammer function above described. The second is a new one which may be used instead of the third to represent unsymmetrical sensibility curves in which the dissymmetry is reversed.

It is to be noted that writing the function $E(\lambda)$, (or any other physical function), in terms of its maxima, E_m and λ_m, gives a function $\frac{E}{E_m}\left(\frac{\lambda}{\lambda_m}\right)$ which is of zero dimensions throughout; i. e., independent of the units employed in the measurement of the quantities involved. Such a relation between pure numbers is often the simplest and always the safest to deal with.

9. CHROMATIC SENSIBILITY AT HIGHER INTENSITIES.

The sensibility of the eye to the same amount of radiation of various wave lengths varies considerably with variations in the intensity of the radiation. The maximum shifts from bluish green over to greenish yellow, the visibility curve at the same time becoming much broader near the maximum. The shift of the maximum toward longer wave lengths with increasing intensity may be easily observed with a small, low-power grating spectroscope and sunlight. With the slit all but closed the brightest part of the spectrum is in the pure green, but on opening the slit wider and wider, the brightest portion moves over nearly to the D lines.

This variation of the visibility of radiation with intensity is of the utmost importance. It shows that the assumption of luminosity proportional to intensity can be true only under special conditions and within certain limitations. In the relation $L = VE$, while L, V, and E are all functions of wave length λ, V is further a function of E. And the greatest variation occurs within the range of illumination in most common use, 1 to 100 meter-candles.

But scant data exists in this field. A. König appears to have been the only one to take up this problem, and his data[5] is complete only for his own eye. His long experience in work on visual sensibility and the fact that his eye appears to have been perfectly normal with sensibilities very nearly those of the average eye make his data of considerable value. König chose a standard of (visual) intensity in the green at 535 $\mu\mu$ and with that compared 13 other parts of the spectrum by widening an auxiliary slit until luminosities matched. These slit widths, corrected for variable prismatic dispersion, gave an inverse visibility curve for the source (a triplex gas flame) used. Then choosing a new standard of intensity, 4 or

König's Data on Visibility—Intensity

Intensity Steps	T	A	B	C	D	E	F	G	H
Int. (meter-candles)	.00024	.00225	.0360	.575	2.30	9.22	36.9	147.6	590.4
Ratio to Preceding Step		9.38	16	16	4	4	4	4	4
Wave Length	Equivalent Slit Widths								
670	296.1	189.0	29.08	14.53	4.56	1.955	1.403	1.23	1.17
650	178.3	87.6	13.9	5.98	1.95	0.991	0.667	0.547	0.420
625	34.8	20.55	6.20	2.93	1.004	0.497	0.392	0.307	0.289
605	12.1	8.60	4.29	2.073	0.869	0.451	0.321	0.289	0.274
590	5.65	4.29	2.75	1.86	0.787	0.523	0.376	0.346	0.330
575	2.76	2.00	1.68	1.42	0.868	0.568	0.449	0.459	0.424
655	1.40	1.232	1.10	1.037	0.876	0.702	0.621	0.603	0.590
535	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
520	1.07	1.09	1.12	1.16	1.40	1.71	1.76	1.77	1.80
505	1.54	1.58	1.59	1.81	2.30	2.78	3.75	4.06	4.46
490	2.34	3.00	3.06	3.33	4.93	7.97	8.91	9.53	10.06
470	5.92	6.49	7.22	7.75	11.0	20.7	22.9	26.6	....
450	19.21	21.5	20.95	24.2	40.7	57.3	62.9	....	....
430	131.6	115.4	85.9	119.2	138.	151.	....	....	....

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[5] A. König, Helmholz Festgrüss, Hamburg and Leipzig, pp. 309–388; 1891.

16 times as great, the process was repeated until he had spectral curves at nine different intensities ranging from the threshold of vision up to about 600 meter-candles. König's data are given below. The different steps of intensity are indicated by *T*, *A*, *B*—*H*; *T* being the threshold value. The actual intensity (approximately in meter candles) and ratio to preceding step are also given.

These data refer to the gas flame spectrum while the visibility curves proper refer to a spectrum of constant (energy) intensity throughout. Fortunately the spectral energy curve of such a gas flame was determined bolometrically by Langley and counter-checked photometrically by König by means of the solar spectrum. His relative values for the energy are

λ =	670	650	625	605	590	575	555	535	520	505	490	470	450	430
E =	13.0	8.88	5.38	3.99	2.97	2.27	1.48	1.00	0.720	0.488	0.370	0.251	0.169	0.114

To obtain the data of the following table each slit width of the preceding table was first multiplied by the spectral energy of the source for that wave length. The reciprocals for these corrected slit widths were then plotted as functions of wave length. These curves all passed through the value unity at 525 $\mu\mu$, but had maxima of different heights. Finally all curves were reduced to the same maximum ordinate unity and the data read from these curves.

Visibility-Intensity, Reduced Data

	T	A	B	C	D	E	F	G	H	C:T	P:T	F:T	H:T
430	.047	.057	.076	.060	.063	0.57	......	......	......	1.28	1.34		
450	.23	.21	.21	.20	.14	.10	.085	......	......	0.92	0.61	0.37	
470	.50	.46	.42	.42	.36	.19	.16	.13	......	0.84	0.72	0.32	0.26
490	.85	(.78)	(.75)	(.70)	(.63)	(.40)	.28	.25	.24	0.82	0.74	0.30	0.34
505	.99	.99	.98	.94	.88	(.63)	.50	.45	.40	0.95	0.89	0.50	0.45
520	.97	.96	.94	.99	.97	(.85)	.72	.69	.67	1.02	1.00	0 74	0.71
535	.75	.76	.76	.83	.98	.98	.91	.88	.87	1.11	1.31	1.21	1.17
555	.36	.42	.47	.54	(.83)	(.97)	.99	.99	.99	1.50	2.30	2.75	2.75
575	.12	.17	.20	.26	(.58)	(.85)	(.97)	(.98)	(.98)	2.17	4.83	8.08	8.17
590	.045	.059	.095	.15	.42	(.71)	(.88)	(.92)	(.92)	3.00	9.40	19.6	20.4
605	.015	.022	.044	.10	.28	.54	.71	.79	.79	6.6	18.6	47.5	52.6
625	......	......	.022	.050	.17	.35	.42	.52	.54				
650	......	......	......	......	.057	.11	.15	.18	.23				
670	......	......	......	......	......	.038	.050	.055	.058				
λ max	511	511	511	515	529	545	560	565	565				

In plotting these curves it is noticeable that those values which lie off from the smooth curve of the probability type (inclosed in parentheses in the table) nearly all lie near the two wave lengths at which the three fundamental sensibility curves of the Young-Helmholtz theory cross. The plot of the wave length of the maximum against the logarithm of the intensity is a remarkable curve.

Some of these curves are reproduced in the accompanying figures. In Fig. 2 are shown three curves of visibility as a function of wave length at constant intensities for the steps *T* (about 0.0002 m-c.), *D* (2.3 m-c.), and *H* (600 m-c.). The curves for steps *B* (0.002 m-c.) and C (0.036 m-c.) are nearly coincident with the curve for step *T*, the threshold value. These curves show the shift of the maximum and the broadening of the visibility curve with increasing intensity.

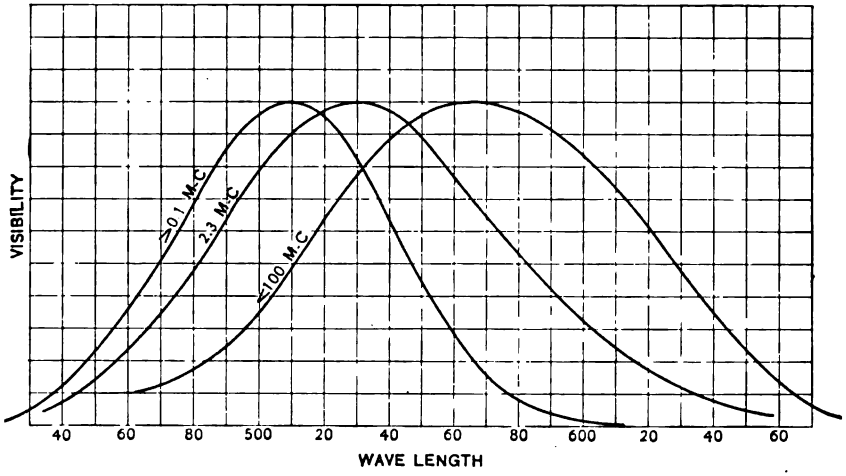

Fig. 2.

In Fig. 3 are shown seven curves of visibility as functions of intensity at constant wave length, the reproductions being for the wave lengths 430, 470, 505, 535, 590, 605, and 670 $\mu\mu$. These monochromatic curves represent the ratio of light to radiation in its relation to intensity under the assumption of maximum ratio equal to unity. The curves for yellow and orange ascend sharply with increasing intensity, while those for blue descend, and those for extreme violet and extreme red as well as for green do not vary much. These results show that the factor (visibility, V) converting radiation into light must vary with both color and intensity, although in the integrated effect for white light that factor may approach a constant value.

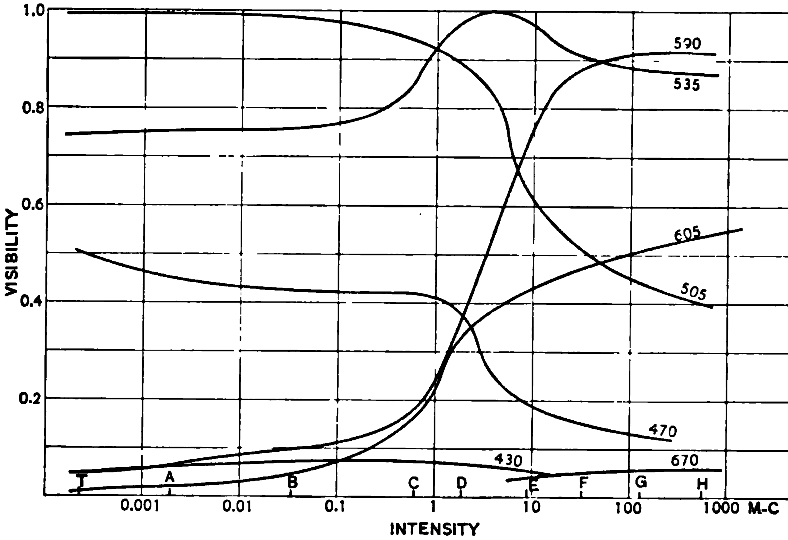

Fig. 3.

10. VARIATION OF THE PARAMETERS OF THE VISIBILITY FUNCTION.

Taking König's data as reduced in the last table as a working basis, let us see how the wave length of maximum sensibility and the parameters giving the visibility curve breadth and height vary with the intensity of the radiation. The wave lengths of the maxima are given in the table as read from the original reduced curves. They are plotted as a function of logarithmic intensity in Fig. 4.

In the plot the uncertainty in each value is indicated by the diameter of the circle inclosing each point. The shift in the position of the maximum of sensibility is seen to occur between intensities of about 0.5 and 50 meter-candles, just the range within which the eye is most commonly used. At very high and very low intensities there appears to be little or no shift. It is very remarkable how closely the shift that does occur is proportional to the logarithmic intensity. It is to be noted that this relation rests directly on experimental data without any assumption as to the form of any subsidiary function.

In Fig. 4 is also a plot of the variation of the parameter κ in

$$V = V_0\, e^{-\kappa(\lambda-\lambda_m)^2}$$

with logarithmic intensity. These values were obtained by calculation from the values of λ and λ_m for the various relative intensities.

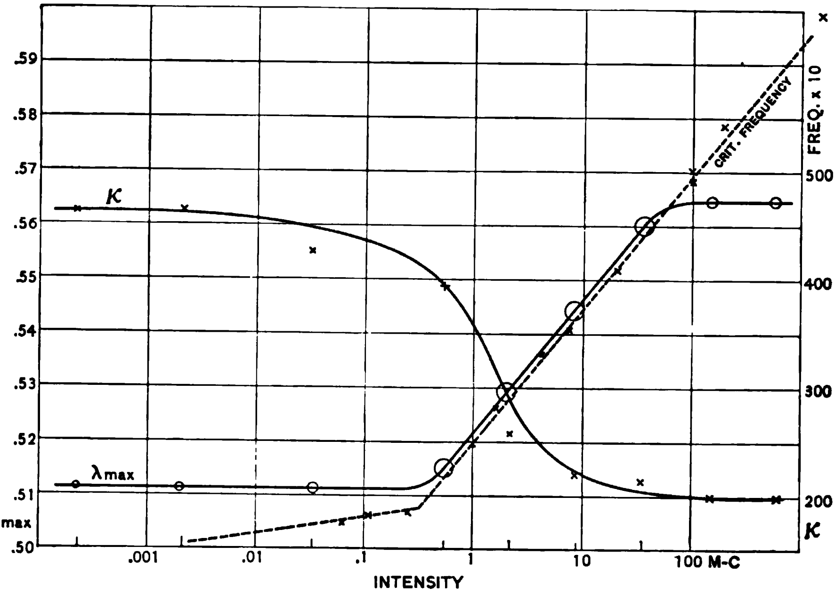

Fig. 4.

Each is the mean of the values at the different wave lengths for each intensity, the two middle values nearest the maximum being

	Luminous Intensity.	Wave Length Maximum Sensibility.	κ Calculated	
			$\text{Log}\,\frac{L}{EV_0}$	$\kappa\,(\lambda-\lambda_m)^2=\text{const.}$
T	0.00024	5.11	4.61	4.6
A	0.00225	5.11	4.66	4.6
B	0.0360	5.11	4.45	4.6
C	0.575	5.15	3.95	4.4
D	2.30	5.29	2.54	3.4
E	9.22	5.45	2.21	2.6
F	36.9	5.60	2.14	2.1
G	147.6	5.65	1.98	2.0
H	590.4	5.65	1.97	2.0

omitted in each case on account of their great uncertainty; they involve logarithms of numbers very nearly equal to unity.

Appended for subsequent discussion are values of κ calculated under the assumption $\kappa(\lambda-\lambda_m)^2=$constant; that is, illumination proportional to radiation.

There remains the parameter V_0 to be determined. This is the value of V for $\lambda=\lambda_m$, or in other words the maximum ratio of light to radiation at a given luminosity. To determine V_0 it is only necessary to measure the same monochromatic radiation as light and radiation, photometrically and radiometrically in meter candles and in meter watts.

The author[6] used a Rubens thermopile which was arranged to be interchangeable with the white screen of a Lummer-Brodhun photometer in determining the spectral intensity of a Nernst lamp. The thermopile was calibrated in watts by direct exposure to a Nernst lamp filament whose potential drop and current were then measured. A correction was then applied for the equatorial distribution of radiation from a linear source. Losses by convection were found to be less than half a per cent at most.

The results obtained are given in the following table. The accuracy is not high, the aim was rather to establish the order of magnitude of V_0 beyond any question.

Wave length	$\frac{\text{Watt}}{\text{cm}^2}$	Meter-candles	$\frac{\text{Meter-candles}}{\text{watt/cm}^2}$	$\frac{\text{Candles}}{\text{watt}}$
620	.000162	64	390000	3.1
566	.000042	64	1520000	12.1
566	.000036	61	1700000	13.5
595	.000123	182	1480000	11.8

These amounts of energy are low to measure radiometrically. The value of V_0 for pure rod vision appears to be almost hopeless of direct determination.

Drysdale[7] determined the ratio of light to radiation units by a somewhat similar method. Using a bolometer and glow lamps, he obtained the value 16.7 cand./watt for "yellow-green" light.

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[6] Electrical World, June 26, 1908.

[7] C. V. Drysdale, P. R. S., **80**, pp. 19–25; 1907.

11. SENSIBILITY TO DIFFERENCES IN WAVE LENGTH.

Of related interest in the theory of visual sensibility and in measurements of color are the minimum perceptible wave length or color differences in different regions of the spectrum and for different intensities. The best recent data is that of Steindler[8] on twelve subjects. The curves differ considerably in detail with the individual, but all are of the same general type. That for Steindler's own eye is reproduced in Fig. 5, the reciprocal of the least perceptible increment being plotted as ordinate to represent sensibility.

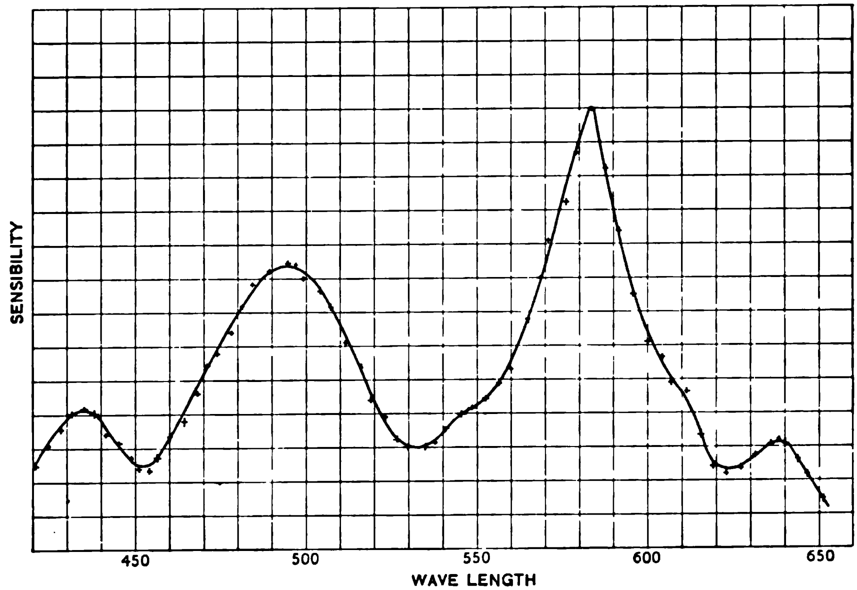

Fig. 5.

The two large central maxima may easily be located with a small spectroscope and daylight. They are supposed to correspond with the intersections of the three fundamental (red, green, and blue) sensibility curves. It would be important in visual theory to know whether these maxima shifted with varying intensity, but no one appears to have studied that phase of the problem.

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[8] O. Steindler, Wien Sitz., IIa, **115**, pp. 1-24; 1906.

III. SUBJECTIVE LIGHT.

12. PHOTOMETRIC SENSIBILITY.

Visual sensibility to variations in intensity decreases continually with increasing intensity. Compared with other physical instruments the eye has a most extraordinary range. It may be used with ease at intensities a million times the minimum perceptible. Its great power of accommodation is due chiefly to the decrease of sensibility with intensity. It is like an ammeter or galvanometer with an automatic shunt continuously variable.

As a measure of relative sensibility the best data theoretically would be the mean errors in a long series of determinations of given fixed intensities in various parts of the spectrum. This has never been done and would be prohibitively tedious. Next best is probably the determinations of the least perceptible increment or "difference limen" by König. These include six different wave lengths and about twenty intensities of each. The least perceptible increment is greater than the mean error of observations of equality but proportional to it, although the exact relation between them is still undetermined.

König referred all his intensities to a single large unit (about 600 meter candles) for all six wave lengths, and then at each wave length stepped down to intensities of 1/2, 1/5, 1/10, etc., of the large standard value. We shall refer to his intensities by the letter I, the threshold value being I_0 measured as a fraction of that high value. But although, on this basis, an illumination of 600 meter-candles in the red is of the same apparent brightness as 600 m-c. in the blue, 1 m-c. in the red is by no means of the same apparent brightness as 1 m-c. in the blue. At lower intensities the divergence is still greater. König's scale of intensities is, however, the only rational one available for experimental work, and from it must be deduced a tempered scale free from such anomalies.

König and Brodhun[9] determined the least perceptible increment for light ranging in intensity from just above the threshold value to about 100,000 meter-candles and in color from violet to deep red. The data relating to König's own eye—a normal trichromat—are reproduced in the accompanying table and figure.

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[9] König and Brodhun, Sitz. Ak. Berlin, July 26, 1888.

Photometric Sensibility.

$\lambda=$ $I_0=$	.670 0.060	.605 0.0056	.575 0.0029	.505 0.00017	.470 0.00012	.430 0.00012	White 0.00072
I	δI:I						
1,000,000							0.0358
500,000							0.0273
200,000		0.0425					0.0267
100,000		0.0241	0.0325				0.0195
50,000	0.0210	0.0255	0.0260				0.0173
20,000	0.0160	0.0183	0.0205	0.0195			0.0175
10,000	0.0156	0.0163	0.0179	0.0181			0.0176
5,000	0.0176	0.0158	0.0166	0.0160			0.0179
2,000	0.0165	0.0180	0.0180	0.0175	0.0180		0.0181
1,000	0.0169	0.0198	0.0185	0.0184	0.0167	0.0178	0.0178
500	0.0202	0.0214	0.0180	0.0194	0.0184	0.0214	0.0192
200	0.0220	0.0225	0.0225	0.0220	0.0215	0.0245	0.0222
100	0.0292	0.0278	0.0269	0.0244	0.0225	0.0246	0.0298
50	0.0376	0.0378	0.0320	0.0252	0.0250	0.0272	0.0324
20	0.0445	0.0460	0.0385	0.0295	0.0320	0.0345	0.0396
10	0.0655	0.0610	0.0582	0.0362	0.0372	0.0396	0.0477
5	0.0918	0.103	0.0888	0.0488	0.0464	0.0494	0.0593
2	0.1710	0.167	0.136	0.0655	0.0715	0.0600	0.0939
1	0.258	0.212	0.170	0.0804	0.0881	0.0740	0.123
0.5	0.376	0.276	0.208	0.0910	0.096	0.0966	0.188
0.2		0.332	0.268	0.110	0.127	0.116	0.283
0.10			0.396	0.133	0.138	0.137	0.377
0.05				0.183	0.185	0.154	0.484
0.02				0.251	0.209	0.223	0.695
0.01				0.271	0.289	0.249	
0.005				0.325	0.300	0.312	
0.002						0.369	

König and Brodhun did not include the increment (δI) in the total light (I) in calculating the values of $\delta I{:}I$. Logically the increment should be included, otherwise the ratio $\delta I{:}I$ becomes meaningless as the threshold value is reached. The values quoted have been recalculated with increment included. In the figure, $\delta I{:}I$ is plotted as a function of the natural logarithm of I, the curves being extended from the data of König and Brodhun to the point $\delta I{:}I=1$ for $I=I_0$, the threshold value, shown as a dotted ordinate in the figure.

We have now to determine the mathematical form of these curves by the methods of synthetic function theory. For brevity, call

$\delta I : I = P\ (I)$ or simply P, the photometric function. Denote the abscissa log I by x and the minimum value of P (about 0.016) by P_m.

All the curves have evidently the threshold value—the abscissa of the dotted ordinate—as natural origin of coordinates; that is, they would be strictly comparable if plotted as functions of $x - x_0$ or log $(I : I_0)$ instead of x or log I. Further, they are all of the general class of functions included in

$$P = a + be^{-z}$$

when a and b are constants (independent of x) and z is a function of x. As a special case we may consider the particular function $z\ (x) = n\ (x - x_0)$ just as we should choose a particular integral of a differential equation as the practical working solution.

Taking, then, the particular function

$$P = a + be^{-n\ (x - x_0)}$$

is a working basis of probably the simplest possible form consistent with observed data; the parameter a is identified with P_m (= about 0.016, a pure number) the minimum value toward which $P\ (I)$ approaches for I very large. The constant b is $1 - P_m$ or about 0.984, since at $x = x_0$, $P = a + b = 1$. Both a and b are of course independent of x and hence of I, and the data shows them to be very nearly, if not quite, independent of wave length as well. And being pure numbers they as well as P are of course independent of the units employed. The parameter n is a fraction varying between ¼ and ½ with different wave lengths.

Since $x - x_0$ is *log* I − *log* I_0 or *log* $(I : I_0)$, P as a function of L is

$$P = P_m + (1 - P_m)\ (I_0 : I)^n$$

to determine n it is convenient to reduce this equation to the form linear in n

$$\log \frac{1 - P_m}{P - P_m} = n \log \frac{I}{I_0}$$

Below is given for each wave length the value of the threshold value I_0 (approximately in meter candles) as determined by König together with the calculated mean values of n and their computed probable errors:

	I_0	n	$P_m^{1/n}$
670	0.060	0.584±0.005	0.000832
605	0.0056	0.388±0.020	0.0000234
575	0.0029	0.364±0.020	0.0000116
505	0.00017	0.335±0.005	0.00000434
470	0.00012	0.323±0.008	0.00000276
430	0.00012	0.323±0.003	0.00000276

In the next table are given for each wave length and intensity:

1. The value of P observed by König.
2. The value of n computed from P with $P_m = 0.016$.
3. The value of P computed from $P_m = 0.016$ and the mean value of n, in the curves reproduced in the accompanying figure discrepancies between observed and calculated values do not show as well as in the table.

Wave Lengths.

	670	605	575	505	470	430
I	P, n calculated and P calculated.					
1000	0.017 .... 0.019	0.020 0.456 0.025	0.019 .... 0.025	0.018 .... 0.021	0.017 .... 0.022	0.018 .... 0.022
500	0.020 .... 0.021	0.024 0.464 0.028	0.018 .... 0.028	0.019 .... 0.023	0.018 .... 0.023	0.021 0.333 0.023
200	0.022 0.591 0.024	0.023 0.464 0.033	0.023 0.445 0.033	0.022 0.348 0.025	0.022 0.282 0.026	0.025 0.333 0.026
100	0.029 0.580 0.029	0.028 0.450 0.038	0.027 0.430 0.038	0.024 0.364 0.028	0.023 0.362 0.028	0.025 0.344 0.028
50	0.038 0.565 0.038	0.038 0.418 0.045	0.032 0.423 0.045	0.025 0.368 0.031	0.025 0.363 0.031	0.027 0.346 0.031
20	0.045 0.602 0.049	0.046 0.418 0.057	0.039 0.431 0.056	0.030 0.372 0.036	0.032 0.350 0.037	0.035 0.327 0.037

Wave Lengths—Continued.

	670	605	575	505	470	430
I	**P, n calculated and P calculated.**					
10	**0.066 0.586 0.065**	**0.061 0.413 0.070**	**0.058 0.388 0.068**	**0.036 0.354 0.041**	**0.037 0.339 0.042**	**0.040 0.327 0.042**
5	**0.092 0.583 0.090**	**0.103 0.357 0.086**	**0.089 0.344 0.082**	**0.049 0.328 0.048**	**0.046 0.327 0.048**	**0.049 0.320 0.048**
2	**0.171 .... 0.142**	**0.167 0.307 0.117**	**0.136 0.323 0.108**	**0.065 0.317 0.057**	**0.071 0.294 0.058**	**0.060 0.320 0.058**
1	**0.258 .... 0.204**	**0.212 0.313 0.147**	**0.170 0.317 0.134**	**0.080 0.315 0.069**	**0.088 0.288 0.069**	**0.074 0.314 0.069**
0.50	**0.376 .... 0.300**	**0.276 0.296 0.187**	**0.208 0.314 0.166**	**0.091 0.323 0.084**	**0.096 0.301 0.073**	**0.097 0.301 0.073**
0.20		**0.332 0.299 0.261**	**0.268 0.323 0.237**	**0.110 0.332 0.108**	**0.127 0.295 0.105**	**0.116 0.309 0.105**
0.10			**0.396 0.270 0.286**	**0.133 0.334 0.131**	**0.138 0.311 0.128**	**0.137 0.312 0.128**
0.05				**0.183 0.314 0.161**	**0.185 0.292 0.155**	**0.154 0.325 0.155**
0.02				**0.251 0.297 0.215**	**0.209 0.320 0.204**	**0.223 0.305 0.204**
0.01				**0.271 0.322 0.265**	**0.189 .... 0.253**	**0.249 0.326 0.253**
0.005				**0.325 0.343 0.331**	**0.300 .... 0.291**	**0.312 0.322 0.291**

The agreement between the function chosen and the data is quite satisfactory in the blue and violet, fair in the red and green, and poor in the yellow. One would expect the poorest agreement at 490 and 575 μμ where the primary sensibility curves intersect. König states that he chose 575 as one wave length purposely to test the color vision theories. Another cause of poor agreement lies in

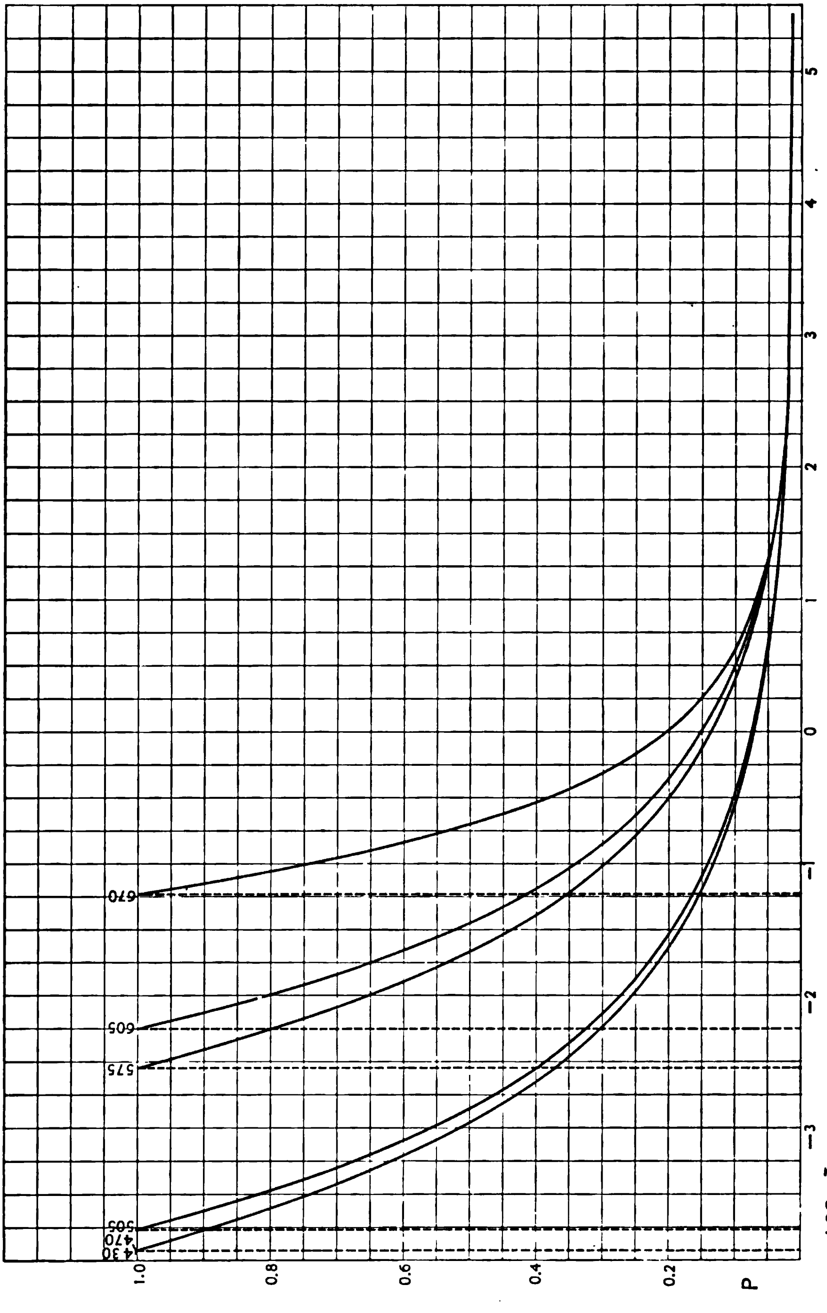

Fig. 6.

the uncertainty in the roughly determined values of the threshold of visibility, I_0 which enters as a vital part of the function. However, the function may be considered as a serviceable first approximation to the intensity sensibility of the average human eye over a wide range of color and intensity.

13. THE LUMINOUS SENSATION, FECHNER'S LAW.

Accepting, then, for the photometric function or least perceptible increment the form

$$P=P_m+(1-P_m)\,(I_o/I)^n$$

as a tentative working basis, we may apply the theory of sensibility to find the scale reading—which is luminous sensation in this case.

P is expressed as a fraction ($P=\delta I/I$) in the above function, so that sensibility, being inversely proportional to δI, is inversely proportional to IP. Calling sensibility S_I and the constant of proportionality K_I, then

$$S_I=\frac{K_I}{\delta I}=\frac{K_I}{IP}=\frac{K_I}{I(P_m+(1-P_m)(I_o/I)^n)}$$

But from the theory of sensibility the sensation of visual brightness B is the general integral of the sensibility with respect to the stimulus, hence

$$B=\int SdI=\int\frac{K_I dI}{I(P_m+(1-P_m)(I_o/I)^n)}$$

$$=\frac{K_I}{P_m}\log(1+P_m(I^nI_o^{-n}-1))^{1/n}$$

plus an integration constant which is zero, since the sensation B approaches zero as the stimulus I approaches the threshold value I_o.

About eighty years ago E. H. Weber observed that the least perceptible increment to a stimulus affecting several of the sense organs, under fixed subjective conditions of attention, expectation, and fatigue, bore a definite relation to the amount of that stimulus.

G. T. Fechner, thirty years later, extended Weber's observations and formulated his results in mathematical terms. Calling the stimulus I and the least perceptible increment δI, then Fechner's statement of what he termed Weber's Law is that $\delta I:I=$constant. This law appears to hold over a wide range of ordinary working intensities, breaking down only for very low and for excessively intense stimulation.

Fechner proceeded further by assuming that the above constant was proportional to the corresponding increment δB to the sensation. Hence $\delta I/I=c\delta B$ and by integration

$$B=c\,(\log I-\log I_0).$$

In this form or in similar forms differing only in the choice of integration constant, Fechner's law has been accepted by phychologists for half a century.

There are two very serious if not fatal defects in this deduction. In the first place, the increments δI and δB are finite quantities and by no means infinitesimal increments approaching zero as a limit, such as would be required for such an integration. The least perceptible increment to the stimulus (δI) is determined by the sensibility of the sensory organ concerned. At the threshold value it is as large as I itself, while at moderate intensities it bears a fixed ratio to I. The value of δB is entirely arbitrary, dependent upon the unit chosen in which to measure it. It may be greater than unity in special cases. In the second place, c is not a constant but a function of I. At low intensities approaching the threshold value it varies rapidly with I. The method used in this paper is free from these defects. Weber's Law $\delta I/I=$constant is extended to apply to high and low intensities, in the form

$$\delta I:I=P_m+(1-P_m)\,(I_0/I)^n$$

or some other form of function not differing greatly from this. Similarly Fechner's Law $B=c\,(log\ I-log\ I_0)$ is more widely applicable in the more general form derived in this paper.

The argument

$$L=(1+P_m(I^nI_0^{-n}-1))^{1/n}$$

of the visual brightness function is of such form that, however the radiation is varied in either wave length or intensity, so long as L

is constant the sensation is constant. This, then, is the proper measure of light measured as light in fractions of some high intensity. Luminosity is light in meter-candles tempered to a rational self consistent scale. But for the Purkinje effect, L would be identical with I and all parameters of both functions would be constant. $Ld\lambda$ is integrable to give total light while I is not.

14. THE GROWTH OF VISUAL IMPRESSION WITH TIME.

The visual sensation as a function of time may be constructed directly from the data of Broca and Sulzer without recourse to sensibility as a function of time. This function has, of course, the value zero at zero and approaches a constant value after a sufficient lapse of time. Broca found that with blue light there was a very pronounced maximum at 0.07 sec., with red a slight maximum farther out (at 0.12 sec.), while with green the curve rose steadily to its final value. These values are for intense illuminations; for lower intensities the maxima are lower and occur after a much greater lapse of time. These results may be represented by very simple functions. Consider the actual sensation as the resultant of a pure positive sensation (no fatigue) represented by the ordinary law of

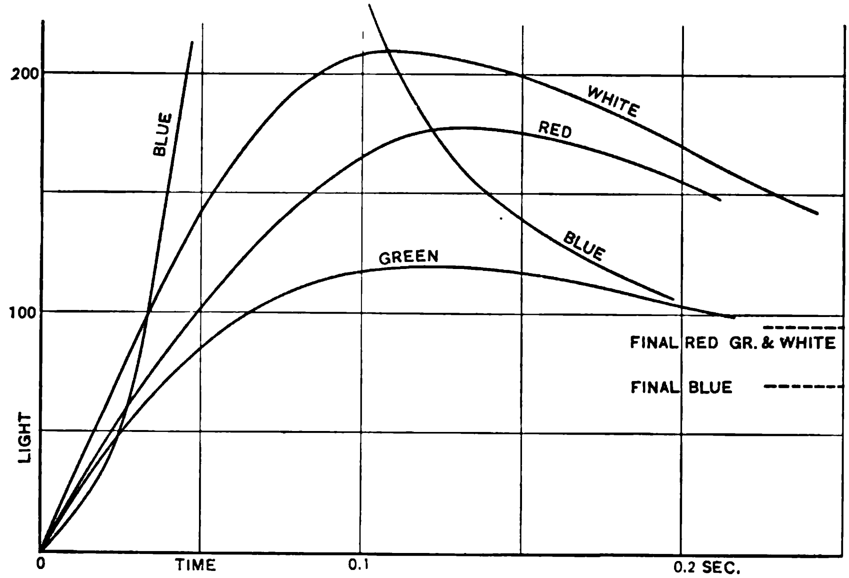

Fig. 7.

growth $(1-e^{-at})$ together with a decrement $1-e^{-\beta(t-t_0)}$ due to fatigue such that the fatigue is greater the greater the intensity of the light to which the eye is exposed, but affected with a lag t_0.

Hence,

$$l=L\,(1-e^{-at})\,(1-e^{-\beta(t-t_0)})$$

will serve as a tentative working basis. This satisfies the conditions $l=o$ for $t=o$ and $l=L$ at $t=\infty$ The derivative dl/dt has the value

$$La\,(1-e^{-\beta t_0})$$

for $t=o$ and has roots at $t=\infty$ and one finite point given by the value of t in

$$\frac{a}{e^{at}-1}+\frac{\beta}{e^{\beta(t-t_0)}-1}=o$$

Hence, for fatigue coefficient β, very small, the maximum will occur only after a lapse of considerable time.

Accordingly Broca's results would indicate very slight fatigue in the green, more in the red, and most of all in the blue and violet.

Work by E. S. Ferry[10] and by T. C. Porter[11] on the critical frequency at which flicker is just imperceptible is of great interest in this connection. Radiation of known intensity and quality is sent through a rotating sectored desk. The frequency at which the flicker disappears appears to be a function of optical intensity alone.

Flicker will evidently vanish when the difference between successive maxima and minima of visual impression becomes less than the least perceptible difference or photometric constant P. Hence, the observation that critical frequency is independent of color is confirmatory to König's Law that the photometric constant is independent of wave length.

Porter found further that the critical frequency is a linear function of the logarithm of the intensity except for a sharp variation at about 0.2 m-c. For intensities above and below this intensity these relations are

$$N_1=12.4\log L+24.9$$
$$N_2=1.56\log L+19.6$$

in meter-candles and common logarithms.

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[10] E. S. Ferry, Am. J. Sci., **44**, p. 192–207; 1892.

[11] T. C. Porter, P. R. S., **70**, p. 313–329; 1902.

These relations are shown graphically by the dotted line in Fig. 4. The elbow corresponds closely with the sharp turn in the curve of wave length of maximum sensibility, the lower branch evidently representing rod vision.

Critical frequency varies from 36 to 50 per second within the ordinary working range of intensities 10 to 100 m-c. Critical periods range from about 0.02 to 0.03 second for these intensities. These short periods of time lie back on the first slope of Broca and Sulzer's curves for $l\ (t)$, the parts that are sensibly straight lines. The corresponding range of intensity is roughly 0.016 × 50 or 0.8 m-c., while from the initial part of any of these curves it is over 40 m-c. Hence we must conclude that at vanishing flicker the eye is operating up near the maximum of a lower curve. It must then be a crude energy integrator but one with very small lag. Much remains to be done in this interesting field, particularly in relation to Talbot's Law. The measurement of intensity in terms of time alone is well worth further study, particularly as to its relation to the Purkinje effect and variation due to different observers.

The relation between critical frequency and luminosity is not at first apparent but may easily be deduced. Let $t_2 - t_1$, be the short interval of time during which the eye is exposed and $t_3 - t_2$ be the period of recovery. Then,

$$\frac{1}{t_3 - t_1} = n = f(L).$$

But,

$$\frac{dB}{dt} = \frac{dB}{dL} \quad \frac{dL}{dt}$$

and

$$\frac{dB}{dL} = \frac{K}{LP} \text{ and } \frac{dL}{dt} = LPns,$$

where s is the fractional sector opening.

Hence,
$$\frac{dB}{dt} = Ksn = Ks\,f(L),$$

where $f(L)$ is some function of L experimentally determined. For instance, that found by Porter to be $a \log L + b$.

15. OTHER FORMS OF SENSATION FUNCTION.

Since no sensation can be directly measured nor more than roughly estimated, the sensation function must always be derived from some subsidiary related function. We have here derived it by integrating photometric sensibility, its derivative, so that it is subject to the same order of uncertainties as the latter. In the photometric function small divergences were smoothed over to obtain a general function of simple form for general use. It may be that this should be broken up into two separate functions, one for cone vision and the other for rod vision, with a different threshold value for each.

There are several forms of photometric function $P(I)$ that would fit the data nearly or quite as well as the one here proposed, but imposing the further conditions of integrability in simple form and that Fechner's logarithmic law must be approached at moderate intensities excludes all forms that the writer has yet been able to construct.

Hertzsprung,[12] in a paper but recently come to my notice, has constructed a radically different function to represent König's data on white light. He chooses as dependent variable not the least perceptible increment but the ratio of the just noticeably different intensities, and as independent variable the geometric mean intensity. Plotting $\log(\log I_2 - \log I_1)$ against $\frac{1}{2}(\log I_2 + \log I_1)$ he notes that the observed points lie near a parabola whose equation is

$$-\log\log\frac{I_1}{I_2} = 1.3138 + 0.43595\frac{\log I_1 I_2}{2} - 0.05796\left(\frac{\log I_1 I_2}{2}\right)^2$$

The agreement is excellent for white light except for low intensities. Unfortunately this function is not integrable nor sufficiently simple to prove very useful. It is approximately equivalent to

$$P = a\,e - (1 + bI - cI^2)$$

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[12] E. Hertzsprung, Zs. Wiss. Phot., 8, p. 468; 1906.

16. THE PURKINJE EFFECT.

If monochromatic radiation, from a slit, be thrown on a screen, it is clear that both the radiation and the illumination produced by it must be proportional to the width of the slit, and hence proportional to each other. The constant of proportionality will, however, in general, vary with the wave length. This is one way of stating the Purkinje effect. If $I_\lambda = V_\lambda E_\lambda$, then, this phenomenon is involved in the fact that the constant of proportionality varies with the wave length.

At illuminations of over 200 m-c. and below 0.2 m-c. down to the threshold of vision, visibility appears from experimental data to vary little, if any, with change of intensity. (See Fig. 4.) In the blue and violet it is nearly or quite independent of wave length; in the green it is slightly greater, while in the yellow and red it increases more and more rapidly. Starting, then, with the constant values at low intensities, an expression for the total integral Purkinje effect may be easily obtained.

At low intensities, calling the maximum unity,

$$V_1 = e^{-4.6(\lambda-5.11)^2}$$

At high intensities where V is again constant

$$V_2 = e^{-2(\lambda-5.65)^2}$$

Hence, V_2/V_1 is approximately

$$\frac{V_2}{V_1} = e^{2.6(\lambda-4.70)^2}$$

These relations are shown graphically in the accompanying figure.

If at low intensity vision is by the retinal rods alone and at high intensity by both rods and cones, then the variation in V is due to the varying proportion of cones to rods in action and the above ratio represents the relative sensibility of rods and cones together to the sensibility of rods alone.

Relative sensibility for slight differences of intensity may be obtained by reading the corresponding values of κ and λ_m from tables or plotted curves such as these in Fig. 4. The assumption of light proportional to radiation for all intensities leads to the expression $\kappa(\lambda-\lambda_m)^2=$constant, and hence to κ as calculated in the last column of the table on page 282. The discrepancy is considerable in the range between ½ and 20 meter-candles, but high and low values agree well.

The relation $L=(1+P_m\,(I^nI_0^{-n}-1))^{1/n}$ between luminosity and illumination gives two more quantitative expressions for the Purkinje effect. This expression is based on I measured downward from a high intensity at which radiation in different spectral regions were brought to the same luminosity. Hence the measured value of the threshold limen I_0 as a function of wave length is an expression for the total Purkinje effect. Again, if luminosity is proportional to illumination at high intensities (over 200 m-c.), then $P_m^{1/n}$ is another expression for the total Purkinje effect. Appended to the table on page 288 are computed values of $0.016^{1/n}$.

In the table below are given series of values for the total Purkinje effect relative to λ 4.70 s-m together with their common logarithms. They are respectively the exponential of 2.6 $(\lambda-4.7)^2$, $P_m^{1/n}$, the observed values of I_0 measured down and H/T from the table on page 279.

λ	$V_2:V_1$	$P_m^{1/n}$	I_0	H:T
4.30	1.52 0.183	1.00 0.00	1.00 0.00	
4.70	1.00 0.00	1.00 0.00	1.00 0.00	1.00 0.00
5.05	1.37 0.137	1.57 0.196	1.42 0.152	1.73 0.238
5.75	17.4 1.265	4.20 0.623	24.2 1.382	31.4 1.497
6.05	144. 2.16	8.46 0.927	46.7 1.67	203. 2.307
6.75	302. 2.48	500. 2.70		

These results show roughly the course of the Purkinje effect The numerical quantities being calculated by such different methods from such widely different and uncertain data, no very close agreement could be expected, particularly since exponentials and ratios of small differences are involved. The Purkinje effect is a direct consequence of the varying of the ratio of light to radiation; no attempt at a quantitative statement of it has heretofore been made.

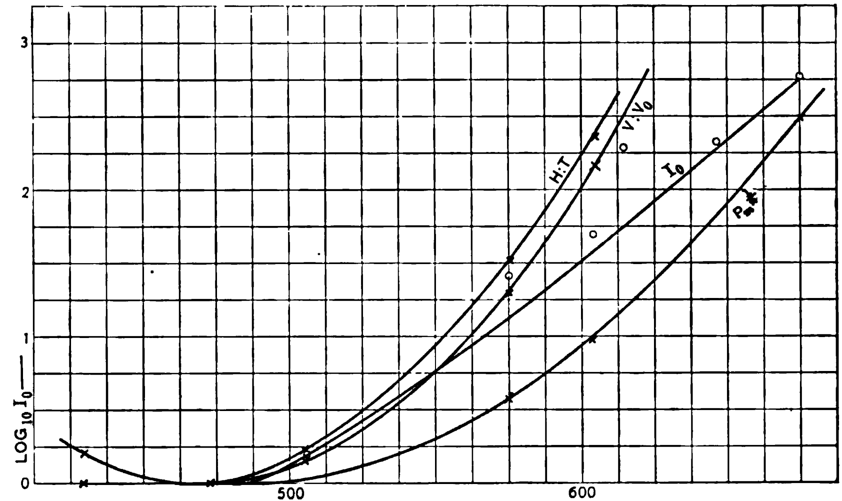

Fig. 8.

The ratio of light to radiation as a function of intensity is shown graphically in Fig. 3, page 281, under the assumption of maximum ordinate equal to unity and intensity step measured at λ 535 $\mu\mu$. It is hardly worth while to plot L as a function of E here. These curves would evidently consist of two straight extremities joined by a short curve for each wave length.

IV. CONCLUSION.

17. COMPLETE LIGHT SCALES.

The logical conclusion of such a paper as this would be tables of numerical values of the luminous equivalent of radiation at each wave length and for all intensities together with the corresponding algebraic expressions for the functions involved. To do this would require a statistical treatment of data relative to the properties of a great number of human eyes. However, fairly complete data on a few eyes have been available, the essential functions have been tentatively constructed, the general problem outlined and the way cleared for incisive experimental work. Eventually it should be possible, as it is highly desirable, to define for general adoption the properties of the average human eye and so completely establish a light scale in terms of radiation.

The functions involved in the luminous equivalent of radiation and its determination are tabulated below:

1. $E(\lambda)$ *Radiation*, spectral energy, in watts per unit wave length.
2. $V(\lambda)$ *Visibility* of radiation, chromatic visual sensibility, a spectral factor converting radiation into (objective) light.
3. $I(\lambda, E)$ *Illumination*, luminous intensity, objective light, proportional to radiation in intensity. $I=VE$.
4. $P(I)$ *Photometric function*, difference limen, least perceptible increment, inverse relative sensibility.
5. $L(I)$ *Luminosity*, subjective corresponding to illumination objective, the argument of the function $B(I)$.
6. $B(I)$ *Visual brightness*, the sensation produced by the illumination I, logarithmic function of L.
7. $l(t)$ *Time function*, lag of sensation behind stimulus.

Of these, E is supposed given; V, P, and $L(t)$ rest entirely upon experimental data, while I, $L(I)$, and B are functions derived from E, V, and P. B is a sensation, hence incapable of precise measurement. Luminosity $L(I)$ is an auxiliary function introduced and defined because of its fundamental importance in practical work as well as in the theory of light.

The assumed and derived forms of these functions are as follows:

$E(\lambda)$, spectral energy curve of source, supposed given.

$V(\lambda)=E/I$ determined experimentally by varying E while holding I (or strictly, L) constant

$$V=V_0 e^{-\kappa(\lambda-\lambda_m)^2}$$

approximately, where V_0, κ and λ_m are functions of E.

$$I=VE$$

for any source once V is known.

$$P=P_m+(1-P_m)\,I_0^n\,I^{-n}$$

where I is measured downward from a high intensity. I_0 (so measured) and n are functions of wave length while Fechner's constant P_m is not.

$$L=(1+P_m\,(I^n\,I_0^{-n}-1))^{1/n},$$

the argument of $B(I)$. Subject to the same conditions and uncertainties as $P(I)$.

$$B=B_0\log\,(1+P_m\,(I^nI_0^{-n}-1))^{1/n}=B_0\log L$$

obtained by integrating $1/IP$ under the assumption that IP is inversely proportional to visual sensibility to intensity differences. Subject to the same conditions and uncertainties as $P\,(I)$.

$$l=L\,(1-e^{-\alpha t})\,(1-e^{-\beta(t-t_0)})$$

The form of V at low intensities (rod vision) is fairly well established, but the value of the constant V_0 in light units per watt is not known to within a factor of ten at best. If Langley's eye was lacking in retinal rods, as the form of his visibility curve would lead us to suspect, then V_0 for rod vision is about twenty times that for cone vision, or roughly, 250 candles per watt. No other data of any kind on this constant is known to the writer. The values of κ and λ_m should rest upon data from hundreds of individuals of diverse nationality instead of upon barely twenty and those chiefly Teutonic.

The form of P and its parameters should be determined for a number of individuals using König's method and the best modern light sources instead of gas flames. Critical frequency and visual fatigue deserve much further study by the most precise methods.

18. THE DEFINITION OF WHITE LIGHT.

Experiment shows that light is sensibly "white" when either (*a*) $E=$constant throughout the visible spectrum or (*b*) light of three or more colors is combined in certain proportions. According to generally accepted theories of color vision, the visibility curve $V\,(\lambda)$ is a composite of three elementary visibility curves such that

$$V=V_1+V_2+V_3.$$

These elementary curves appear to be indeterminate at present chiefly because there are more variables to be determined than equations to determine them. By assuming them to be of equal area (König and Dieterici) or equal maximum ordinate or of exactly the same form, they may be determined roughly by mixing known amounts of arbitrary colors to produce white light. Assuming them to be known and taking $I_1 = EV_1$, etc., evidently the criterion for white light is that

$$\int I_1 d\lambda : \int I_2 d\lambda : \int I_3 d\lambda = \int V_1 d\lambda : \int V_2 d\lambda : \int V_3 d\lambda$$

whatever the form of the spectral energy curve E (λ).

One of the best recent determinations of these curves[13] is reproduced in the accompanying figure (Fig. 9), where equality of area has been assumed. All these curves overlap, to some extent, the red and green very widely. From ordinary experience one would say that the blue curve is somewhat too narrow and much too high and too steep at the foot of the short wave length side. From these curves it would appear that the most nearly monochromatic source to give a sensation of white light would be a heavy line located between λ560 and λ580 balanced up with a little greenish blue of wave length 480 $\mu\mu$.

The question of the best source of white light often arises. No economical source of pure white (E = const. or as defined above) light is known or perhaps possible. Sunlight and the light from carbon at a temperature of 4000 degrees are approximately white, since their spectral energy curves are nearly constant throughout the visible spectrum, but contain a large proportion of radiation of low visibility. The whitest discontinuous sources are the incandescent vapors of sulphur and uranium and helium gas, but these emit a very large proportion of waste energy.

An emission curve, the shape of the visibility curve (compare Fig. 2), would not correspond with white but with decidedly greenish white light.

The best source would probably be not pure white but slightly greenish or yellowish consisting of radiation nearly uniform in intensity between 530 and 610 $\mu\mu$ or else of a spectrum line at or near 570 balanced by a fainter one at 480 $\mu\mu$. It is remarkable to what extent light may differ from white and yet with steady exclusive use appear but slightly tinted. In other words, the visual color accommodation is large.

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[13] F. Exner, Wien. Sitz., **111** (IIa), p. 837; 1902.

Highly colored illumination, like that from the mercury glow lamp, although not injurious to the eyes, appears to be of low visual economy, that is a higher luminosity is required to produce the same comfort and acuity of vision. There is a real gain in focal achromatism for which the eye is, however, already well provided. Fig. 9 shows that at least two of the three sets of cones are in use with monochromatic illumination in every part of the spectrum except the violet and extreme red.

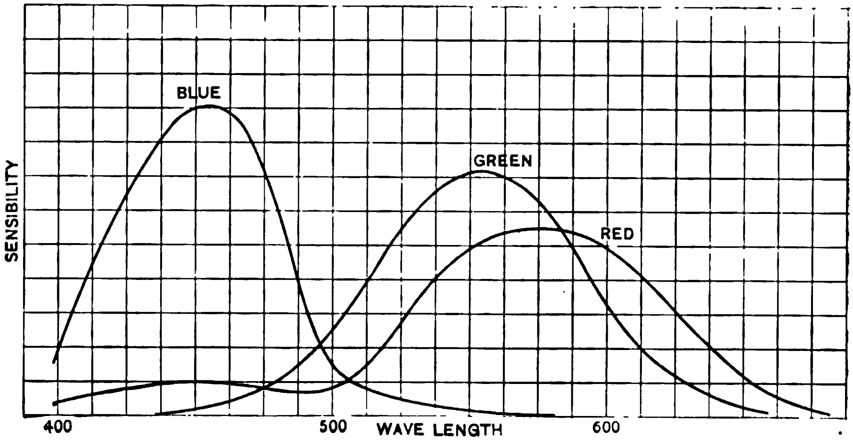

Fig. 9.

19. UNITS OF ILLUMINATION.

Neither subjective nor objective light is expressible in purely mechanical units since the magnitude of each is determined by a pure sensation. The units involved being entirely arbitrary, may be chosen according to convenience. Four units of illumination commend themselves:

1. The threshold value is from a mathematical standpoint by far the most suitable, since measuring in terms of this would greatly simplify the mathematical expressions involved. It is much too small, however, for use as a practical unit; it is difficult or impossible of precise determination and is separated from the region of common use by the whole Purkinje effect.

2. The meter candle, lux, and lumen now in ordinary use have the advantage of familiarity, but are small and within a region of intensity heavily burdened by the Purkinje effect.

3. A much higher intensity of, say, 1000 meter candles has much in its favor. This is about half that on a horizontal plane in the open on a cloudy day. It is a good working intensity and safely above the Purkinje effect. It is objectionable chiefly in that ordinary artificial illuminations expressed in terms of it would be fractions.

4. Another unit of similar magnitude, but different character from (3), would be one-thousandth of a watt per square cm of radiation lying in the spectral region between 560 and 580 $\mu\mu$. The last two units would be large enough for accurate measurement as radiant energy as well as light.

20. STANDARDS OF LUMINOSITY.

Evidently the logical primary standard of illumination is one in which the measured quantity is radiation measured as radiation by some form of radiometer and not a rate of combustion, a linear dimension, or an electric current. Such a primary standard was recently proposed by Steinmetz,[14] who advocated the use of the three mercury lines, blue 435, green 546, and red 691 $\mu\mu$. Referring to Fig. 2, the red and blue are seen to be rather too far out, while Fig. 9 indicates that far from equal energy proportions would be required to produce white. The principal objection to Steinmetz's proposition is the difficulty in getting sufficient intensity for radiometry, particularly in the red.

The least intensity of radiation that can be measured with the necessary certainty is about 0.0001 watt/cm^2. This is approximately 200 meter-candles at 570 (green-yellow), 70 m-c. at 640 in the red just beyond the orange, and an equal amount in the blue green at 500 $\mu\mu$. This means quite intense sources, but sources safely above the Purkinje effect. A glow lamp burning in air like the Nernst would appear to be the most practicable. Two or three segments of selected wave lengths from the continuous spectra of such a source could easily be obtained of sufficient intensity and purity.

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[14] C. P. Steinmetz, Proc. A. I. E. E., March, 1908.

21. LUMINOSITY AND TEMPERATURE.

For perfect radiators, the so-called black bodies, $E(\lambda)$ is known as a function of temperature, hence with $V(\lambda)$ known and constant (at high intensities), the light emitted by such a body may be expressed as a function of temperature. Being concerned with the visible spectrum we may safely use the simple Wien-Paschen formula

$$E(\lambda)=C_1\,\lambda^{-n}e^{-C_2/\lambda T} \tag{1}$$

This multiplied by visibility gives the optical intensity of the radiation. Using the probability curve visibility function

$$V=V_0e^{-\kappa(\lambda-\lambda_m)^2} \tag{2}$$

$$I=EV=C_1V_0\,\lambda^{-n}e^{-C_2/\lambda T-\kappa(\lambda-\lambda_m)^2} \tag{3}$$

Hence the isochromatic relation is of the form

$$\log I=A-B/T \tag{4}$$

where

$$A=\log\,(C_1V_0\,\lambda^{-n})-\kappa(\lambda-\lambda_m)^2$$

and

$$B=C_2/\lambda$$

The value of the constant B is seen to be independent of the form of visibility function assumed while the value of A is not. Further, the form of function (4), namely $I\,(T)$, is independent of the form of the visibility curve being determined solely by the form of radiation function $E\,(\lambda)$ chosen.

Equation (4) assumes a single, constant wave length, hence it can apply to but a very limited spectral region. It further assumes that the form of the visibility function does not change with intensity, so that if used at moderate and low intensities account must be taken of variations in κ and λ_m. It might be found experimentally to hold approximately for the total light emission because about 90 per cent of the light is confined to a region only $0.1\,\mu$ broad, but it could never be considered of general validity for such a case.

The relation between total light emission and temperature is obtained by integrating $EVd\lambda$ from zero to infinity. Function (3) above is not integrable, but using the alternative form (see page 277) of visibility function

$$V = V_0 \left(\frac{\lambda_m}{\lambda}\right)^{\nu} e^{\nu\left(1-\frac{\lambda m}{\lambda}\right)} \tag{5}$$

the product EV is readily integrable in the form

$$I = \int_0^{\infty} EV d\lambda = C_1 V_0 \lambda^{\nu}{}_m e^{\nu} \Gamma(n + \nu - 1) \left(\frac{C_2}{T} + \nu\lambda_m\right)^{1-n-\nu} \tag{6}$$

which is in the functional three constant form

$$I = A\left(\frac{B}{T} + 1\right)^{-s} \tag{7}$$

when

$$S = n + \nu - 1$$
$$B = C_2 / \nu\lambda_m$$
$$A = C_1 V_0 \lambda^{\nu}{}_m e^{\nu} \Gamma(n + \nu - 1)(\nu\lambda_m)^{-s}$$

function (7) is subject to the same high intensity limitations as (4); it can be safely used only at intensities of over about 100 meter candles unless variations in the parameters are taken into account.

22. PRACTICAL APPLICATIONS.

To obtain the maximum sensibility in many measurements involving visual observation, some of the above results may be applied to advantage. A few of these applications are briefly outlined below. If necessary, they may be considered more in detail in a later paper.

Photometry, as a determination of luminous intensity, may be accomplished either by bringing known and unknown to equality by a zero method (ordinary photometry) or by the direct measurement of intensity by an empirical method. Only when the spectral distributions in the two sources are identical is the ordinary photometric method strictly a zero method. Readings are then independent of the observer, be he color-blind or of normal vision, and of the absolute intensity at which comparisons are made. Uncertainties in readings vary with the individual, and are least at intensities of illumination of from 500 to 50,000 m-c., but at intensities as low as 5 m-c. are only three times the minimum. When the sources compared are not spectrally identical, the ordinary photometric method is not strictly applicable, for zero methods can not be used. With corrections for the observer's eye according to a fixed average luminous equivalent, the method could still be used or observations could be made with rod vision and similar corrections applied.

The direct measurement of intensity by observing critical frequency or visual acuity is less accurate than the zero methods but has the advantage of simplicity and freedom from color effects.

In pyrometry or wherever color screens are used, the shift with intensity of the center of luminosity of the transmitted band must be taken account of. Here the transmitted light is a resultant of three curves; the spectral energy of the source, visibility, and the transmission curve for the screen used.

In chemical work, the so-called colorimeters used in comparing the depths of the colors of two solutions are ordinarily used at too low illuminations of field to attain maximum sensibility. An illumination equivalent to a thousand meter candles in the resultant field is still too low for the highest sensibility.

In astronomy the measurement of star magnitude depends upon the color of the star and the visual sensibility of the observer to the extent of a large fraction of a magnitude. Corrections should be applied for color and observer according to visibility and visual properties.

In polarimetry the light intensities used are necessarily low. With a simple analyzing nicol it is, of course, merely a question of the use of the brightest possible source, for the maximum sensibility is the square root of the ratio of the intensity of the source to the threshold value. For a half shade instrument it has been shown[15] that for maximum sensibility the analyzing angle ($=2\delta$) is given by

$$\frac{P}{I}+\frac{dP}{dI}=0 \text{ where } I=E^2\delta$$

I being the light intensity in the visual field, E that at the source, and P being the photometric function as used in this paper. Substituting the value of P, I is given by

$$I^n P_m=(n-1)(1-P_m)I_o^n$$

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[15] This Bulletin, 2, p. 258; 1906.

from which the analyzing (or polarizing) angle may be calculated when the intensity E of the source used is known.

In spectroscopy it is customary to record impressions of the relative intensities (visual or photographic) of spectrum lines, some bright line being 10 or 100 and some very faint line unity. These are of course but rough estimates of purely subjective sensations. It was in a search for a rational and practicable intensity scale that this work was undertaken. We have here to deal, not with fixed quantities like stellar magnitudes, but with intensities varying with every variation in either source or set up as well. For the more precise work the best procedure appears to be a radiometric measurement through photometric comparison with an intermediate source of known spectral energy distribution. In less accurate work the simpler method of critical frequency for no flicker might be employed in preference.

It has been the aim of the writer in this paper to establish a scientifically precise yet practically manageable relation between light and its sole measurable constituent, radiation. It is hoped that this preliminary paper may pave the way toward more and better data and to a more complete solution of the general problem in all its details. The next step is the general adoption of a suitable nomenclature. Ultimately we may perhaps be able to define and adopt the visual properties of the average normal human eye. When that has been established, light may be precisely defined in terms of radiation and all values of the luminous equivalent of radiation, as a function of wave length and intensity, completely established.

WASHINGTON, September 3, 1908.

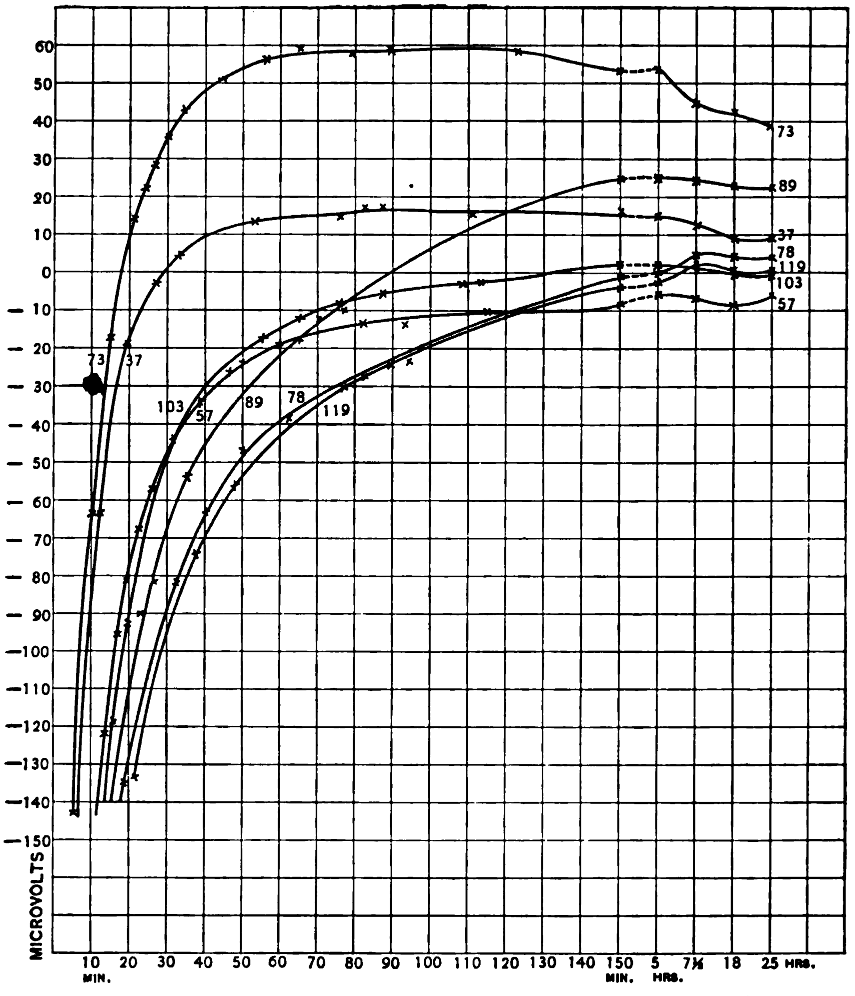

Fig. 1.—*Differences of individual cells from normal values after an abrupt change of temperature from 25° to 4°.8.*

52839—08. (To follow page 308.)

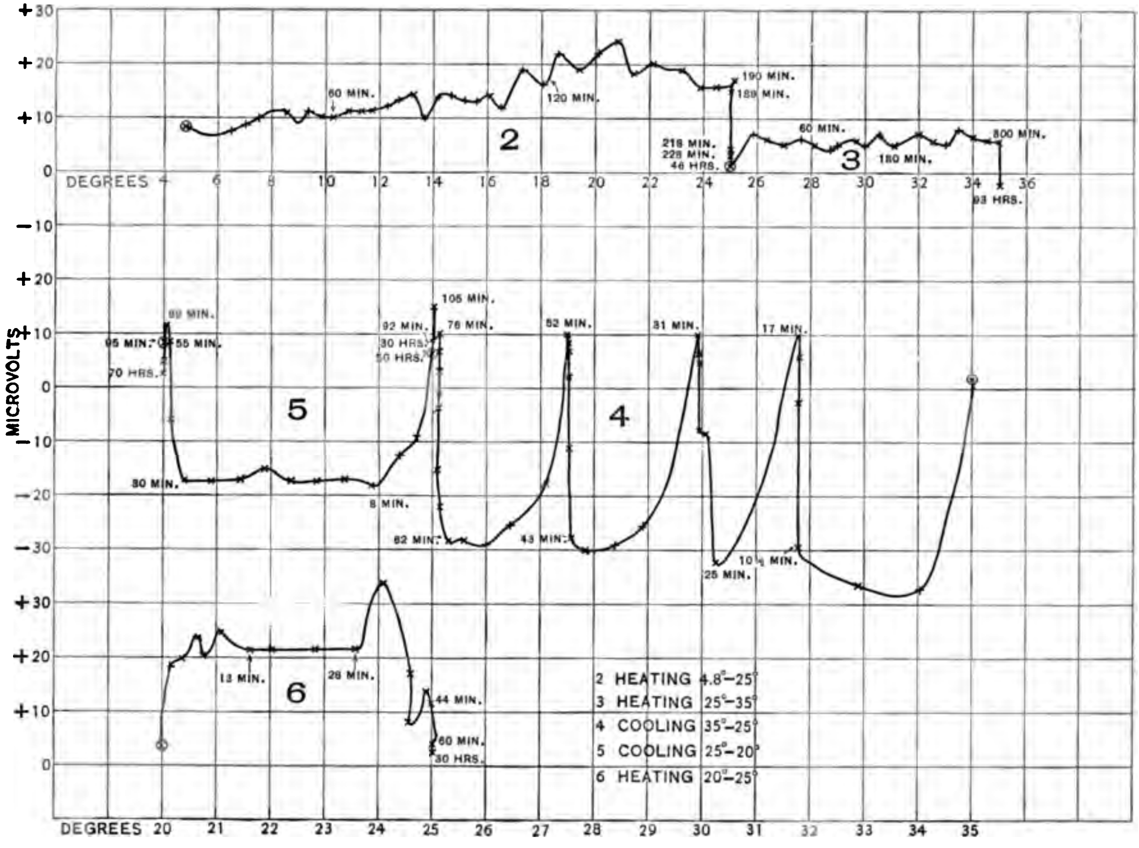

Fig. 2.—(*Curves 2–6.*) *Average differences of seven cells from normal values during temperature changes.*

THE TEMPERATURE FORMULA OF THE WESTON STANDARD CELL.*

By F. A. Wolff.

INTRODUCTION.

The more recent work on the standard cell has demonstrated that the Clark and Weston types are both reproducible to a very high degree of accuracy, and that when properly set up both may be depended on within a few parts in 100,000 for a year at least. As these qualities, in connection with the concreteness and ease of reproduction of the standard cell, seem to point to it as the logical choice for the second fundamental electrical unit, a redetermination of the temperature formulæ has been undertaken. On account of the very small effect of temperature variations on its electromotive force and the absence of cracking at the amalgam terminal, distinct advantages over the Clark cell, an investigation of the Weston cell was first made. A redetermination of the temperature formula of the Clark cell will, however, also be undertaken.

HISTORICAL REVIEW.

The first statement regarding the temperature coefficient of the Weston cell is found in the specifications of the patent issued to Dr. Edward Weston, who justly claimed as a special advantage of the cell its very small variation in electromotive force with temperature changes.

The results of Dearlove,[1] obtained with several lots of cells in which apparently no excess of cadmium sulphate or mercurous sul-

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*Read at meeting of American Electrochemical Society, Apr. 30, 1908. Trans., **13**, p. 187.

[1] Electrician, **31**, p. 645; 1893.

phate was employed, were quite irregular. Three cells of the first lot, in which the amalgam "was estimated to contain one part of cadmium to five parts of mercury," had an E. M. F. greater than 1.03 volts, and differed from each other by as much as 2 millivolts. In addition the E. M. F. of the cells was variable, and all had abnormally high temperature coefficients. The irregularities were attributed by Dearlove to the amalgam, the composition of which was found by him to exert a very decided influence on the E. M. F. Five cells set up with 12.5 per cent amalgam in 1893 gave a temperature coefficient ranging between 0.003 and 0.007 per cent per degree in the interval between 15° and 35°, and 0.007 to 0.010 per cent between 15° and 64°. Observations made while the cells were being brought back to room temperature and others on cooling to 0° showed that the results were influenced by marked hysteresis. Seven presumably similar cells set up by Dr. Muirhead were found by Dearlove to have a "temporary temperature coefficient" of 0.008 per cent between 2°.5 and 22°.

The generally accepted temperature formula for the Weston cell with excess of cadmium sulphate crystals, which alone will be considered in this paper, is based on a series of measurements by Jaeger and Wachsmuth,[1] extending over a period of five days, during which the four cells employed were subjected to large and sudden temperature changes, no attempt being made to maintain the temperature constant for any considerable time. Besides the comparisons made at room temperature, 30 measurements in terms of two additional cells employed as reference standards are given, which may be summarized as follows:

On the first day two of the four cells were measured at 14°.9; on the following day the same cells were compared with each other at the same temperature. On the third day one cell was measured at 0° and two others at 9°, 9°.1, and 9°.6. During the forenoon of the fourth day two cells at 1°.7 and two at 12°.6 and 12°.8 were compared with the reference cells, while during the afternoon two sets of measurements were made at 25°.3 and 25°.9 on one cell of each lot. On the fifth day one cell at 1°.5 and two at 20° were measured. Additional observations not given in the paper, and presumably check measurements at the above temperatures, are referred to in a footnote.

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[1] Wied. Ann., **59**, p. 575; 1896. Elektrotechn. Zs., **15**, p. 507; 1894.

The cells in question were all made up with 14.3 per cent amalgam, with which irregularities observed at lower temperatures have since been connected. The following formula was derived from *all* of the observations:

$$E_t = E_{20} - .000038\,(t-20) - .00000065\,(t-20)^2$$

This formula has since been in general use. On account of irregularities observed between 0° and 5°, the authors limited its application to temperatures between 5° and 26°, and still later Jaeger[3] raised the lower limit to 10°.

These irregularities were first attributed by Kohnstamm and Cohen[4] to an inversion of cadmium sulphate similar to that observed in the case of zinc sulphate in the Clark cell, but it may be regarded as now established that no such inversion takes place between 0° and 60°. Irregularities have, however, been shown to be introduced by the employment of cadmium amalgams containing more than 15 per cent by weight of cadmium. Jaeger[5] found that at room temperatures the E. M. F. of the Weston cell is almost independent of the composition of the amalgam employed, provided it contains between 5 and 14.3 per cent cadmium. In the case of more concentrated amalgams the E. M. F. was found to be variable, the changes amounting to 1 millivolt for a 15.4 per cent amalgam, 11 millivolts for a 20 per cent amalgam, and 44 millivolts for amalgamated cadmium rods. Jaeger, therefore, recommended the use of a somewhat more dilute amalgam than that first employed, since its composition was too close to the limit at which irregularities were observed. No observations were, however, made by Jaeger at other than room temperatures. The subject has been further investigated by Bijl[6] and Puschin,[7] and it has been shown that cadmium amalgams consist, within certain temperature limits, of a liquid phase and a solid phase, the latter composed of mixed crystals of cadmium and mercury, having a composition depending on the temperature. With both phases coexistent the E. M. F. toward cadmium sulphate at a given temperature is, therefore, independent of the relative amounts of the metals present, as found by Jaeger, while in the absence of the liquid phase irregularities in behavior are found. The composition of the amalgam to be employed in the standard cell must therefore be such that both the liquid and solid phases will be present within the usual temperature range, a condition probably satisfied by the 12.5 per cent amalgam employed at the Bureau of Standards and elsewhere.

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[3] Zs. Instk., **20**, p. 317; 1900. Ann. d. Phys., **4**, p. 123; 1901.

[4] Wied. Ann., **65**, p. 344; 1898.

[5] Wied. Ann., **65**, p. 106; 1898.

[6] Zs. f. Phys. Chem., **41**, p. 641; 1902.

[7] Zs. Anorg. Chem., **36**, p. 201; 1903.

A temperature coefficient more than twice as large as that obtained by Jaeger and Wachsmuth was found by Barnes,[8] who measured three cells set up with 1 : 6 amalgam, two of which were filled with moist cadmium sulphate crystals instead of crystals and solution. Measurements were made between 0° and 40° "allowing ample time at each interval for the establishment of a steady and uniform temperature." The observations at the various temperatures were completed in one day, the cells having been kept at 0° during the night previous. The values obtained above 15° were fairly well represented by a linear coefficient of 0.0086 per cent per degree, but under 15° Barnes states that no regularity can be said to exist, the difference between 0° and 15° being 2.1, 4.3, and 3.6 millivolts. From the observations made immediately before and after the above measurements, changes of +.11, −0.35, and +0.21 millivolt were found for the cells, indicating that trustworthy results are hardly to be expected after the relatively rapid temperature changes on account of hysteresis. Large differences were also found by Barnes in the values of four other cells rapidly brought from 15° to 0°.

A further study of all the Reichsanstalt cells with 1 : 6 amalgam was made by Jaeger[9] in 1896, but the results were not published until after the appearance of Barnes's paper. Thirty-four cells were measured in an oil bath at 0°, at which temperature 14 were found to differ from the values calculated by the formula of Jaeger and Wachsmuth by less than 0.02 per cent, while the observed values of the remaining 20 were from 0.02 to 0.23 per cent (mean, 0.09 per cent) too high. The observations were made several hours after the bath had been cooled to 0°. In addition, three of the cells, one belonging to the first group and two to the second, were measured at temperatures between −16°, the freezing point of saturated cadmium sulphate solution, and 40°. The two "abnormal" cells showed maximum differences from the temperature formula of +4.5 and +3.2 millivolts, while the cell with "normal" behavior showed a maximum difference of −0.57 millivolt (at − 11°.5). The observations extended over a period of two weeks, during which the cells were generally allowed to come to room temperature overnight. After the first observation on each day, usually at room temperature, the cells were subjected to relatively large temperature changes. An inspection of the tables and the plotted results, particularly consecutive measurements on the same day, in which the temperature was changing slowly after a relatively large change, seems to indicate that the discrepancies are due to hysteresis.

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[8] Jour. Phys. Chem., 4, p. 339; 1900.

[9] Ann. d. Phys., 4, p. 123; 1901.

A very extensive series of observations between 0° and 30° was made by Jaeger and Lindeck[10] in 1899, with the object of verifying the formula obtained by Jaeger and Wachsmuth. Fifty-six series of measurements were made on eight old cells with 14.3 per cent amalgam and twelve new cells with 13 per cent amalgam, together with five unsaturated Weston cells and five Clark cells with which we are at present not concerned. Six of the old Weston cells had already been classed by Jaeger as abnormal at lower temperatures, while the two others had apparently not been previously investigated at such temperatures. As might have been expected, these all showed very marked irregularities, particularly at 0°, at which all the cells were kept for two periods of eight and eleven days, twenty-three series of measurements being made. The new cells, however, showed much smaller changes under these conditions. A second series of measurements was later made on five cells with 13 per cent amalgam, set up at the same time as the twelve above referred to, and on two lots of twenty new cells set up with 12 and 13 per cent amalgam. The results obtained at the Reichsanstalt will be discussed below in their relation to those found in the present investigation, particularly as the data obtained by Jaeger and Lindeck were not employed in the derivation of a temperature formula, being merely used for verifying the previous work at the Reichsanstalt.

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[10] Zs. Instk., **21**, p. 33 and p. 65; 1901. Ann. d. Phys., **5**, p. 1; 1901.

Two further series of measurements by Barnes and Lucas[11] on four new "moist crystal" cells are summarized below. In the published tables only the uncorrected potentiometer readings are given, so that the reductions were made on the assumption that the values of two portable cells of the Weston Electrical Instrument Company, measured at the same time, were the same for both sets of observations.

TABLE I.

Results of Barnes and Lucas.

Cells	Series	Temperature range	Mean temperature coefficient		
			Observed	Range	Calculated
Cd 16	1	14°6–27°9	−.0050%	.0045–.0057%	−.0040%
17		14°6–19°9	.0030	.0024–.0030	.0035
18		19°9–27°9	.0064	.0064–.0081	.0043
19					
Cd 16	2	13°6–29°5	−.0058	.0057–.0060	−.0040
17		13°6–18°7	.0024	.0020–.0028	.0033
18		18°7–29°5	.0075	.0070–.0078	.0044

It will be seen that the results can not be represented by a linear temperature coefficient; that the differences in temperature coefficient between the individual cells are large, and that the values for the lower range are *smaller* than those called for by the generally accepted temperature formula (column 6). The results can, therefore, not be regarded as confirming the previous work of Barnes as to the temperature coefficient being linear, nor do they establish the opinion expressed by Barnes that the differences in results are due to the greater rapidity with which saturation equilibrium is established in the case of the "moist crystal" cells in comparison with cells of the usual construction.

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[11] Jour. Phys. Chem., 8, p. 196; 1904.

Smith[12] at the English National Physical Laboratory has recently found, as the result of measurements on six selected cells between 10° and 30°, the formula:

$$E_t = E_{17} - 0.0000345\,(t-17) - 0.00000066\,(t-17)^2$$

The temperature was changed at the rate of 1° per hour and maintained constant by means of a thermo-regulator at least one hour before the measurements were made, the cycle of temperatures being repeated three times. The data on which the formula is based are not published, but judging from the results obtained at the Bureau on cells obtained by exchange from the National Physical Laboratory, some hysteresis may be expected under the above conditions. On account of the close agreement, in the range covered, of Smith's formula with that of the Reichsanstalt, the universal adoption of the latter is recommended by him.

THE PRESENT INVESTIGATION.

The Cells.—In the present investigation it was thought desirable to include a large number of cells, so as to determine whether any differences in the temperature coefficient are introduced by variations in the methods of preparation and purification of the materials. Altogether 200 cells were studied, including nearly all those previously reported on,[13] a number of exchange cells obtained from other investigators, and a considerable number of other cells set up at the Bureau to determine the influence of variations in the specifications which might possibly affect the E. M. F.

The Comparing Baths.—The cells were contained in two automatically regulated petroleum baths, each provided with an efficient stirrer, a thermo-regulator sensitive to within 0°.01 C, resistance coils for electric heating, and coils for water circulation for temperatures below that of the room. As described in the article referred to, the cells were mounted in such a way as to be free from direct influence of the heating or cooling coils. The oil in which the cells were immersed covered them to a depth of several centimeters.

The Temperature Cycle.—The measurements were made at every 5° for a complete cycle from 0° to 40°, beginning and ending at

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[12] Phil. Trans. Roy. Soc., **207**, p. 411; 1908.

[13] Wolff and Waters, this Bulletin, **4**, p. 1; 1907. (Reprint No. 70.)

25°, the lower temperatures being taken first, and a smaller cycle from 15° to 25°. As it was deemed desirable to repeat the 0° measurements with ascending temperatures, both baths were maintained at a temperature of approximately −4° for twenty-four hours between the two series of observations made at 0°. The regulator was set so that the temperature at which the baths regulated was within ± 0°.02 of the temperature desired, except for 10 sets in bath 1 and 4 sets in bath 2, so that, by applying a slight correction, the values observed could be reduced to the temperatures decided upon, thus making possible the application of a least-squares formula to the reduction of the results.

Regulation.—The temperature of the baths was maintained constant well within 0°.02 C for at least forty hours at each temperature. No difficulty was of course experienced in regulating the baths at temperatures above that of the room (about 23°), but at 20° and 15° great difficulty was at first found, mainly on account of the cooling coils of both baths being in parallel. In the first measurements at these temperatures the cooling was effected by a very slow stream of tap water which had been passed through a lead coil surrounded by crushed ice. (The use of ice, even for these temperatures, was made necessary on account of the temperature of the tap water rising above 20° during the night.) The difficulty was traced to the separation of air bubbles from the water, which, collecting in the cooling coils, gradually obstructed the circulation and thus disturbed the regulation. Various schemes were tried, but the most satisfactory method consisted in pumping water from a tank maintained several degrees below the temperature at which the baths were to be regulated. For this purpose a slow flow of ice water into the tank sufficed, the level being maintained by means of an overflow. A direct circulation of ice water was employed for the auxiliary cooling at 10°, while the lower temperatures, 5°, 0°, and −4°, were obtained by circulating a brine solution of suitable concentration, the brine being kept approximately at its freezing point by means of a coil in the same tank through which much cooler brine was circulated at a desired rate by the aid of a by-pass. In the few cases of defective regulation, occasioned by the separation of air bubbles, broken belts, etc., a new start was made and the cells were maintained at the desired temperatures for at least forty hours longer.

Method of Observation.—As the apparatus and method of observation have already been fully described,[14] only an outline will be here given. Each of the 200 cells was measured in opposition to a particular cell in a separate bath maintained at 25°. The latter was compared in a similar manner with 14 cells in the same bath, composing two sets of 7 to which all the results were reduced. In addition, the 7 cells of one of the reference sets, connected in series, were directly compared by the opposition method with five sets of 7 cells in each of the two baths. The slight differences found by the two methods, averaging ±2 microvolts per cell, exceeded 3 microvolts in only ten series.

Altogether 58 observations were made on each of the 200 cells, and 19 others kept at 25° in completing the two cycles. At each step at least two series of measurements were made, the first on the day after the temperature was changed, and the last after the cells had been at constant temperature at least 40 hours.

Including the series measurements over 12,000 separate observations were therefore made, about 3500 being employed in the calculations for the cells tabulated below.

Temperature Measurements.—On account of the relatively small temperature coefficient of the Weston cell all temperature measurements were made with sufficient accuracy by means of mercurial thermometers, graduated to tenths of a degree, for which the corrections were determined to 0°.01 by another division of the Bureau.

Reduction of Results.—In working up the results three corrections were applied to each observation. The first to reduce all results to a common basis, the mean value of the reference cells; the second to allow for the very slight temperature differences of the baths from their nominal values, and the third one-half the mean difference found by comparing the cells separately and in sets of seven in series. The two latter corrections rarely exceeded 2 microvolts.

A preliminary temperature formula was first calculated by the least-squares method from the means of all the observations on 10 cells selected on account of their small hysteresis as indicated by the agreement of the observations with ascending and descending temperatures. Although these values agreed within 5 microvolts of each other it was found that they could not be represented as well as might be expected by a quadratic formula. A three-term formula was therefore tried, which was found to give residuals of ± 2 microvolts. With the approximate temperature formula thus found,

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[14] Wolff and Waters, this Bulletin, **4**, p. 39; 1907.

$$E_t = E_{20} - 0.0000406\,(t - 20) - 0.000000939\,(t - 20)^2 + 0.000000009\,(t - 20)^3$$

the residuals for each cell at each temperature could be computed, and from these, corrections to the above coefficients for the individual cells were obtained. The residuals found at each temperature for the ten cells by the employment of the corrected coefficients thus obtained agreed with each other to within ±2 microvolts. This, with the close agreement of the third-term coefficients of the individual cells, seemed to fully justify the use of a three-term formula.

The results for a considerable number of cells were worked up by taking into account all the observations made, but an inspection of the latter indicated the presence of hysteresis in some of the cells, successive readings at the same temperature, in ascending and descending portions of the cycle, in both cases approaching the same common value, as was shown also in a preliminary study of some of the cells when subjected to rapid as well as prolonged temperature changes. It was therefore thought best to reject all but the last measurements made at each step of the temperature cycle. These measurements, made at least 40 hours after the temperature was changed by 5°, were given equal weight in computing the corrections to the coefficients of the trial formula.

DISCUSSION OF THE RESULTS.

In Tables II to VIII below the results for 178 of the 200 cells are given. The 22 omitted are made up of 7 exchange cells, 5 cells containing oil admitted through defective seals, 8 cells of a series set up to determine possible electromotive differences with samples of cadmium sulphate obtained from different sources and purified by different methods, together with 2 cells set up with white mercurous sulphate paste which had been employed in special cells,[15] constructed to make possible a study of the influence of a thorough and continuous mixing of the ingredients of the paste by rotation of the cell about its axis. In all these cases the differences between the values obtained with ascending and descending temperatures were very large and irregular, and in addition the majority exhibited abnormal values before beginning the temperature cycle. The results seem to indicate a lag in the establishment of concentration equilibrium of the mercury ions, as abnormally high values are obtained at lower temperatures. This behavior is particularly marked in the cells set up to determine differences in the electromotive properties of different samples of cadmium sulphate, in which, in order to eliminate differences arising from the mercurous sulphate, relatively small quantities of this material had to be employed, owing to the supply of the particular sample being limited. The paste, in most cases, therefore, contained a relatively large excess of cadmium sulphate crystals. In two of the exchange cells set up at the University of Wisconsin, and which also exhibited abnormal and variable values before beginning the temperature cycle, there does not appear to be an excess of cadmium sulphate crystals in the paste. Among the other exchange cells excluded are one of the two obtained from the Reichsanstalt, in both of which the paste was probably disturbed in transit, two cells obtained from the National Physical Laboratory in 1904, and two set up by Prof. Hulett, in both of which oil was admitted through defective seals. All these cells exhibited abnormal values, particular attention being called to the behavior of F–7, set up by Prof. Hulett in 1904, and which belonged to the series, the behavior of which led him to suspect that the chemical system of the Weston cell is in unstable equilibrium.

In the following tables the cells are grouped according to the method of preparation or treatment of the mercurous sulphate employed in the cell.[16] Column 1 gives the number of the cell; column 2 the designation of the sample, the letter indicating the method employed; column 3, the mean of the total range in value of each cell at each temperature, all observations being included; column 4, the mean range, rejecting all but the last measurements of each set at each temperature; column 5, the maximum range corresponding to column 4; column 6, the square root of the mean squares of the residuals obtained from the temperature formula deduced for the particular cell, and columns 7, 8, and 9, corrections[17] in microvolts to be applied algebraically to the coefficients of the final formula given below.

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[15] Wolff and Waters, this Bull., 4, p. 81; 1907.

[16] This Bulletin, 4, p. 41; 1907.

TABLE II.

Cells Set Up with Electrolytic Mercurous Sulphate Made with Old Apparatus.

[Results expressed in microvolts.]

Cell	Mercurous Sulphate	Mean Range All Observations	Mean Range	Maximum Range	$\sqrt{\frac{\Sigma r^2}{9}}$	$\Delta\alpha$	$\Delta\beta$	$\Delta\gamma$
			Last Observations at Each Temperature					
*75	a_1	75	37.2	100	11.2	+ 8.02	−.037	−.0034
*76	a_4	23	8.3	17	2.6	+ 1.55	−.026	+.0005
77	a'_4	7	2.6	7	1.1	− .50	−.054	+.0028
*64	a_5	50	22.5	66	7.2	+ 4.23	−.047	+.0006
*65	a_6	27	12.8	30	3.1	+ 1.98	−.026	+.0010
*40	a_7	127	63.3	145	....	+19.67	−.463	−.0209
*8a	a_7	99	58.3	100	13.0	+ 6.15	+.062	−.0048
*66	a'_7	33	13.7	39	4.1	+ 2.30	−.014	+.0010
*41	a_6	84	38.6	82	8.5	+ 5.28	−.079	+.0010
*9a	a_6	42	26.2	48	8.8	+ 2.20	+.057	−.0038
67	a'_8	20	8.5	20	11.0	+ 1.31	−.005	−.0024
*42	a_9	52	35.0	46	7.2	+ 3.35	+.008	−.0010
11a	a_9	11	2.4	4	1.9	+ .79	−.018	+.0002
68	a_{10}	21	9.5	20	3.0	+ 1.29	−.016	+.0007
70	a'_{10}	6	4.2	9	2.4	− .56	+.004	+.0008
71	a_{11}	6	3.2	9	2.7	− .44	+.015	.0000
*72	a'_{11}	28	12.9	25	3.1	+ 1.62	+.019	+.0008
73	a_{12}	7	3.0	5	2.4	− .03	−.001	+.0008
74	a_{13}	14	10.0	20	2.3	− .26	+.012	+.0007
110	a_{13}	7	2.8	10	1.4	− .45	+.010	+.0002
112	a_{13}	14	9.6	11	1.8	− .09	.000	.0000
173	a_{13}	4	2.0	4	1.7	− .65	+.006	+.0004

* Excluded in taking means.

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[17] These were first calculated as corrections to the coefficients of the trial formula and were so given in a paper read before the American Electrochemical Society, Trans., **18**, pp. 197 et seq; 1908.

TABLE III.

Cells Set Up with Electrolytic Mercurous Sulphate Made with New Apparatus.

[Results expressed in microvolts.]

Cell	Mercurous Sulphate	Mean Range All Observations	Mean Range	Maximum Range	$\sqrt{\frac{\Sigma r^2}{9}}$	$\Delta\alpha$	$\Delta\beta$	$\Delta\gamma$
			Last Observations at Each Temperature					
*26	b_1	30	12.9	25	3.9	+ 1.44	−.039	+.0012
51	b_1	9	5.2	10	2.4	+ .15	−.009	+.0005
27	b_2	4	1.5	3	2.5	− .13	−.008	+.0006
52	b_2	15	9.2	22	3.8	+ .84	−.039	+.0001
*28	b_3	28	11.1	26	3.6	+ 1.70	−.024	+.0006
61	b_4	19	6.9	21	2.5	− .25	+.013	+.0004
62	b_5	11	7.4	16	2.2	− .05	−.006	.0000
63	b_6	10	5.6	11	2.1	− .20	−.001	+.0008
29	b_7	17	6.9	12	2.4	+ .52	−.014	+.0008
3a	b_7	8	3.2	7	2.2	− .36	+.014	+.0004
167	b_7	5	2.2	7	4.2	− .32	+.009	−.0004
168	b_7	8	3.6	10	2.2	− .08	−.012	+.0006
168a	b_7	11	6.6	15	2.7	− .44	+.018	−.0003
30	b_8	23	7.4	12	2.0	+ 1.11	−.004	+.0007
*193	b_8	42	22.9	38	5.3	+ 2.14	.000	+.0008
46	b_8	9	3.6	5	9.7	− 1.30	+.010	+.0050
46a	b_8	6	3.2	8	2.1	− .31	+.002	+.0007
*46b	b_8	30	15.8	35	9.3	+ .79	+.092	−.0051
31	b_9	11	5.8	10	2.3	+ .52	−.023	+.0004
47	b_9	6	2.6	4	2.3	− .39	−.010	+.0011
32	b_{10}	3	1.8	5	1.8	− .83	−.002	+.0010
4a	b_{10}	9	5.1	11	4.2	− .32	+.022	−.0007
33	b_{11}	11	3.8	9	1.9	+ .25	−.005	+.0006
48	b_{11}	14	6.3	9	4.0	+ .27	−.027	+.0002
34	b_{12}	11	5.1	8	3.2	+ .66	−.026	+.0010
49	b_{12}	15	9.2	19	2.4	+ .18	−.005	+.0006
6a	b_{13}	4	2.2	5	2.1	− .54	+.005	+.0006
176	b_{13}	12	4.5	12	2.5	− .36	+.008	+.0004
151	b_{14}	6	2.6	8	1.8	− .60	+.004	+.0008
152	b_{14}	6	2.1	4	1.9	− .61	+.002	+.0007
153	b_{14}	4	2.6	7	2.1	− .48	+.007	+.0004
154	b_{14}	3	1.1	2	2.1	− .15	+.004	+.0010
155	b_{14}	4	1.7	6	2.1	− .18	+.007	+.0007
156	b_{14}	4	2.3	6	2.0	− .27	+.006	+.0006
164	b_{14}	10	3.2	9	2.1	− .39	+.009	+.0010
165	b_{14}	4	2.5	8	2.6	− .62	+.014	+.0003

* Excluded in taking means.

TABLE III—Continued.

Cells Set Up with Electrolytic Mercurous Sulphate Made with New Apparatus—Continued.

Cell	Mercurous Sulphate	Mean Range All Observations	Mean Range	Maximum Range	$\sqrt{\frac{\Sigma r^2}{9}}$	$\Delta\alpha$	$\Delta\beta$	$\Delta\gamma$
			Last Observations at Each Temperature					
166	b_{14}	10	5.2	10	2.4	− .29	+.017	−.0004
111	b_{14}	8	4.1	13	2.2	− .51	+.012	+.0002
113	b_{14}	7	2.5	8	1.7	− .39	+.002	+.0006
171	b_{14}	5	2.1	5	2.1	− .73	+.001	+.0008
172	b_{14}	5	2.0	5	1.6	− .64	+.004	+.0003
37	b_{15}	5	2.5	8	1.6	− .55	.000	+.0009
*15a	b_{16}	25	12.5	35	7.7	+ .76	+.050	−.0043
69	b_{17}	8	3.8	11	2.2	− .51	+.007	+.0005
59	b_{18}	7	2.6	9	1.8	− .05	+.006	+.0006
60	b_{19}	8	4.2	13	1.9	− .19	+.004	+.0006
*50	b_{20}	34	13.3	46	4.0	+ 2.97	−.016	−.0004
*53	b_{21}	18	11.2	26	3.3	+ .25	+.002	−.0004
54	b_{22}	12	7.8	13	2.4	− .14	+.001	+.0001
55	b_{23}	9	5.6	12	2.3	− .46	+.017	.0000
56	b_{24}	5	2.4	6	1.9	− .45	+.018	−.0001
57	b_{25}	8	3.1	8	2.3	+ .17	+.016	−.0002
178	b_{23}	8	3.3	12	4.2	− .34	+.032	−.0016
*P9	b_{26}	32	12.1	39	6.6	+ .08	+.076	−.0027
*P11	b_{26}	99	20.9	125	13.5	+ .33	+.177	−.0063
*P12	b_{26}	23	7.1	30	8.9	+ .37	+.069	−.0046
179	b_{27}	7	4.1	17	4.1	− .48	+.030	−.0014
187	b_{27}	6	2.4	5	1.8	− .41	+.008	+.0002
188	b_{27}	7	2.6	5	3.7	− .36	+.031	−.0016
180	b_{28}	9	2.9	8	5.9	− .09	+.046	−.0028
190	b_{29}	11	5.5	9	2.0	− .09	−.006	+.0002
191	b_{30}	9	4.9	11	1.6	− .30	+.024	−.0010
192	b_{31}	10	3.0	7	2.5	− .22	+.027	−.0011

* Excluded in taking means.

In view of the unmistakable hysteresis found in some of the cells for which the data are tabulated above, not all were included in calculating the coefficients of the final temperature formula. An arbitrary criterion for rejection was decided upon, those cells being excluded in which the mean and maximum difference with ascending and descending temperatures, taking the last measurements of each set at each temperature, exceeded 10 and 25 microvolts respectively. (Columns 4 and 5.) Even on this basis 137 cells, over two-thirds of the whole number taken, passed inspection. Among

TABLE IV.

Cells Set Up with Chemically Prepared Mercurous Sulphate.

[Results expressed in microvolts]

Cell	Mercurous Sulphate	Mean Range All Observations	Mean Range	Maximum Range	$\sqrt{\frac{\Sigma r^2}{9}}$	$\Delta\alpha$	$\Delta\beta$	$\Delta\gamma$
			Last Observations at Each Temperature					
86	c	3	1.3	3	1.8	− .46	−.002	+.0007
99	d_1	7	4.3	8	2.6	+ .45	−.030	+.0007
99a	d_1	10	5.9	8	1.8	+ .14	−.005	+.0005
84	d_2	7	4.4	11	1.7	− .67	+.002	+.0011
84a	d_2	4	1.9	5	1.6	− .53	+.001	+.0010
*43	e_1	30	13.5	31	5.4	+ .79	+.042	−.0026
1a	e_1	20	8.8	16	3.5	+ .82	+.007	−.0008
157	e_1	15	7.8	20	7.1	+ .39	+.067	−.0042
158	e_1	8	2.6	8	2.6	− .41	+.015	−.0002
160	e_1	9	6.2	25	1.7	− .29	+.001	+.0004
161	e_1	5	2.9	11	2.3	− .51	+.002	+.0002
162	e_1	5	1.7	8	2.5	− .41	+.012	−.0002
103	e_2	7	3.4	6	1.1	− .69	+.015	+.0008
2a	e_2	17	9.8	17	2.3	− .12	−.014	+.0012
44	e_3	14	5.8	11	5.5	+ .19	+.043	−.0029
104	e_3	4	2.1	9	1.9	− .63	+.009	+.0004
105	e_4	6	2.6	4	2.3	− .25	+.098	+.0002
106	e_5	7	2.9	8	2.0	− .24	+.004	+.0003
106a	e_5	5	1.2	3	1.7	− .41	+.010	+.0002
107	e_6	6	3.1	9	1.5	+ .23	+.097	+.0004
98	f_1	10	2.5	8	4.0	− .08	+.036	−.0021
97	f_2	4	1.1	2	2.3	− .21	+.002	+.0001
*100	g_1	56	23.6	53	7.0	+5.09	−.030	−.0011
*100a	g_1	68	37.6	66	9.2	+3.34	−.022	+.0004
*101a	g_2	127	90.6	160	11.7	+4.83	−.020	−.0020
*163	g_3	40	16.8	29	7.1	+3.06	−.044	−.0023
*95	h_1	41	21.1	46	6.5	+1.91	+.038	−.0042
*96	h_2	49	27.2	43	10.6	+3.56	+.046	−.0079
85	h_3	5	2.1	5	1.9	− .49	−.004	+.0006

* Excluded in taking means.

TABLE V.

Cells Set up With Specially Treated Commercial Samples of Mercurous Sulphate.

[Results expressed in microvolts.]

Cell	Mercurous Sulphate	Mean Range All Observations	Mean Range	Maximum Range	$\sqrt{\frac{\Sigma r^2}{9.}}$	$\Delta\alpha$	$\Delta\beta$	$\Delta\gamma$
			Last Observations at Each Temperature					
93	i_1	14	5.5	8	2.6	+ .72	+.001	−.0007
94	i_2	5	1.4	3	2.1	− .19	−.006	+.0005
78	k_1	4	2.3	7	1.5	+ .18	+.012	+.0007
79	k_2	4	1.6	4	1.1	− .06	+.011	+.0006
80	k_3	4	1.6	4	1.6	− .46	+.003	+.0004
81	k_4	6	2.2	5	0.6	− .16	+.011	+.0006
82	k_5	7	3.9	8	2.2	− .09	−.002	+.0003
108	l_1	7	3.1	8	1.6	+ .72	+.010	+.0002
116	l_2	16	7.1	24	3.0	+ .23	.000	+.0002
109	l_3	6	2.2	4	1.5	+ .85	−.003	+.0007
114	l_4	6	2.8	5	2.6	+ .09	−.005	+.0002
115	l_5	5	1.7	4	1.9	+ .93	+.005	−.0001
117	m_1	8	5.1	8	2.8	+1.35	+.001	−.0006
119	m_2	8	3.2	9	2.1	.00	+.004	.0000
118	m_3	8	4.0	25	3.0	+ .22	−.001	−.0004
120	m_4	9	4.0	16	3.8	− .11	+.009	−.0005
90	n_4	7	2.2	6	2.3	+ .28	+.015	+.0003
91	n_5	6	1.8	3	1.7	+ .58	−.027	+.0004
92	n_6	6	2.1	8	2.1	+ .03	−.013	+.0008

TABLE VI.

Exchange Samples of Mercurous Sulphate.

[Results expressed in microvolts.]

Cell	Mercurous Sulphate	Mean Range All Observations	Mean Range	Maximum Range	$\sqrt{\frac{\Sigma r^2}{9}}$	$\Delta\alpha$	$\Delta\beta$	$\Delta\gamma$
			Last Observations at Each Temperature					
39	Hulett I	15	6.2	12	2.4	−.01	−.002	+.0007
141a	Hulett II	5	2.6	6	1.7	−.45	+.009	+.0003
142	"	7	3.2	7	1.5	−.20	.000	.0000
143	"	4	2.4	9	1.9	−.46	−.007	+.0008
144	"	5	3.6	14	2.1	−.35	−.011	+.0009
145	"	5	3.0	4	4.0	−.33	.000	+.0004
146	"	11	6.2	19	1.9	−.27	+.016	−.0003
147	"	5	1.9	6	1.8	−.45	.000	+.0004
148	"	8	3.5	7	1.7	+.01	−.008	+.0001
149	"	6	2.7	5	1.9	−.07	−.005	+.0001
150	"	5	1.6	5	2.9	−.59	−.034	−.0001
†169	"	5	2.2	10	1.7	−.23	−.009	+.0008
†170	"	11	7.4	18	3.3	−.15	+.028	−.0004
*189	Guthe	23	13.3	28	6.4	+.79	+.060	−.0040

* Excluded in taking means.

† Clear crystal cells.

TABLE VII.

Exchange Cells.

[Results expressed in microvolts.]

Cell	Mercurous Sulphate	Mean Range All Observations	Mean Range	Maximum Range	$\sqrt{\frac{\Sigma r^2}{9}}$	$\Delta\alpha$	$\Delta\beta$	$\Delta\gamma$
			Last Observations at Each Temperature					
O_2	P.T.R.	13	7.8	12	4.3	+1.70	−.059	+.0012
*4	L.C.E.	92	37.6	108	11.6	− .43	−.158	+.0044
*6	L.C.E.	94	36.3	86	7.9	+1.01	−.180	+.0063
*H26	N.P.L.	40	25.0	46	3.1	+1.08	−.055	+.0019
*C19	N.P.L.	147	82.5	214	6.8	+2.90	−.405	+.0154
P53	N.P.L.	19	8.0	22	8.5	− .65	−.050	+.0016
*P54	N.P.L.	48	26.5	63	2.5	+ .98	−.021	+.0042
K11	N.P.L.	4	1.7	5	1.7	− .15	+.007	+.0002
K15	N.P.L.	4	1.5	5	1.4	− .21	+.005	+.0005
*H.B.9	N.P.L.	46	11.3	28	7.6	+1.70	−.070	+.0041
57	Wold	9	2.4	5	1.5	− .14	+.004	+.0011
†*O_4	Hulett	43	23.9	82	5.7	−2.58	+.016	+.0045
†A_8	Hulett	14	8.5	13	3.2	− .05	+.010	.0000
†*A_{62}	Hulett	26	12.3	28	3.1	+1.79	−.016	+.0005
†C_1	Guthe	25	9.5	20	4.7	− .63	+.023	+.0005
†*5	Carhart	19	10.6	57	4.1	+ .14	+.039	−.0014
†*8	Carhart	39	23.6	36	3.0	+1.98	−.038	+.0011
1	Strong	21	9.1	15	2.1	+ .33	+.008	+.0010
*2	Strong	31	12.8	29	5.3	+1.12	+.012	+.0007
3	Strong	15	5.7	14	2.5	− .05	+.017	+.0010

* Excluded in taking means. † Clear crystal cells.

TABLE VIII.

Cells Set Up with Different Samples of Cadmium Sulphate.

[Results expressed in microvolts.]

Cell	Mercurous Sulphate	Mean Range All Observations	Mean Range	Maximum Range	$\sqrt{\frac{\Sigma r^2}{9}}$	$\Delta\alpha$	$\Delta\beta$	$\Delta\gamma$
			Last Observations at Each Temperature					
121	b_{15}	6	2.6	9	3.1	−.48	+.001	+.0006
122	b_{15}	5	2.1	7	2.1	−.61	+.011	+.0003
123	b_{15}	4	2.1	6	2.3	−.57	+.006	+.0006
124	b_{15}	10	5.4	11	2.3	−.20	−.005	.0000
125	b_{15}	11	5.6	14	2.7	+.24	−.007	−.0004
126	b_{15}	6	1.9	4	2.2	+.53	+.011	−.0002
127	b_{15}	7	4.4	10	1.9	−.50	+.007	+.0002
128	b_{15}	8	4.0	9	1.8	−.57	+.016	+.0003
133	b_{15}	10	5.5	13	2.0	+.21	−.010	+.0007
*136	b_{15}	33	16.2	44	13.8	+.46	−.106	+.0001
138	b_{15}	14	8.3	22	2.9	+.31	−.015	−.0004

* Excluded in taking means.

those excluded are 11 of the 22 cells given in Table II, in all of which relatively coarse-grained white samples of mercurous sulphate were employed; 10, including 3 portables of special construction, of the 63 of Table III, all set up with gray samples of electrolytic mercurous sulphate; 7 of the 29 cells of Table IV, nearly all set up with coarse-grained white samples of chemically prepared mercurous sulphate; 1 of the 14 cells set up with exchange samples of mercurous sulphate; 11 of the 20 exchange cells and 1 of the 11 cells of Table VIII.

The corrections to the coefficients of the trial formula were calculated from the means of the separate residuals of the 137 cells, equal weight being given to each cell. Applying the corrections thus found, we obtain

$$E_t = E_{20} - .00004075\,(t-20) - .000000944\,(t-20)^2 + .0000000098\,(t-20)^3$$

the final temperature formula in terms of which all the results are expressed.

Table IX shows the agreement of the corrections to the coefficients of the final formula, for the separate groups corresponding to Tables II to VIII inclusive, excluding the 41 cells just referred to.

Columns 3, 4, and 5 were obtained from the means of the residuals of the individual cells composing the group, as calculated from the final temperature formula, each cell being given equal weight.

TABLE IX.

Mean Corrections in Microvolts to Coefficients of Final Formula.

Table	Number of cells	$\Delta\alpha$	$\Delta\beta$	$\Delta\gamma$
II	11	+.15	−.007	+.0003
III	53	−.09	+.003	−.0001
IV	22	−.11	+.003	−.0001
V	19	+.36	−.002	.0000
VI	13	−.14	−.003	.0000
VII	9	+.22	−.006	+.0007
VIII	10	−.04	−.001	−.0001
II–VIII	137	+.01	.000	.0000

The agreement of mean corrections for the separate groups is excellent. Particular attention should be called to the results for the exchange cells (Table VII), which do not differ appreciably from the rest.

The agreement of the separate cells with the final temperature formula is further shown by Tables X and XI, which give respectively the mean residuals and mean deviation for each group at each temperature. The last line of each table gives the corresponding values for all the cells on which the final temperature formula is based.

TABLE X.

Mean Residuals for Each Group given by Final Temperature Formula.

[Results expressed in microvolts.]

Table	Number of cells	0	5	10	15	20	25	30	35	40
II	11	+8.4	+6.4	+4.5	+1.9	− 0	−2.0	+0.5	+1.9	−3.4
III	53	−3.7	+1.8	+2.2	−0.3		+0.9	+3.4	+6.1	+1.5
IV	22	−3.7	+2.0	+1.2	+0.4		+1.8	+4.0	+5.3	+2.5
V	19	+6.9	+9.3	+5.1	+1.8		−2.3	−1.3	−1.7	−6.8
VI	13	−1.6	+1.9	0.0	−0.3		+0.5	+2.9	+5.3	+2.9
VII	9	+9.9	+8.5	+3.2	−0.3		−1.0	−0.3	−1.0	−6.4
VIII	10	−2.4	+3.2	+0.8	−1.1		−0.1	+2.8	+3.8	+1.8
	137	−1.4	+2.4	+1.0	−1.2	−1.4	−0.7	+1.0	+2.4	−1.6

TABLE XI.

Mean Deviation of the Individual Cells from Final Temperature Formula.

[Results expressed in microvolts.]

Table	Number of cells	0	5	10	15	20	25	30	35	40
II	11	16.4	10.8	6.0	4.4	0	3.3	4.8	7.4	9.0
III	53	12.6	7.8	4.4	2.3	0	2.0	4.2	6.4	4.8
IV	22	12.4	5.2	3.0	1.5	0	2.6	4.5	5.6	4.8
V	19	8.9	9.0	5.8	2.3	0	2.8	3.7	5.8	8.4
VI	13	5.9	3.6	2.3	1.1	0	1.4	2.9	5.4	5.0
VII	9	16.9	9.1	6.2	3.4	0	3.3	5.1	5.9	7.5
VIII	10	7.5	6.9	4.0	2.7	0	2.3	4.6	5.8	5.8
II-VIII	137	11.6	7.3	4.4	2.3	0	2.4	4.2	6.1	5.9

Since all the cells of the Bureau of Standards were set up with 12.5 per cent amalgam, it seems probable that the cause of hysteresis observed in almost one-third of the whole number taken lies elsewhere. The agreement of the cells set up with large crystals of cadmium sulphate (marked † in the tables) with the remainder in which finely crushed cadmium sulphate crystals were employed, indicates that the irregularities are connected with the paste limb. This conclusion is strengthened by the behavior of the cells in which scant quantities of mercurous sulphate were employed in making up the paste. A further study of the abnormal cells will be undertaken with the object of establishing definitely the cause of their irregularities by an analysis of their ingredients, by setting up new cells to verify the conclusions which may thus be reached and by a redetermination of the temperature coefficient and behavior of amalgams, particularly at lower temperatures.

After making an analysis of the results here given, it seemed probable that the cells exhibiting hysteresis should also have shown abnormal initial values, as the laboratory temperature generally differed from 25°, the temperature of the comparing baths. An examination was therefore made of the observations on the particular cells immediately after they were set up, and in nearly every case evidence of hysteresis was indeed found. This was commented on in the paper on the Clark and Weston cells frequently referred to, where attention was called to the fact that such cells tended to approach normal values. The exceptions to this rule are nearly all found in the cells of Table V, set up with commercial samples of mercurous sulphate treated with sulphuric acid under various conditions, some of which exhibited relatively high initial values, which, however, have persisted without appreciable change. Not one of these cells shows appreciable hysteresis, indicating that the high values are probably due to residual impurities.

TEMPERATURE OF MAXIMUM ELECTROMOTIVE FORCE.

In making the first sets of measurements at 5° and 0° it was found that the temperature coefficient unmistakably changed sign between these limits, which was verified by the subsequent observations with ascending temperatures.

The differentiation of the corrected temperature formula gave + 3°.0 as the temperature of maximum electromotive force and zero temperature coefficient. Since this corresponds approximately to the temperature of saturation of the cadmium sulphate solution employed in the portable cells of the Weston Electrical Instrument Company, which have a practically negligible temperature coefficient, the solubility curve of cadmium sulphate was computed by the least squares method from the data of Mylius and Funk.[18] The observations, while not as concordant as might have been desired, ranged from −10° to +60° and led to a formula giving a minimum solubility at +1°. The very close agreement of these values has suggested further experimental work which it is hoped can shortly be undertaken.

THE BEHAVIOR WITH RAPID TEMPERATURE CHANGES.

In March, 1907, some preliminary measurements were made on over 90 cells contained in one of the baths to determine the influence of relatively rapid temperature changes. The temperature was reduced from 25° to 20° in 2½ hours, and while the temperature was falling the difference between 7 of the cells connected in series and 7 of the reference cells contained in another bath was measured. In addition, one partial and three complete sets of observations were made on the separate cells while at the lower temperature, at which they were left for several weeks. The temperature was then rapidly increased to 25° and three further observations made, the first within two hours. Additional observations on a number of normal cells transferred back and forth from the 25° bath to the 20° bath showed that, even with very abrupt changes in temperature, most of the cells reached a practically constant value well within an hour, with ascending temperatures. On lowering the temperature a longer time was required for the attainment of the normal values.

It was found that nearly all the cells agreed quite as well at 20° within 24 hours after the temperature was changed as at 25°; that the corresponding difference in electromotive force was approximately 20 microvolts greater than that given by the formula of Jaeger and Wachsmuth, and that nearly all the cells assumed their normal values within 24 hours after they were brought back to 25°. The series observations, made while the temperature was being changed and since reduced by the new formula above proposed, show further that the mean value of the 7 cells selected agreed with the calculated values to within ten microvolts, even though the temperature was changed at the rate of 2°.5 per hour.

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[18] Ber., **30**, p. 824; 1897.

In December, 1907, further preliminary measurements were made over a wider range on 7 other cells selected as representative. They were first separately transferred from a 25° bath to another regulated at 4°.8. The results are given in the curves of Fig. 1, in which the differences between the individual cells and the values calculated by the proposed formula are given. From these it will be seen that one of the cells showed considerable hysteresis, while the remainder practically attained their normal values within 5 hours after an abrupt change in temperature of over 20°.

Five further sets of observations on the same cells were made under the following conditions:

While bringing the cells back to 25°, Curve 2, Fig. 2.
While changing the temperature from 25° to 35°, Curve 3, Fig. 2.
While changing the temperature from 35° to 25°, Curve 4, Fig. 2.
While changing the temperature from 25° to 20°, Curve 5, Fig. 2.
While changing the temperature from 20° to 25°, Curve 6, Fig. 2.

As will be seen from Fig. 2, in which the ordinates represent the departure of the mean values of the 7 cells from the proposed formula, the differences are mainly due to the rapidity with which the temperature was changed, as indicated by the rapid approach to normal values as soon as the heating or cooling was interrupted. The agreement with ascending temperatures was somewhat better than with descending temperatures, which was also found to be the case in the subsequent measurements made to determine the temperature formula.

The departures of the individual cells from the proposed formula are given in Table XII, which shows that one of them, No. 73, exhibited considerable hysteresis. This would of course partly account for the magnitude of the mean departure of the 7 cells (Curves 2 to 6).

TABLE XII.

Departure of Individual Cells from Final Temperature Formula.

[Results expressed in microvolts.]

Date	Temperature	Hours at Temperature	W 73	W 57	W 103	W 78	W 27	W 119	W 89	Mean
*Dec. 3.	25°		− 1	0	− 1	− 1	− 1	- 1	− 2	− 1
*Dec. 5.	25°		+ 1	0	0	+ 1	0	0	− 4	0
Dec. 5, 5 P. M.	4°8	2	+60	−9	− 1	−22	+18	−26	−13	+ 1
Dec. 5, 8.30 P. M.	4°8	5½	+54	−7	+ 3	+ 1	+17	− 3	+11	+11
Dec. 5, 10 P. M.	4°8	7	+55	−5	+ 3	+ 2	+16	− 1	+12	+12
Dec. 6, 9 A. M.	4°8	18	+45	−6	+ 2	+ 7	+13	+ 3	+11	+11
Dec. 6, 4 P. M.	4°8	25	+43	−8	0	+ 6	+10	+ 1	+ 9	+ 9
Dec. 7, 9 A. M.	4°8	42	+40	−5	0	+ 6	+10	+ 1	+10	+ 9
Dec. 7, 4 P. M.	25°	½	0	0	− 1	+ 8	0	+ 4	+10	+ 3
*Dec. 9, 9 A. M.	25°	41½	0	−1	0	+ 1	0	+ 1	+ 5	+ 1
Dec. 9, 4.20 P. M.	35°	½	− 7	0	0	− 1	0	− 1	+ 4	− 1
Dec. 10, 9 A. M.	35°	17	− 4	−2	− 2	− 8	− 1	− 2	+ 2	− 2
Dec. 11, 9 A. M.	35°	41	− 2	−2	+ 1	− 6	+ 1	0	+ 1	− 1
Dec. 11, 12.20 P. M.	25°	1	+55	0	+10	− 6	+19	+ 2	+ 2	+12
Dec. 11, 1.05 P. M.	25°	2	+53	0	+10	− 4	+17	+ 3	+ 2	+12
Dec. 11, 4 P. M.	25°	5	+47	0	+10	− 2	+15	+ 5	+ 2	+11
Dec. 11, 8 P. M.	25°	9	+41	0	+ 9	− 2	+13	+ 5	0	+ 9
Dec. 12, 10 A. M.	25°	23	+32	−4	+ 8	− 2	+ 8	+ 3	0	+ 6
Dec. 12, 4 P. M.	25°	29	+31	−4	+ 8	+ 1	+ 6	+ 3	+ 1	+ 7
Dec. 13, 11 A. M.	25°	48	+29	−6	+ 7	− 1	+ 7	+ 1	− 1	+ 5
Dec. 13, 1.15 P. M.	20°	1½	+59	−6	+ 6	−11	+13	− 3	− 4	+ 8
Dec. 13, 4.25 P. M.	20°	3½	+55	−6	+ 6	− 6	+ 8	− 2	− 4	+ 7
Dec. 14, 9 A. M.	20°	20	+42	−7	+ 2	− 4	+ 5	+ 2	− 4	+ 5
Dec. 14, 4.15 P. M.	20°	27	+40	−8	+ 1	− 3	+ 5	− 2	− 4	+ 4
Dec. 16, 9 A. M.	20°	68	+33	−7	+ 1	− 1	+ 3	0	− 4	+ 4
Dec. 16, 12.09 P. M.	25°	4 min	+24	−3	+ 3	+ 4	+ 1	+ 4	− 2	+ 4
Dec. 16, 12.27 P. M.	25°	22 min	+25	−2	+ 4	+ 4	+ 1	+ 3	− 2	+ 5
Dec. 16, 1 P. M.	25°	51 min	+25	−3	+ 4	+ 4	+ 2	+ 2	− 2	+ 5
Dec. 17, 4.30 P. M.	25°	28 hrs	+24	−4	+ 4	+ 3	+ 2	+ 1	− 2	+ 4

*Taken as basis of reference.

The results obtained in the preliminary measurements above described indicated the advisability of keeping the cells at least 24 hours at each temperature employed in the determination of the temperature formula, even though a much shorter period would have been sufficient for most of them.

RESULTS OBTAINED BY OTHER INVESTIGATORS.

A brief outline of previous work on this subject has been given above, and it therefore only remains to compare the results obtained with those found in this investigation. Since the results obtained by Dearlove, Barnes, and Barnes and Lucas are quite irregular and indicate the existence of large hysteresis effects as has been pointed out above, this discussion will be confined to a comparison of those obtained at the Reichsanstalt and at the Bureau of Standards.

The original measurements by Jaeger and Wachsmuth,[19] upon which the generally accepted formula is based, were therefore reduced by the author, and it was found that if the obervations at 0°, 1.5°, and 1.7° were excluded the average deviation of the measurements from the new formula amount to ±13 microvolts as compared with ±8 microvolts by the generally accepted formula. When it is recalled that the results of Jaeger and Wachsmuth were obtained from a relatively small number of measurements on four cells, set up with 1 : 6 amalgam, and that no attempt was made to maintain the temperature at a constant value for any length of time, the agreement is quite remarkable. The exclusion of the observations at the lower temperatures, at which the differences from the proposed formula are from 100 to 200 microvolts, is fully justified by Jaeger's admission of the existence of irregularities below 10° for the 1 : 6 amalgam.

As pointed out in the introduction to this paper, the results obtained by Jaeger in 1898 also apply to cells with amalgam of the same composition, and further establish the existence of irregularities at lower temperatures. The differences between individual cells of the same lot indicate, however, that the abnormal values may not be entirely due to the 1 : 6 amalgam.

The results obtained by Jaeger and Lindeck[20] in 1899–1901 have also been carefully worked over. No particular attention was paid to the eight old cells with 1 : 6 amalgam, six of which were among those previously reported as abnormal. The 56 series of measurements on twelve new cells with 13 per cent amalgam were first tabulated. The two series at 0°, extending over two periods of 8 and 11 days, show relatively large differences between the individual cells, as well as a not inconsiderable hysteresis. Instead of taking the mean of all such measurements, the mean of the last three in each series was taken. Series 15 and 54 were also excluded on account of relatively sudden temperature changes. The observations were otherwise grouped in the same manner as was done by Jaeger and Lindeck, and from these the mean values of the twelve cells at twelve temperatures in the range 0° to 30° were obtained. The mean deviation from the Reichsanstalt formula was found to be ±40 microvolts, while the mean deviation from the formula here proposed amounted to ±20 microvolts. This suggested the desirability of computing a temperature formula from the Reichsanstalt results. It was found that the mean values of the twelve cells, the observations grouped as above, could best be expressed by the formula

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[19] Wied. Ann., **59**, p. 575; 1896.

[20] Ann. d. Phys., **5**, p. 1 (1901). Zs. Instk., **21**, p. 33–65 (1901).

$$E_t = E_{20} - .00003988\ (t-20) - .000000941\ (t-20)^2 + .0000000146\ (t-20)^3$$

with a mean residual of ±10 microvolts. In addition, the coefficients are in excellent agreement with those of the temperature formula proposed.

The results of further measurements made on five cells set up at the same time and with the same material as the twelve to which the above results apply, together with those obtained with two new lots of 20 each, set up with respectively 12 per cent and 13 per cent amalgams, are given in Table XIII. The cells were only measured at 0°, at which they were kept for several days, and at room temperatures. The b and β groups were freshly set up and changed considerably between the measurements at room temperatures, made before and after the observations at 0°, so that the differences between 0° and 20° were calculated on the assumption that the rate of change was uniform in the interval.

TABLE XIII.

Summary of Results—Jaeger and Lindeck.

Cells	Number	Amalgam Per cent Cadmium	Mean Difference in Microvolts 0°–20°
a_1–a_5 a_{11}–a_{17}	12	13	318
a_6–a_{10}	5	13	230
b_1–b_{15}	15	13	433
b_{16}–b_{20}	5	13	379
β_1–β_5	5	12	345
β_6–β_{20}	15	12	355
All	57	—	358

The agreement of the individual cells as well as the separate groups is not entirely satisfactory and hardly establishes the existence of a difference in the mean temperature coefficient between 0° and 20°, arising from the employment of 12 and 13 per cent amalgams. The mean difference in value between 0° and 20° for the 137 cells, all set up with 12.5 per cent amalgam, on which the proposed formula is based, is 359 microvolts, while that given by the formula of Jaeger and Wachsmuth is 500 microvolts, so that it may safely be concluded that the heretofore accepted formula can not be applied beyond the limits, 10°–30°, indicated by Jaeger.

Since the meeting of the American Electrochemical Society last April, at which this paper was first presented, there have appeared two short papers on "The Cadmium Cell at Low Temperatures" by Henry Tinsley,[21] and on "The Influence of Temperature on the Electromotive Force of Cadmium Cells," by R. Jouast.[22] Tinsley reports on some observations made on a number of cells set up in accordance with the specifications of the National Physical Laboratory, containing 12.5 per cent amalgam and agreeing within 5 parts in 100,000 with the cells of the National Physical Laboratory.

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[21] London Electrician, **61**, p. 321; June 12, 1908.

[22] C. R., 147, 42; July 6, 1908.

During a severe cold spell last winter a difference of nearly 2 millivolts was observed between these and a master cell, of considerably larger size, containing 10 per cent amalgam, although the former agreed well with one another. One of them was therefore selected and measured against the master cell, both being placed in an oil bath, the temperature of which was allowed to rise from 3° to 21°. The differences between the two cells, amounting to nearly 2 millivolts at 3°2, are attributed by Tinsley to the amalgam, the cell with 12.5 per cent giving lower values at the lower temperatures.

Jouast, at the Laboratoire Central d'Électricité, found that cells containing 12.5 per cent amalgam gave a value at 0° 100 microvolts less than that given by the formula of Jaeger and Wachsmuth, but agreeing with the results of this investigation. On the other hand, cells with 10 per cent amalgam, while agreeing with each other at 10°, differed at 0° by several parts in 10,000 among themselves and had values 1 millivolt higher than the 12.5 per cent cells. On abrupt cooling, the electromotive force increased almost instantly by approximately 2 millivolts, the cells attaining a constant value after several days. The abnormal behavior is again attributed to the composition of the amalgam, but while Tinsley finds this to occur with the 12.5 per cent amalgam, Jouast finds irregularities with the 10 per cent amalgam. The author found only a few of the entire 200 cells employed, all of which were set up with 12.5 per cent amalgam, which differed by more than 50 microvolts from the temperature formula proposed, so that it appears quite probable that the irregularities are not due to the amalgam.

In Table XIV are given the differences in E. M. F. calculated by the various formulæ, for every 5° from 0° to 40°, in terms of the value at 20° taken as standard.

From the table it will be seen that the agreement of all the formulæ is very good between 10° and 30°. In work of the highest precision, however, the differences, amounting to 20 microvolts in the interval 20° to 25°, between the generally accepted formula and that proposed may have to be taken into account.

TABLE XIV.

Differences in Microvolts from 20° Value by the Various Formulæ.

Temp.	Jeager and Wachsmuth	Jaeger and Lindeck *	Smith	Wolff
0	+ 500	+ 304	+ 505	+ 359
5	+ 424	+ 337	+ 428	+ 366
10	+ 315	+ 290	+ 319	+ 304
15	+ 174	+ 174	+ 176	+ 179
20	0	0	0	0
25	− 206	− 221	− 209	− 226
30	− 445	− 478	− 451	− 492
35	− 716	− 761	− 725	− 791
40	−1020	−1057	−1033	−1114

*Computed by author.

SUMMARY OF CONCLUSIONS.

From the results obtained in this investigation the following conclusions may be drawn:

1. That Weston cells can be set up which exhibit extremely little hysteresis in the range 0° to 40°, even when subjected to large and rapid temperature variations.

2. That such cells will maintain their values under prolonged heating or cooling.

3. That the relation between E. M. F. and temperature is best represented by the formula

$$E_t = E_{20} - .00004075\,(t - 20) - .000000944\,(t - 20)^2 + .0000000098\,(t - 20)^3.$$

4. That no indications were found of differences arising from the methods of preparation of the mercurous sulphate employed.

5. That the results of Jaeger and Lindeck on cells with 12 and 13 per cent amalgams are in better agreement with the formula given above than with the generally accepted temperature formula, which, however, agrees fairly well with the former between 10° and 30° C.

6. That some of the cells showed marked and persistent hysteresis, particularly at lower temperatures for which the values are in general greater than those corresponding to the formula above given.

7. That the majority of cells in which hysteresis was found were set up with relatively coarse-grained white samples of mercurous sulphate.

8. That a similar behavior was found in most of the cells having abnormal values at room temperatures, and in cells in which oil had been admitted through defective seals.

9. That nearly all the cells exhibiting abnormal initial values showed evidence of decided hysteresis.

10. That according to the above formula the E. M. F. of the Weston cell has a maximum value at 3°, corresponding approximately to the temperature of minimum solubility of cadmium sulphate, and to the saturation temperature of the cadmium sulphate solution employed in the portable cells of the Weston Company for which the temperature coefficient is almost negligible.

The author wishes to acknowledge the valuable assistance given him by Mr. M. P. Shoemaker and Mr. J. H. Dellinger in making about half of the observations, and to thank Mr. Shoemaker for the regulation of the baths and for his painstaking labors in the computation of the results.

WASHINGTON, September 1, 1908.

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RADIATION CONSTANTS OF METALS.

By W. W. Coblentz.

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INTRODUCTION.

When a substance becomes heated, it is generally supposed that, however small the consequent rise in temperature above the absolute zero, energy will be lost by radiation from its surface. For solids and liquids, in which the adjacent molecules affect each other, the radiant energy so lost consists of all wave lengths from zero to infinity; but at a given temperature a certain frequency of molecular vibration is more energetic than all the others, which are less and less energetic the farther they are removed from the preponderating vibration. With rise in temperature the maximum of this preponderating vibration shifts toward the shorter wave-lengths, and eventually energy is emitted which is of a frequency affecting the eye.

Only solids and liquids are supposed to emit energy of all frequencies, but of variable intensity, in different parts of the spectrum. The energy emitted may therefore vary uniformly in its distribution throughout the spectrum, or the partition of energy may be in the form of sharp emission bands superposed upon a weak continuous spectrum.

Gases, on the other hand, emit discontinuous spectra, or spectral lines. It is a well-known property that the energy required for excitation is different for different spectral lines, being independent of the wave length of the line.[1]

In the case of spectral lines the energy emitted is proportional to the excitation, while in the case of solids the spectral distribution follows some complex law. In the case of the oxides, etc., it was shown in a previous paper[2] that the partition of energy in the spectrum is generally in the form of sharp emission lines superposed upon a weak continuous spectrum. Some substances were found to emit only a continuous spectrum, with no marked emission bands, but no substance was found in which the spectrum consisted entirely of sharp emission bands. The maximum of the continuous spectrum was found to shift toward the short wave-lengths with rise in temperature. The discontinuous spectrum of many substances was found to merge into a continuous one at high temperatures. On the other hand, the energy emitted by the sharp emission bands was proportional to the energy supplied, and there appeared to be no tendency for a sharp emission band to merge into the continuous spectrum upon which it is superposed. In this respect several of the oxides exhibited an emission spectrum similar to that of a flame, which is a combination of the radiation from solid particles and from a gas.

Furthermore, it was found that the maxima of sharp emission bands do not shift with rise in temperature. Several emission bands were found to be in common with the different oxides studied, but no conclusion was reached as to their cause other than the presence of the oxygen atom. Nothing further than a qualitative proof of Kirchhoff's law of proportionality of emission and absorption could be established in this search for a law governing the radiation from the oxides, etc., which are known as "transparent media" or "electric insulators."

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[1] See a fuller discussion in this Bulletin, **5**, p. 178.

[2] This Bulletin, **5**, p. 162.

On the other hand, a uniformly heated cavity, or so-called "black body," and also platinum, have been found, by others, to emit a continuous spectrum of radiant energy. The distribution of this energy is unsymmetrical about the maximum of the energy curve, having the appearance of the probability function modified by suitable constants. With rise in temperature the maximum of the energy curve shifts toward the short wave-lengths. The solids heretofore investigated,[3] and the uniformly heated cavity, or "black body," in which it has been possible to determine the approximate temperature, have spectral energy curves which are represented by the function:

$$E = c_1 \lambda^{-a} e^{c_2/\lambda T} \qquad (1)$$

The shifting of the maximum of the energy curve with rise in temperature is expressed by the law—

$$\lambda_{max} T = \text{const.} \qquad (2)$$

With these two equations it is possible to determine the constants of radiation of platinum and of the "black body," provided we know the temperature. For platinum $a = 6$ and for the "black body" $a = 5$. There are theoretical reasons[4] for believing that all metals behave like platinum in their emissive properties, since they are electrical conductors and exhibit a high reflecting power in the visible spectrum. The fact that they are electrical conductors of

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[3] Paschen, Ann. der Phys. (3) p. **58**, 455; **60**, p. 663, 1897; (4) **4**, p. 277; 1901.

Lummer and Pringsheim, Verh. d. Deutsche Phys. Gesell., **1**, p. 215, 1899, et seq.

[4] Kirchhoff's Law says that for a given temperature, T, and wave-length, λ, the emissivity, E, the absorptivity, A, and the reflectivity, R, of a substance is related to the emissivity, e, of a complete radiator by the equation:

$E = Ae = (1 - R)e$ since $A = 1 - R$. For electrical conductors of the metallic type, the reflective power is related to the electrical conductivity by the relation,

$$100 - R = 36.5\sqrt{\frac{w}{\lambda}}$$

where w is the specific resistance. Since this relation is true only for long wave-lengths (beyond 10μ), and since the value of R is low and of A is high in the visible spectrum, such metals as gold, copper, platinum, etc., must show selective emission, however small, in the visible spectrum.

This subject has been thoroughly treated by Aschkinass, Ann. der Phys. (4), **17**, p. 960; 1905.

the metallic type indicates that the reflecting power, which increases with the electrical conductivity, is high throughout the infra-red. Hence, it may be assumed tentatively that the emissivity function of such metal filaments as tungsten, tantalum, and osmium is similar to that of platinum and of a complete radiator. The appearance of the contours of the energy curves for various temperatures will give some clue to the admissibility of this assumption, which is nothing more than has been tacitly made by previous observers; and until a better one is suggested, the present method is the only one available, without a knowledge of the temperature of the radiator. Since it is at present impossible to obtain these metals in thin strips, similar to platinum, by means of which it is possible to determine the temperatures, there is certainly no objection in making this assumption and then testing it experimentally by the one remaining method, to see what happens. This method consists in observing the spectral energy curve, in which the temperature, T, is kept constant, without knowing the actual temperature. This gives the value of a, and in this manner we can obtain some idea of the probable emissive power of the filament, as to whether the emissivity (total radiation) is proportional to the fourth power ($a-1=4$ for a black body, $a-1=5$ for platinum) or to some higher power of the absolute temperature.

The value of a is determined from the equation

$$(3) \qquad \frac{E}{E_{max}} = \left\{ \frac{\lambda_{max}}{\lambda} e^{\frac{\lambda-\lambda_{max}}{\lambda}} \right\}^{a}$$

for it can be shown from equation (1) that the ratio of the emissivities (the observed galvanometer-bolometer deflections, E and E_{max}) for any two wave-lengths is as given in this equation. How far the above assumption falls short of the observed facts was shown in a previous paper[6] on the radiation from the Nernst glower, in which the radiation curve at high temperatures is apparently continuous, but which in reality is the composite of numerous sharp emission bands, which increase in intensity and broaden out with rise in temperature. For the Nernst glower this constant was found to decrease from $a=7$ at low temperatures to $a=5$ at high temperatures.

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[6] This Bulletin, 4, p. 253; 1908.

An investigation of the emissive properties of the metals is of interest in connection with the numerous conjectures offered to account for the high efficiency of the new metal filament incandescent lamps. Some of these conjectures, quoted in a recent paper on this subject,[6] would seem to be but little more than mere guesses, made without considering the experimental data already at hand, from which we can form a reasonable estimate of what to expect of similar substances operated under similar conditions.

From our present knowledge of the similarity of the electrical conductivity and of the absorption[7] and reflection[8] of a majority of the metals extending over the whole range of the Mendelejeff series, it is reasonable to suppose that the few metals remaining uninvestigated will behave similarly in their emissive and other physical properties. Such metals as gold, silver, copper, platinum, etc., which have a low reflecting power (see Fig. 1) throughout or in narrow regions of the visible spectrum, may well be expected, and are known, to have a marked selective emission in this region of the spectrum, but there has not been a single metal investigated which does not have a uniformly high reflecting power throughout the infra-red, from which one could expect to explain a high efficiency due to the entire absence of radiant energy in large regions of the infra-red, such as was found in the oxides, which are electrical insulators. On the other hand, the high efficiency of the new metal filaments has been attributed to the higher melting points, and consequently the higher temperature, at which they can be operated as compared with platinum.[9] This is true in part, since the temperature at which platinum can be successfully operated is only about 1200 to 1400° C, at which temperature even a black body emits only a yellowish-white light.

The great distinguishing characteristics of metals are their uniformly high reflecting power (with the absence of absorption bands) throughout the infra-red, and their low reflecting power, with bands of almost no reflection, in the visible spectrum. The energy not reflected must, of course, be absorbed; hence, from Kirchhoff's law the emissivity, E_λ, at a given temperature and wave length, must be proportional to the emissivity, e_λ, of a black body, i. e., $E_\lambda = e_\lambda (1 - R_\lambda)$ where R_λ is the reflecting power of the metal. From this, it is evident that the metals must emit selectively in the short wavelengths (up to 1μ) as shown by the reflection curves in Fig. 1. A notable and easily observed example is copper, which has a band of low reflectivity at 0.5μ and a second band at 0.3μ, in the ultraviolet.

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[6] Waidner and Burgess, this Bulletin, **2**, p. 328; 1907.

[7] Hagen and Rubens, Ann. der Phys. (4) **8**, p. 432; 1902.

[8] Hagen and Rubens, Ann. der Phys. (4) **1**, p. 352; 1900.
Coblentz, this Bulletin, **2**, p. 470; 1906.
Minor, Ann. der Phys. (4) **10**, p. 581; 1903.

[9] Lummer, Ziele der Leuchttechnik; Elektrotech. Zs. **28**, pp. 787, 806; 1902.

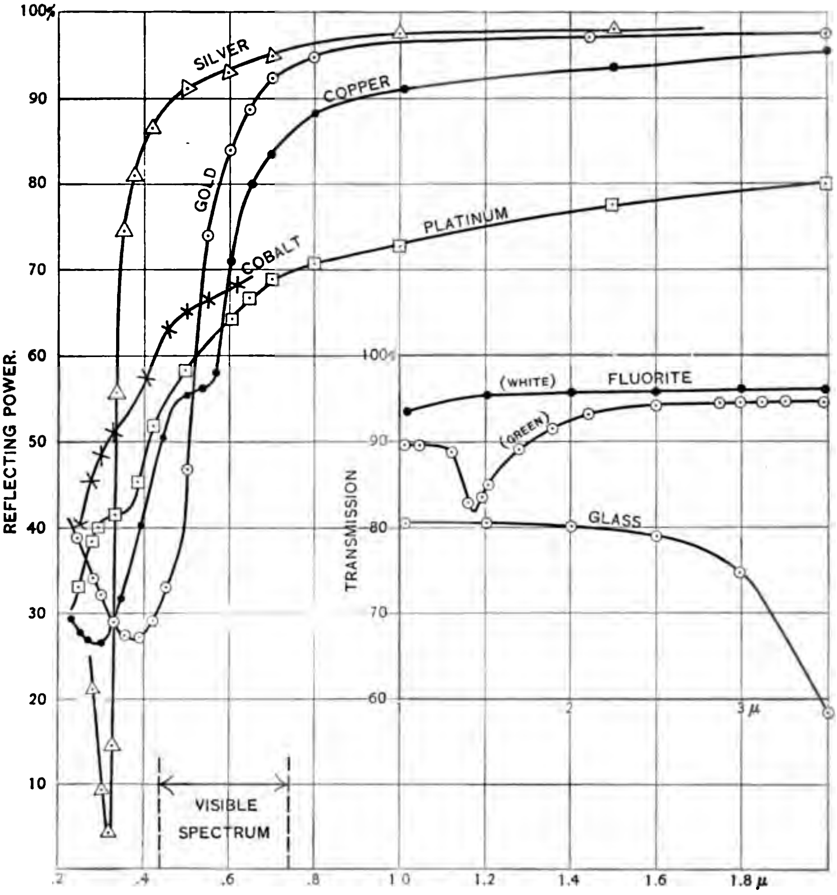

Fig. 1.

The band at 0.5μ gives rise to the green color of molten copper, the intensity of the emission being almost half as great as that of a black body at the same temperature. On the other hand, such metals as nickel, cobalt, palladium, platinum, iridium, etc., which have a low reflecting power in the visible (and ultraviolet spectrum) which increases rapidly to a high and uniform reflecting power beyond 1μ but which have no sharp bands of low reflectivity (see platinum, Fig. 1; data from Hagen and Rubens, and from Minor), must also emit an abnormal amount of energy in the visible spectrum as compared with the infra-red. In this case the extra emission is greatest in the violet and decreases uniformly but rapidly toward the infra-red. At high temperatures this will give a more bluish tint to the emitted light than would be the case if the reflecting power were uniformly high throughout the whole spectrum. The latter is a condition for producing a "gray body," which, by definition, is one in which the emissivity is reduced by the same amount throughout the spectrum. The fact that the reflecting power is uniform in the infra-red indicates that there can not be any emission bands It is evident that the spectral emissivity function of metals must be far more complex than that given in equation (1). Since the emissivity, E_λ, is dependent upon the reflecting power, which is a function of the index of refraction and the absorption coefficient, which, in turn, as will be shown later on, are functions of the temperature and the electrical conductivity, it is evident that the emissivity function must contain factors which take account of these phenomena.

To the writer it has seemed that an explanation of the high efficiency of the new metal filament lamps is to be sought for in the value of the so-called emissivity constant a (which, as will be shown presently, is most effective in suppressing the energy in the long wave-lengths) and in a possible selective emission (i. e., abnormally high but uniform emission, due to the higher value of the absorption coefficient) in the visible spectrum, so that the operating temperatures may or may not be so very different for filaments made of various metals. All the metals investigated show a low reflecting power in the visible and ultraviolet spectrum. This lowering of the reflecting power is much greater than can be accounted for by the scattering due to lack of polish of the surface. This is particularly true of the "white" metals such as platinum, magnesium, aluminum, cadmium, zinc., etc., to which class osmium appears to belong. In the case of platinum, Hagen and Rubens[10] found that the absorption coefficient is unusually high in the visible spectrum, and decreases uniformly toward the infra-red, being only four-fifths its former value at 1.4μ. Such a large variation in the absorption coefficient (and index of refraction) must necessarily cause a selective emission, however small, in the visible spectrum, where the absorption is the greatest. This selective emission is augmented by the rapid rise in the reflecting power beyond 1μ (absorption unknown) which suppresses the radiation in proportion to the reflecting power, and which is uniform throughout the infra-red, thus making it highly improbable that there are bands of selective emission in the infra-red.

The fact that these metal filaments emit a white light indicates that the high emissivity, if present, must extend over a wide region of the visible spectrum, as compared with zinc oxide which emits a yellowish light when heated, and with (molten) copper which emits a greenish light.

On the other hand, if the emission function of the metals is of the form given in equation (1), as has been shown to be true for platinum, then the exponent a is more effective in suppressing the infra-red radiation than one might suppose. This is shown in Table I, in which the first horizontal line gives the wave lengths for which the computations are made. In the second, third, and fourth lines these wave lengths are raised to the power of $a = 5$, 6, and 7, corresponding, respectively, with a complete radiator, platinum, and tungsten. In the fifth and sixth lines, the computations are given for the logarithmic function. The seventh line is obtained by dividing the values in the third line by the corresponding values in the second line; similarly, the eighth line is obtained from the fourth and second lines, and the ninth line from the sixth and the fifth lines.

The point of interest in this table is the preponderence of the λ^{-a} factor over the $e^{-c_2/\lambda T}$ factor in suppressing the infra-red radiation. For example, the ninth line shows that at 5μ the $e^{-c_2/\lambda T}$ factor reduces the emissivity of platinum to 0.87 that of a complete radiator, while (seventh line) the λ^{-a} factor reduces the emissivity to 0.2 its former value. In the case of tungsten (eighth line) the

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[10] Hagen and Rubens, Ann. der Phys. (4) 8, p. 450; 1902.

λ^{-a} factor reduces the emissivity to 0.04 that of a complete radiator at 5μ. In other words, if the complete radiator gives a bolometer deflection of 100 mm at 5μ, then, for platinum and tungsten, at the same temperature, the λ^{-a} factor reduces the deflections to 20 mm and 4 m respectively.

TABLE I.

This table shows the preponderance of the factor λ^{-a} over the factor $e^{-c_2/\lambda T}$ in decreasing the emissivity, E_λ, beyond 1μ. These values are computed from the equation $E_\lambda = c_1 \lambda^{-a} e^{-c_2/\lambda T}$, using $T=2000°$ Abs.

1. **Wave length,** λ..........	0.5μ	1μ	2μ	3μ	4μ	5μ
2. **Value of** λ^{-a} **for** $a=5$ **(complete Radiator)** ...	32	1	.0313	.00412	.000978	.00032
3. **Value of** λ^{-a}, **for** $a=6$ **(Platinum)**	64	1	.01565	.00137	.000244	.000064
4. **Value of** λ^{-a} **for** $a=7$ **(Tungsten)**	128	1	.0078	.000457	.0000611	.0000128
5. **Value of** $e^{-c_2/\lambda T}$ **for** $a=5$ **(complete Radiator,** $c_2=14500$**)**	5.05 $\times 10^{-7}$	.000638	.027	.0894	.164	.235
6. **Value of** $e^{-c_2/\lambda T}$ **for** $a=6$ **(Platinum,** $c_2=15600$**)** .	1.43 $\times 10^{-7}$	.000378	.0195	.073	.139	.204
7. **Reduction of** E_λ **by** λ^{-a} **in changing from** $a=5$ **to** $a=6$ **(complete Radiator to Platinum)**......	2	1	.5	.33	.25	.20
8. **Reduction of** E_λ **by** λ^{-a} **in changing from** $a=5$ **to** $a=7$ **(complete Radiator to Tungsten)**	4	1	.25	.11	.067	.04
9. **Reduction of** E^λ **by** $e^{-c_2/\lambda T}$ **in changing from** $a=5$ **to** $a=6$ **(complete Radiator to Platinum)**......	.284	.596	.723	.824	.848	.87

From this it is evident that even though the various metal filaments may not have a marked selective emission in the visible spectrum, the one having the highest value of a will radiate the least in the infra-red. It would therefore appear that unless the value of C_1, equation (1), is very different for the different metals, the amount by which the temperature must be raised may be very small in order to bring the metal, having the largest value of a, to the same intensity E, in the visible spectrum, as that of another metal having a smaller value of a; and at the same time the former metal may radiate considerably less in the infra-red. The operating temperatures may therefore not be so very different in the metal filament lamps, especially if the one having the higher value of a has also a selective emission in the visible. The spectrophotometric curves published by Nichols[11] show that for tungsten and tantalum, set to equal intensity at 0.589μ, the intensity is almost 50 per cent greater in the tungsten curve than it is in the tantalum at 0.45μ, with correspondingly high values intervening. On the other hand, at 0.7μ, conditions are just the reverse, the tantalum curve being 12.5 per cent greater than that of tungsten. This makes the slant of the curve of tantalum steeper than the tungsten. It should be the steeper (aside from effect due to differences in temperature) at any wave length, for, as will be shown presently, the value of a is smaller for tantalum than it is for tungsten at the same temperature (estimated by the position of E_{max}, see Figs. 9 and 10); and as may be seen from eq. (1), the value of $dE/d\lambda$ is greater the lower the value of a.

Although much has been written on the possible selective emission in the visible, no really conclusive data was at hand[12] until experiments were undertaken recently by Drs. Hyde and Middlekauff in this Bureau.

It is quite impossible to make radio-metric observations in the visible spectrum because of the smallness of the radiant energy involved. Recourse must, therefore, be had to the spectrophotometer. Heretofore, these spectrophotometric measurements of one source have been compared with that of another used as a standard, of which the distribution of energy is in even greater uncertainty than the one under investigation. The proper method of procedure is to refer all measurements to the black body at a given temperature, which we know, from theory and from experiment, must have a uniformly increasing distribution of energy with increase in wave length, and of which the energy curve may be computed with greater accuracy than it can be observed in the visible spectrum. This is the manner in which the investigation is being carried out in this Bureau.

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[11] Edw. L. Nichols, Trans. Illuminating Eng. Soc., **8**, p. 325; 1908.

[12] Except a few experiments by Leder, Ann. der Phys. **24**, p. 305; 1907.

It is, therefore, a great satisfaction to be able to include here some preliminary spectrophotometric measurements (referred to a black body at a fixed temperature, as a standard of comparison) of the distribution of energy in the spectra of the filaments described in this paper. The black body employed is one constructed with especial care, to be used in the redetermination of the radiation constants. Every precaution was taken to insure accuracy in the temperature measurements, as well as in the uniformity of the temperature distribution within the black body.

In the spectrophotometric comparison of the incandescent lamp with the black body, the difference in the temperatures of the front and the rear surfaces of the radiating wall was less than 0°.3 throughout the series of measurements. The temperature of the black body was kept "normal" at 1432° C, with a maximum variation of 0°.3 during the whole series of observations. On the other hand, the "substitution method" employed by Drs. Hyde and Middlekauff reduces spectrophotometric errors to a minimum, so that it is felt that the results are reliable and that explanations of disagreements with others are to be sought elsewhere.

The spectrophotometric comparison of the metal and carbon filaments with the black body shows no selective emission. On the other hand, the acetylene flame has a marked selective emission in the region of 0.72μ.

Although the spectrophotometer shows no bands of selective emission, one can not determine by means of such an instrument whether or not the emissivity throughout the visible spectrum is abnormally high as compared with the infra-red, which high emissivity must result from the abnormally low reflecting power in the visible and ultra-violet as compared with the infra-red, as shown in

Fig. 1, and which must merge gradually into the normal emission beyond 1μ where the reflecting power is uniform.

II. DETERMINATION OF THE α RADIATION CONSTANT.

It is beyond the scope of this paper to fully discuss the relative merits of previous determinations of this "constant." It must be said, however, that present-day investigators working in this field are prone to forget how little was known of the physical properties of prisms, of reflecting mirrors, of the transparency of the atmosphere, etc., at the time (about 1890) when renewed interest in the laws of radiation was awakened by the invention of a workable bolometer by Langley, and a reflection diffraction grating by Rowland, which in turn was applied by Langley to calibrate his rock-salt prism for the infra-red spectrum. At that time nothing very accurate (if at all) was known beyond 3μ concerning the transparency of the air, fluorite, water vapor, etc., nor of the reflecting power of silver mirrors and of reflection gratings, nor of the dispersion of fluorite, nor of the emission and absorption of gases and water vapor, all of which investigations were carried out (and have served as a starting point by all subsequent observers) by Paschen, before he could proceed with his main problem, viz, the investigation of the laws governing the radiation from solids by means of spectral energy curves.

In his first paper[13] on this subject he examined the radiation from iron oxide heated upon a platinum strip. He found that, with rise in temperature, the maximum of the energy curves shifted according to the often previously announced relation $\lambda_{max}T=$ Const., and that the total radiation followed a law requiring a variation greater than the fourth power of the absolute temperature, as deduced by Boltzmann. The various radiation laws proposed up to that time were found untenable, and he set up a complicated empirical formula which fitted his observations. He found, experimentally, that the value of E/E_{max} was best represented by equation (3). At the end of his paper he announced that the simple empirical relation $E=c_1\lambda^{-a}e^{-c_2/\lambda T}$ with $a=5.6$ was fairly well fulfilled by his results, and concluded that it is probably not far removed from the true emission function. He communicated these results to Wien,[14] who, a month later, published a theoretical deduction of equation (1), with $a=5$ for a complete radiator. In his second paper,[15] in which he examined the radiation from copper oxide, lampblack, gas carbon, and platinum, Paschen found equation (1) and its subsidiary relations verified in unexpected places, while in expected places the law was not verified. He found that the values varied from $a=5.5$ to $a=6$, depending upon the substance investigated. From the fact that for a given substance the value of a obtained by three methods gave concordant results when using "blackened bodies," he concluded that the same will be true for an "absolutely black body," when in equation (1) and equation (3) the value of $a=5$, as demanded by theory.

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[13] Paschen, Ann. der Phys. (3) 58, p. 455; 1896.

The practical realization of a black body by Lummer and his colleagues came with the development of electrically heated furnaces, when it was shown that $a=4$ for a uniformly heated cavity, or complete radiator, and $a=6$ for platinum.[16]

It seems to have been overlooked by those who have criticised Paschen's work because the scale of temperature is in doubt, that the value of the constant a was in all cases computed from equation (3) in the form;

$$(4) \qquad a=\frac{\log E-\log E_{max}}{\log e-\frac{\lambda_{max}}{\lambda}\log e+\log\frac{\lambda_{max}}{\lambda}}$$

which does not contain the temperature factor.[17]

The values of a for iron oxide alone were obtained from an untold number of computations based upon 47 energy curves, the temperature being varied from 117° to 1124° C. In spite of the thoroughness of the work, Paschen did not find a systematic variation of *a* with rise in temperature (see Fig. 2), which has been found in subsequent investigations, which, as will be mentioned presently, must occur at some stage, and which his other observations (see Ann. der Phys. **60**, p. 707, and Table XIX) seem to show.

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[14] Wien, Ann. der Phys. (3) **58**, p. 662; 1896.

[15] Paschen, Ann. der Phys. (3) **60**, 662; 1897.

[16] Lummer and Pringsheim, Verh. d. deutsch. Phys. Gesell. **1**, pp. 23 and 215; 1899.

[17] Of course the value of a was also computed, using the observed temperatures, the results usually being in close agreement with those obtained from equation (4), but only a few computations were made using temperatures.

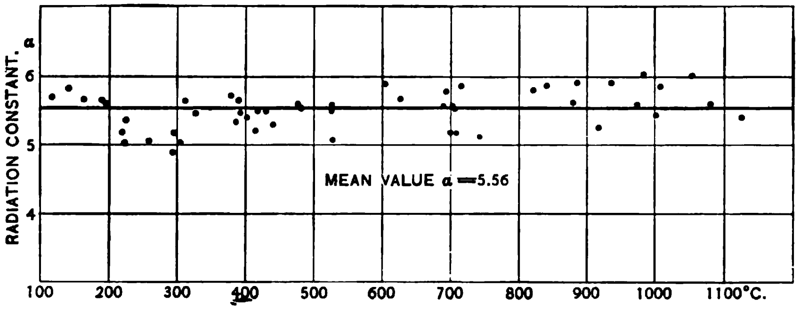

Fig. 2.—*Iron Oxide* (*Paschen*).

The determination of *a* by this method is an extremely tedious process, rendered more so by the presence of the atmospheric absorption bands. To overcome the latter difficulty, Paschen computed the position of the E_{max} in his energy equations by obtaining λ_{max} from the equation:

$$(5) \qquad \lambda_{max} = \frac{(\log\lambda_2 - \log\lambda_1)\lambda_1\lambda_2}{(\lambda_2 - \lambda_1)\log\epsilon}$$

selecting for the purpose wave lengths on the energy curve where two ordinates were of equal height ($E\lambda_1 = E\lambda_2$), but which did not fall within the region of the atmospheric absorption bands.

In all cases, except for platinum, the values of *a* determined by equation (3) and those from equation (1) using the temperature were in close agreement. As already mentioned, there is an inconsistency in that some of his results show no variation in *a* with rise in temperature. On the other hand, Lummer and Kurlbaum[18] found from the measurement of the total radiation of platinum and of iron oxide that the value of *a* underwent a very marked decrease with rise in temperature (see Table IV). This fact seems to have escaped the attention of Lummer and Pringsheim in their investigation of the spectral radiation of platinum.[19] (See a discussion of their results on a subsequent page, under "platinum.") The present results confirm the observations of Lummer and Kurlbaum.

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[18] Lummer and Kurlbaum, Verh. Phys. Gesell. Berlin, **17**, p. 110; 1898.

[19] Lummer and Pringsheim, Verh. d. deutsche Phys. Gesell., Berlin, **1**, p. 215; 1899.

Although it seems to have been overlooked heretofore, it is obvious that the so-called constant, *a*, must decrease in value (either abruptly at some temperature or uniformly throughout the range in temperature) and approach that of a complete radiator, otherwise a point would be attainable at which the total radiation is greater than that of a complete radiator at the same temperature. The writer's attention was called to this fact by Dr. C. W. Waidner some years ago, and the subject has since been discussed by Mendenhall and Ingersoll,[20] on the basis of the radiation laws, especially the Stefan law. This same conclusion follows from a consideration of Maxwell's electromagnetic theory of light, which in its original form does not consider the vibratory periods of the molecules, and which demands the existence of analogous relations between the electric conductivities and the transparencies of the metals to radiant energy.[21]

The electromagnetic theory of light requires the "optical constants" (the refractive index, n, and the absorption coefficient, k), to have a temperature coefficient which increases with the wave length. In the visible spectrum the change in the optical constants of the metals with the temperature is exceedingly small as shown by various observers.[22]

On the other hand, Hagen and Rubens[23] have shown that at 25μ the absorption coefficient undergoes a change which is proportional to the change of resistance which the metals show with increasing temperature. In the case of platinum, in which the temperature coefficient is large, the observed emissive power $(100-R)$ increased from 3.36 (arbitrary units) at 170° C to 9.78 at 1500° C, which agrees with the same values computed using the specific resistance of the metal.

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[20] Mendenhall and Ingersoll, Phys. Rev., **25**, p. 1; 1907.

[21] Drude, "Physik des Aethers," p. 574; 1894.
E. Cohn, "Des electromagnetische Feld," p. 444; 1900.
M. Planck, Sitzber. Akad. Wiss., Berlin, p. 278; 1903.

[22] Drude, Ann. der Phys. (3) **39**, p. 538; 1890.
Pflüger, Ann. der Phys. (3) **58**, p. 493; 1896.
Laue and Martens, Phys. Zs., 8, p. 853; 1907. The latter observers, working over a range from 800° to 1500° C, found no marked change in the optical constants of platinum.

[23] Hagen and Rubens, Ann. der Phys. (4) **9**, p. 873; 1903. Phil. Mag., p. 157; Feb., 1904.

In view of the fact that the reflecting power, R, of the metals increases with the electrical conductivity, which decreases with rise in temperature, one would expect to find a corresponding increase in the emissivity, i. e., a decrease in the value of a, as demanded by the electromagnetic theory of light. This variation of the "optical constants" with temperature being extremely small in the visible spectrum, and very marked at 25μ, is evidently a function of the wave length. It is therefore of interest to investigate their behavior in the intervening region.

It will be shown presently that, without a single exception, the metals herein examined show a decrease in a with rise in temperature, as is to be expected. For carbon (a non-metal) which decreases in resistance with rise in temperature, the results are inconclusive, the a decreasing uniformly for the "untreated" carbon and apparently increasing for the "flashed" carbon. For the latter the measurements should extend over a wider range of temperature in order to be conclusive.

III. METHODS OF PRESENT INVESTIGATION.

In the present investigation it was found that the determination of the value of a is beset with difficulties which precludes great accuracy. The observations of the prismatic distribution of energy can be made with an apparent accuracy of 0.2 to 0.5 per cent, the errors being introduced subsequently in reducing the observations.

The energy spectrum was obtained from about 15 spectrometer settings, between 0.8 and 3μ (settings every 1′ to 2′ of arc). The galvanometer deflections were plotted to a scale commensurate with the observations, and from this the normal energy curve was obtained by dividing the observed galvanometer deflections by the width of the bolometer strip expressed in wave lengths,[24] using the first two terms in the function for the slit-width correction, viz:

$$(6) \qquad ef(\lambda)=E(\lambda)-\tfrac{1}{6}E_1(\lambda)+\tfrac{2}{45}E_2(\lambda)\ldots\ldots$$

where E is the observed galvanometer deflection and λ is the wave length. The slit-width correction is difficult to determine in the region where a dispersion curve passes through a point of inflection. In this region (at 1.5μ) where, unfortunately, the maxima of the energy curves also occur, the dispersion of fluorite is extremely difficult to observe and probably not known with as great accuracy as in other parts of the spectrum. The dispersion curve (refractive index and wave length) was obtained by plotting all the known observations, and it coincided precisely with the latest and probably the most accurate determination[25] of the refractive indices of fluorite. From this dispersion curve the prism calibration curve (minimum deviation setting and corresponding wave length) was computed, and from the latter the so-called slit-width correction curve was found by reading from the calibration curve the width which the bolometer strip (0.6 mm) subtended in wave lengths in different parts of the spectrum.

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[24] Paschen, Ann. der Phys., **60**, p. 714; 1897.

The value of the bolometer strip of 4′ of arc is given in Table II, third column, from which it will be noticed that in precision work the slit-width correction for fluorite is a tedious and somewhat uncertain factor at 1.5 to 1.7μ.

In this respect rock salt (or quartz) would be better adapted to work of this type, because the double curvature in the dispersion curve falls at 3μ. However, the lack of purity of rock salt, the smaller dispersion at 1 to 2μ, and particularly the difficulty in keeping the surfaces polished, prohibit its use.

Other errors that enter into the determination of the value of a are "stray" or "field light," which is superposed upon the spectrum, and the loss of energy by absorption at the mirrors which is quite marked in the wave lengths from 0.5 to 1.0μ. The latter may be corrected, thus reducing the values of a by 2 to 5 per cent. In a few cases in the present work the reflecting power correction was applied, but the uncertainties (introduced after making the observations) in obtaining E and E_{max}, especially the latter, which amounted to 5 per cent, did not seem to justify the procedure. The "field light," which comes from the prism surfaces and from scattering internally, can not be eliminated from the energy curve by absorption screens, which it is hoped to apply in finding isochromatic

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[25] Paschen, Ann. der Phys. (4) **4**, p. 302; 1901.

TABLE II.

Calibration Curve of Fluorite Prism, 60°

Spectrometer Setting	Wave Length	Slit-width Correction
20°00′	0.5893μ	0.034
58	.625	.0395
55	.694	.054
53	.754	.071
19°50	.8875	.102
49	.9407	.1128
48	1.002	.1230
47	1.063	.1318
46	1.1318	.1432
45	1.206	.1542
44	1.286	.1640
43	1.370	.1720
42	1.457	.1745
41	1.545	.1760
19°40	1.633	.1743
39	1.719	.1715
38	1.8040	.1682
37	1.887	.1647
36	1.9685	.1617
35	2.048	.1585
34	2.127	.1540
32	2.277	.1475
19°30	2.421	.1407
28	2.558	.1345
26	2.689	.1285
24	2.813	.1240
22	2.934	.119
19°20	3.051	.1147
15	3.328	.105
10	3.578	.098
5	3.806	.0915
19°00	4.035	.086
50	4.445	.0774
40	4.825	.0707
30	5.169	.066
15	5.642	.059
18°00	6.082	.056
30	6.864	.049
17°00′	7.546μ	.042

energy curves. The only applicable test is to measure the energy in the ultraviolet where silver has a minimum reflecting power. For the most intense radiation from a Nernst glower the stray radiation determined in this manner was extremely small, and hence negligible in the metal filaments.

The adjustment of the apparatus has previously been described.[26] The adjustments of the spectrometer (the "zero setting" of the bolometer) was made with the yellow helium line.

Great precaution had to be taken to prevent radiation from more than one filament entering the spectrometer slit. To this end the first spectrometer mirror was covered except a narrow portion through the center. The center of the cone of light, which was only about 3 mm wide for the metal filaments, was brought to a fixed mark upon a ground-glass screen placed within the spectrometer; and the adjustment of the lamp, covered with a hood containing a narrow slit, was continued until no light from side (or line of sight) filaments entered the spectrometer. This adjustment is emphasized because the curious results of Sartori[27], in which he found that a 3.5-watt-per-HK-carbon-filament lamp had the maximum of its energy spectrum at a shorter wave length than that of a 1-watt-per-HK-osmium lamp, seem (to the writer) to be due to a lack of centering the narrow cone of light upon the prism face. The spectrometer-bolometer adjustments were made before and after each series of observations, and the bolometer sensibility was frequently tested while making the series of observations. The energy was taken from a storage battery. The observations were repeated in different parts of the spectrum after measuring through the spectrum, and in general immediately after passing the maximum emission, in order to test any change in emissivity of the filaments due to deterioration.

Every effort was made to avoid errors which would interfere with the accuracy attainable, and it is felt that the explanation of some of the results obtained, which are apparently at variance with those of others—for example, the lack of constancy of the exponent, a—must be sought elsewhere, and not in faults in the apparatus.

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[26] This Bulletin, **4**, p. 533; 1908.

[27] Sartori: Illuminating Engineer (London), **1**, p. 601; 1908, contains an abstract of this paper.

Although, as previously described, the apparatus was practically free from absorption, a very weak, narrow absorption band seemed to be present at 1.45 μ, which may be due in part to the prism. Although it was perfectly free from flaws and almost colorless, the prism belonged to what is known as the "green" variety of fluorite, which has been found to have a very narrow, shallow absorption band at 1.42μ (see Fig. 1, which gives the transmission of a very light-yellowish-green sample 3.85 mm thick, and also the transmission of a clear white fluorite 2.28 mm thick). To simplify the computations, which were long and tedious, the value of a was computed (see equation (3)) for those wave lengths which were used in computing λ_{max} in equation (5). The wave length λ_1 was selected, which would not occur in possible absorption bands, which was quite removed from the correction for reflection, and which did not place λ_2 too far into the infra-red (beyond 2.5μ) where, for the incandescent lamps, the glass walls begin to absorb heavily (see Fig. 1 for the transmission of a sample of glass 0.75 mm thick). The value of λ_{max} found in this manner from equation (3) would vary from the mean by 1 to 3 (very rare) units in the second decimal place. The main difficulty was in the uncertainty in taking the values of E and E_{max}, especially the latter, from the energy curves, so that in spite of all the precautions taken the variation in the value of a for any one energy curve, using different values of λ_1 and λ_2, is of the order of 3 to 5 per cent. However, the values are consistent, all being high or low, which is the point of principal interest in this investigation. Moreover, the values of a so determined are compared with platinum, of which the remaining constants are fairly well known.

IV. COMPARISON OF THE SPECTRAL ENERGY CURVES OF VARIOUS SUBSTANCES.

One method of demonstrating the great difference in the emissivity in the infra-red as compared with the visible spectrum, due to the difference in the value of a, is to set the various filaments to the same intensity in the visible and observe their energy curves. In order to do this it is necessary to keep the bolometer at the same sensibility throughout the examination, and to have the same energy flux, i. e., the same area of the prism face covered by the different filaments. The latter will depend upon the diameter of the radiating filament and its distance from the spectrometer slit. Then, with these two factors constant, the energy supply is regulated in the various lamps so that they all give the same deflection at a certain wave length; for example, 0.5μ. The problem is simplified and sufficiently approximated by setting the different lamps to a color match with an ordinary photometer, or, better still, with a spectrophotometer, and noting the voltages of the different lamps. The radiation curves are then observed at these voltages, regardless of the area of the prism face covered or the sensibility of the bolometer, and the deflections in the various energy curves are reduced by a fixed amount, which is determined by the factor which must be applied to make the deflections equal for all the lamps for a certain wave length—say the yellow or red. With the cooperation of Dr. E. P. Hyde, a series of energy curves were made for several commercial incandescent lamps, including untreated carbon (curve *a*), flashed carbon (curve *b*), tungsten (curve *c*), and osmium (curve *d*), all shown in Fig. 3. In all the lamps the radiation was observed through the glass walls, hence no comparison can be made of the curves beyond 2.5μ. Great care was taken to shield the lamps so that the radiation from only one filament entered the spectrometer slit. The cone of energy covered only about 3 mm of the prism face, thus greatly reducing the deflections, which consequently could not be determined with great accuracy in the visible. Hence, while the observations demonstrate conclusively the very different emissivity of these lamps, the curves can not be used to estimate the value of the so-called constant α. For example, the curves for tungsten and osmium, according to the other determinations of α, should be closer together, with the tungsten curve perhaps the lower of the two. (See, however, discussion under osmium.) The suppression of the infra-red radiation in the two metals with a high value of α is very marked in comparison with the radiation from the carbon filaments. A mere glance at Fig. 3 shows that the carbon lamps emit about one-third more infra-red energy than the metal filaments. This alone is sufficient to explain the high efficiency of the latter, without introducing the question of a selective emission

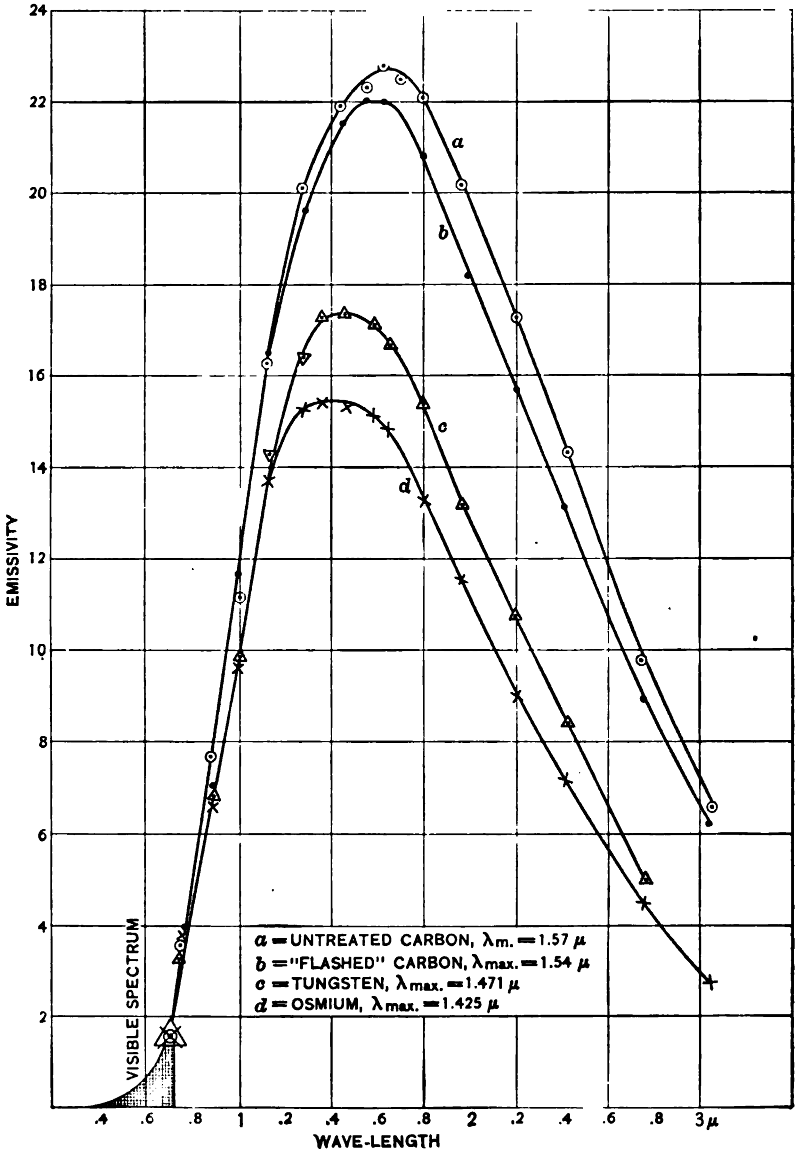

Fig. 3.—*Spectral Radiation Curves.*

in the visible, the term "selective emission" being used to indicate an abrupt change in the energy curve with change in wave length, such as was found in the oxides.[28]

The estimation of the temperature of these filaments based upon the shift in the maximum is hardly admissible, for a change from $a=5$ to $a=6$ introduces a change of 20 per cent in the maximum, λ_{max}, while the value of C_2 changes but 7 per cent. Increasing the value from $a=5$ to $a=7$ (tungsten) introduces a change of almost 40 per cent in the maximum ($\lambda_{max}=C_2 \div Ta$).

As a further illustration, in the present curves the value of $a=5.9$ for the flashed carbon ($\lambda_{max}=1.54\mu$) and $a=7.1$ for tungsten. The change in a is 18 per cent, which for the same temperature would make $\lambda_{max}=1.25\mu$ for tungsten, when in reality the value is $\lambda_{max}=1.47\mu$. Evidently, it is impossible to say what the temperature is without measurements with a thermocouple, to determine the other constants which enter the energy equations.

V. RESULTS OF THE DETERMINATION OF THE "CONSTANT, a," FOR VARIOUS METALS.

In this investigation the filaments to be examined were in the form of commercial incandescent lamps with glass walls, and also in the form of especially made lamps with fluorite windows, secured with Khotinsky cement, as shown in Fig. 4. The latter lamps were about 15 cm long, 5 cm in diameter, and admitted a filament about 5 to 7 cm long. The filaments for the lamps with fluorite windows were taken from commercial lamps. All the filaments were heated to incandescence by means of direct current from storage batteries, hence there was no difficulty in keeping the current constant.

The results obtained with the lamps with fluorite windows are not very satisfactory for the metal filaments, owing to the fact that they could not be properly treated in exhausting them. The exhausting was done with a mercury pump, and was continued for several days, the pressure being reduced to less than 0.01 mm. The results are of interest in showing that the energy curves are

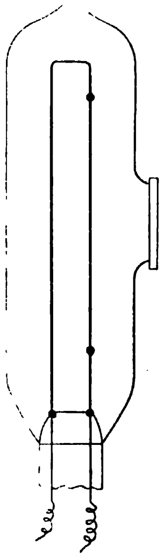

Fig. 4.

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[28] This Bulletin, 5, p. 159.

smooth and continuous throughout the infra-red examined; they also show the great influence of the methods of exhaustion upon the constant a and upon the life of the filament.

In the commercial lamps care must be taken to operate them at a sufficiently high temperature so that the maximum emission is far removed from the increased absorption of the glass wall. In Fig. 4 is given the transmission curve of a fragment of the glass wall, 0.75 mm thick, of an incandescent lamp. The absorption increases very rapidly beyond 2.5μ, so that without a knowledge of the absorption of each lamp no comparisons beyond 2.5μ can be made of the various energy curves.

In the following illustrations the values of the so-called radiation constant, a, are platted for a given energy consumption. They are characterized by various signs (circles, dots, triangles, etc.), to indicate at what wave length the E_{max} was taken to compute a.

The wave lengths (λ_{max}) of E_{max} are also shown as being of value in estimating the probable value of a from a knowledge of the location of E_{max} with respect to a possible small absorption band at 1.45μ. The various lamps were operated on a storage battery and no difficulty was experienced in maintaining a constant radiation.

1. UNTREATED CARBON.

The so-called radiation constant a was found for a filament about 5 cm long and 0.1-mm diameter, mounted in an exhausted glass vessel, as shown in Fig. 4. At the highest energy consumption the filament appeared as bright as a normal burning incandescent carbon lamp. The apparent black-body temperature, corresponding to the different values of energy consumption, was measured by Doctors Waidner and Burgess with an optical pyrometer for red, green, and blue light. The values given in Table III were obtained from their watt-temperature curve extrapolated for high temperatures. The observed and the computed temperatures are at greater variance the greater the energy supplied. It is of interest to note that the selective emission as found by the optical pyrometer (which is in reality a photometer in which the temperature of the comparison lamp is varied) is highest in the red, which is just the opposite of the metals and of the Nernst glower.[19] On the other hand the acetylene flame emits selectively like an oxide.

From the values of *a* shown in Fig. 5, it appears that there should be a better agreement in these temperatures at the higher temperatures. The various computations of *a* for any given energy consumption are in close agreement, are uniformly high at low temperatures, and decrease with rise in temperature. At about 1200° $a=6.5$, at about 1500° $a=5.7$, and at about 1700° $a=5.2$.

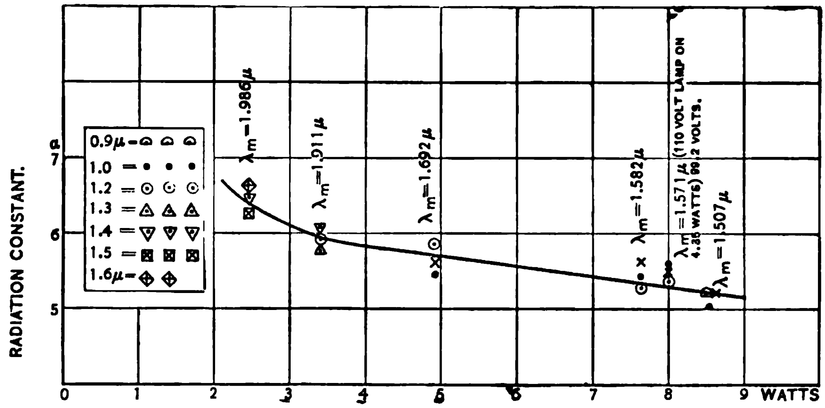

Fig. 5.—*Untreated Carbon.*

The value of *a* was also determined for a 110-volt incandescent lamp with glass walls when operated on 99.2 volts, 43.5 watts (see Fig. 5). The energy curve is shown in Fig. 3, curve *a*. The value of $a=5.49$, which is in agreement with the previous results when the energy curve has its maximum at 1.58μ, i. e., when the filament was at the same temperature.

In this series of measurements a commercial lamp with glass walls, and one with a fluorite window and short filament were used. Nevertheless, the value of *a* for the same λ_{max} are in excellent agreement.

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[19] This Bulletin, **2**, p. 319; **4**, p. 548.

TABLE III.

Energy Consumed	Temp. K. Red	Temp. K. Green	Temp. K. Blue	λ_{max}	$T_{max}=\frac{2930}{\lambda_{max}}-273^\circ$	$T_{min}=\frac{2620}{\lambda_{max}}-273^\circ$	Radiation Constant, a
			CARBON (Untreated)				
2.47 Watts .085 Amp.	1200°	1160°	1120°	1.986μ	1202°C	1044°C	6.5
3.38 Watts .1 Amp.	1390	1340	1284	1.911μ	1262	1097	6.0
4.94 Watts .125 Amp.	1675	1610	1557	1.692μ	1457	1275	5.7
7.63 Watts .16 Amp.	2030	1950	1895	1.582μ	1572	1372	5.35
8.5 Watts .17 Amp.	2135	2040	1985	1.507μ	1672	1462	5.2
			OSMIUM (Short filament, fluorite window)				
.96 Watts .8 Amp.	1314	1322	1327	1.556μ	1607	1407	
1.53 Watts .9 Amp.	1422	1435	1440	1.521μ	1652	1447	
2.44 Watts 1.1 Amp.	1607	1632	1640	1.414μ	1797	1581	
3.36 Watts 1.2 Amp.	1692	1724	1733	1.371μ	1862	1637	
5.04 Watts 1.4 Amp.	1850	1895	1910	1.303μ	1977	1742	
7.38 Watts 1.6 Amp.	2000	2060	2080	1.267μ	2042	1797	
			ACETYLENE				
Flat,	1357	1450	1475				
End on	1522	1762	(=1593 and 1825°, respectively, when placed in front of a block of magnesium oxide.)				

2. METALLIZED AND FLASKED CARBON.

The "metallized" carbon examined consisted of a 110-volt "Gem" incandescent lamp (glass walls) which at normal operation required 50 watts. The mean value of the "constant," $a = 5.85$. There seems to be a slight tendency for the value of a to increase with energy consumption (see Fig. 6), but this may be due to the fact that on

36 watts the E_{max} occurs at 1.47μ, where there is a small absorption band which prevents the accurate measurement of the maximum deflection. The value may therefore be more nearly $a=6$.

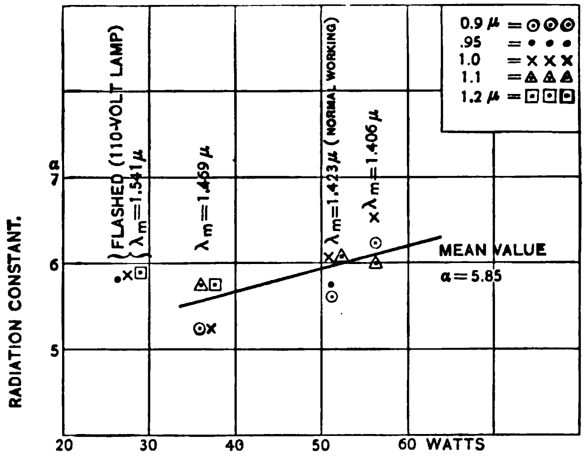

Fig. 6.—*Metallized Carbon.*

A 110-volt "flashed" carbon lamp with glass walls (on 84.75 volts—watts not determined; see curve *b*, Fig. 3) was also investigated. The E_{max} occurs at $\lambda_{max}=1.54\mu$ as compared with $\lambda_{max}=1.571\mu$ for the untreated carbon filament (Figs. 3 and 5) having the same color. The mean value of $a=5.88$ was found from three closely agreeing computations. (See Fig. 6.)

Paschen found that the value of this "constant" for carbon rods heated in the air was about $a=5.6$, while for the same carbons heated in a vacuum a varied from 5.0 to 5.3. Since we do not know the condition of the surfaces, the results obtained by various observers can not be compared.

3. THE HELION FILAMENT.

The helion filament consists of a carbon filament covered with a thin coating of silica. The samples examined were unmounted experimental ones (single loop 10 cm long), kindly presented by the inventor, Prof. H. C. Parker.

The filaments were mounted in bulbs with fluorite windows, and an examination was made without seasoning them. It was found that the filaments became covered with black spots, so that an extensive examination could not be made. On first starting, the value of $a = 8.2$, which is unusually high. On 17 watts (87 volts) the filament was as bright as an ordinary 110-volt 16-c p flashed carbon lamp, and $a = 7$, Fig. 7. On 21 watts the value of $a = 6.4$, and on examination it was found that the filament was covered with black spots where the silica had disappeared. But even in this condition the value of a is high and comparable with platinum. The results show that if the covering of silica can be made stable the efficiency of such a filament will be unusually high, due to the high value of a; but, unfortunately, on account of chemical decomposition of the silicon carbide surface it will probably not be possible to operate the filament at such high temperatures as is possible with tungsten. Hence, when perfected, the two may not be so very different in their efficiency as a light producer, which in the case of the helion filament is due to the fact that its operating temperature must be kept low, and in the case of the tungsten filament is due to smaller value of a, which places it closer to a complete radiator, although its operating temperature may be higher than for helion.

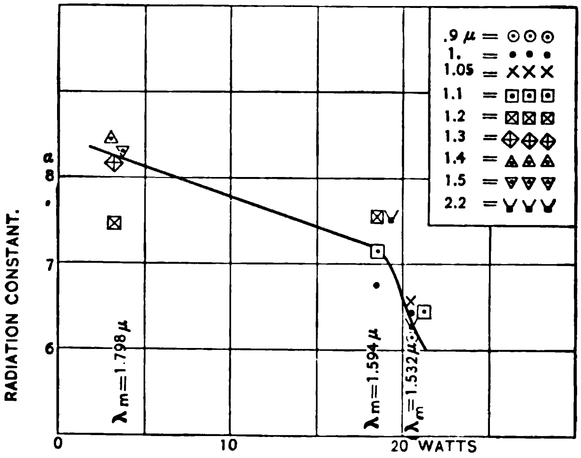

Fig. 7.—"*Helion*" (*Silica coated carbon*).

4. PLATINUM.

In order to obtain an estimate of the probable value of a of the various metal filaments, it is necessary to compare them with platinum, of which the "constant," a, was supposed to be fairly well established.

Paschen's[30] values of a seem to increase slightly with temperature, which may be due to an increase in smoothness of surface. Lummer and Pringsheim also record a change in the emissivity of platinum due to a change in the surface. The values of a for platinum found by Paschen (in marked contrast with all the other substances examined by him) differ by 18 per cent, being $a = 5.536$ when he used equation (3) and $a = 6.423$ when computed from two equations requiring a knowledge of the temperature. For the oxides of iron and of copper the values, computed by these same three methods, are in agreement within 2 to 3 per cent. On the other hand, Lummer and Pringsheim[31] did not make the tedious computations, but assumed a value of $a = 6$ and computed the temperatures corresponding with the observed temperatures, evidently assuming that the temperature scale is sufficiently reliable for that purpose. Although there is no systematic variation there is in some instances (especially at high temperatures), a considerable discrepancy between the computed and the observed temperatures. This method of procedure does not show how constant the value of a is with rise in temperature, and it is unfortunate that the computations of a were not made for each energy curve. They concluded that the observed temperatures are too low and that the more probable value is $a = 5.85$. The difference between their observed and their computed (using $a = 6$) energy curves differ more and more, at 2 to 5μ, the higher the temperature. This would seem to indicate that in addition to the fact that equation (1) does not hold for long wave-lengths, unless the temperatures are seriously in error the value of a decreased with rise in temperature, thus increasing the values of E at 3 to 4μ. In the total radiation measurements by Lummer and Kurlbaum[32] the constant, σ, of the Stefan law was found to increase from $\sigma=4.28$ to $\sigma=19.64$ (in arbitrary units), while for a "black body" investigated at the same time the value of $\sigma=109$ (about) throughout the whole range of temperatures. (See Table IV.)

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[30] Paschen, Ann. der Phys., (3) **60**, p. 662; 1897.

[31] Lummer and Pringsheim, Verh. d. deutsche Phys. Gesell., Berlin **1**, p. 215; 1899.

[32] Lummer and Kurlbaum, Verh. Phys. Ges., Berlin, **17**, p. 106; 1898.

TABLE IV.

Absolute Temperature		$\sigma=\left(\frac{E}{T^{a-1}\ \ T_0^{a-1}}\right)$			
T	T_0	Black Body	Polished Platinum		Iron Oxide
		$a=5$	$a=5$	$a=6$	$a=5$
372.8	290.5	108.9	....	....	....
492	290	109.0	4.28	8.24	33.1
654	290	108.4	6.56	9.65	33.1
795	290	109.9	8.14	10.06	36.6
1108	290	109.0	12.18	10.94	46.9
1481	290	110.7	16.69	11.01	65.3
1761	290	....	19.64	11.16	....

In Table IV the value of σ is obtained by dividing the observed total radiation by the difference between the observed temperatures of the radiating substance and the shutter raised to the fourth power ($a-1=4$ for a black body). This test of the constancy of the value of σ for platinum is not a fair one, since it is known that its emissivity is of the order of the fifth power of the temperature. In the fifth column of Table IV, under platinum, the writer has computed (from the values of σ and the corresponding temperatures which gave the value of E) the value of σ on the assumption that the total radiation of platinum is as the fifth power of the absolute temperature, i. e., for $a=6$. The variation in σ is much less than for $a=5$, and it appears that for a value of $a=6.5$ (about) the value of σ would remain constant throughout this range of temperature. But a value of $a=6$ seems to be the upper limit and for this value the coefficient, σ, increases 40 per cent, which seems to indicate quite conclusively that part of this variation in σ is due to an actual variation (decrease) in the "constant" a.

From this it is evident that the value of σ is decreasing with rise in temperature. It would therefore appear that the change in a ought to be detected in the spectral energy curve, for it is inconceivable that, assuming a selective emission in the extreme infrared, the energy can increase sufficiently to affect the total radiation and not be observable in the spectral radiation curve at 1 to 3μ. This inconsistency, of a variation of a in their observations on total radiation and of its assumed constancy in the spectral radiation curves, seems to have been overlooked by Lummer and his collaborators.

The variation in a with change in temperature was entirely unexpected, and was not found until most of the observations had been made and the computations plotted, as shown in the accompanying illustrations.

At first it was suspected that the variation is due to an increase in stray light, but the observations of E_{max} for platinum lie at such long wave lengths (as compared with those of the other metals) that the change in a appears to be real. The platinum examined was the purest made by Heraeus. The strip was 50 by 1.5 by 0.02 mm in a glass bulb with fluorite window, as shown in Fig. 4. The strip was welded to thick platinum wires, which were in turn welded to the copper supports. This procedure was found necessary to prevent the hot platinum strip from fluxing with the copper leads. In order to work under constant conditions the platinum was aged by heating it for a short time to a higher temperature than it was used in the experiments. The small deflections at the low temperatures introduces great variations in the value of a, but the important point brought out is that the values of a are uniformly high or uniformly low, depending upon the temperature. This is illustrated in Fig. 8. A slight variation in taking the E_{max} and the E from the energy curve causes a variation of 2 to 5 per cent in a, as shown at different parts of the curve where more than one value of a for any given wave length (for example at 0.9 and 1.6μ) is plotted. The abscissa is represented by the observed current in amperes (squared) instead of the usual "watts." Estimating temperatures from the equation $\lambda_{max}T = 2620$, it may be said that the constant decreases from $a = 8.5$ at 900° C to $a = 6.3$ at 1100° C. For some unexplained reason all the values are higher than than those found by Paschen using the same method. This, however, is of minor importance, for we are concerned with this value relative to that of the other metals.

In Fig. 1 is shown the reflecting power of platinum,[33] which increases rapidly from a low value of 65 per cent at 0.5μ to 94 per cent at 5μ, beyond which the reflecting power is uniform and equal to 96.4 per cent at 14μ.

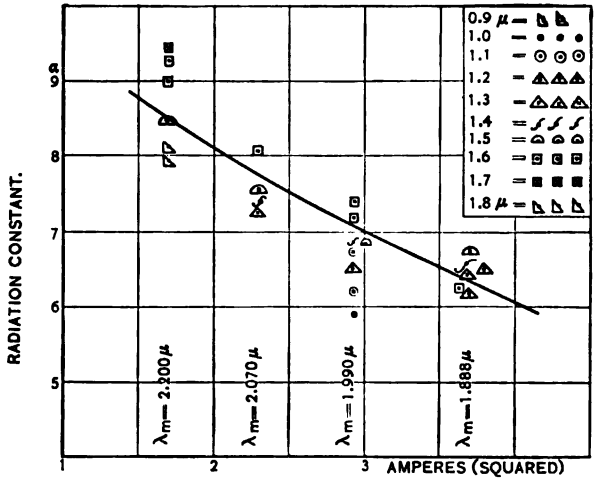

Fig. 8.—*Platinum.*

The absorption of a thin film of platinum[34] decreases rapidly with wave length, being much less than that of gold and silver in the infra-red. From the rapid change in the absorption (and reflection) curves it seems evident that platinum must show selective emission (being uniformly high but not necessarily broken into bands) in the visible spectrum, as compared with the infra-red.

5. OSMIUM.

One series of measurements of a was made on a short (3-cm) piece of osmium, which was taken from an osmium lamp. A series of five energy curves, the λ_{max} varying from 1.26μ to 1.52μ, gave concordant values of $a = 6.3$ (mean of 14 computations), with no definite decrease with rise in temperature. In Table III are given the apparent temperatures of this filament as found with an optical pyrometer and the corresponding λ_{max}. In fact, the value of a seemed to increase on first operation, which is in accord with general experience in the manufacture of new filaments.

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[33] Hagen and Rubens, Ann. der Phys., (4) **11**, p. 873; 1903.

[34] Hagen and Rubens, Ann. der Phys., (4) **8**, p. 432; 1902.

The color of the filament remained a silver-gray in appearance, but the surface was pitted and rough, showing bright patches. These defects were of sufficient magnitude to affect the emissive power, making it lower than one would expect from a highly polished surface of the same material. In addition to this, the filament was short and affected by heat conductivity from the ends. The results can not be given great weight in considering the value of a. The work was repeated, using a Siemen & Halske's commercial 55-volt, 32-cp lamp ("Auer Oslampe"). The energy curve when operated on 38.6 volts (28 watts) is given in Fig. 3. Here the E_{max} falls within the region of a possible absorption band, and $a = 6.77$ (mean of six computations; see Fig. 9 for the variation in a)

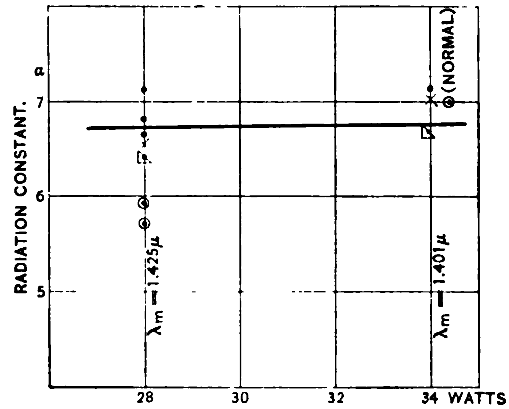

Fig. 9.—*Osmium.*

which is greater than usual for any given wave length, e. g., at 1μ. This same lamp, when operated on 50 volts (10 per cent below normal) 34 watts, gave a very concordant value of $a = 6.96$ as the mean of four computations. Here the E_{max} falls outside of the questioned region of absorption. Two computations (equation 5) gave $\lambda_{max} = 1.4002\mu$ and $\lambda_{max} = 1.4015\mu$, and on the whole it is believed that the latter value is the more nearly correct. These two series of measurements give a mean value of $a = 6.86$. This lamp was calibrated for photometric work, and hence was not operated at higher temperatures.

The value of a of osmium appears to be somewhat larger than that of tungsten (thick filament) for the same λ_{max}, and is of the same magnitude as that of tungsten when the filaments are thin, hence probably more homogeneous. The emissivity of osmium should therefore be equal or slightly less than that of tungsten throughout the infra-red, as indicated in Fig. 3.

According to Lummer and Pringsheim,[25] the high efficiency of the osmium lamp is due to its selective emission in the visible spectrum. It is a "whiter" metal than tungsten, but whether or not this has a marked effect the emission remains undetermined.

The measurements with the spectrophotometer do not show a selective emission in the visible spectrum. The present measurements indicate (for a color match in the visible spectrum, Fig. 3), that osmium has its $\lambda_{max} = 1.425\mu$ and tantalum has its $\lambda_{max} = 1.472\mu$. From this it appears that the so-called "radiation constant, a," of osmium must be higher than that of tungsten; and hence the working temperature of osmium must be higher than that of tungsten (unless the constant c_1 is greater in the former, which can hardly be possible). This alone would account for the much-discussed cause of the high light-efficiency of osmium filament lamps.

6. TUNGSTEN.

The first measurements were made upon a thin filament of tungsten, 3 cm long, placed in a lamp with a fluorite window, as shown in Fig. 4. The conduction from the ends caused it to be spindle-shaped when heated. The examination was made to show that the spectrum is not discontinuous, as mentioned on a previous page. The mean value of $a = 7.28$ was found from four computations, which is unusually high considering the fact that the $\lambda_{max} = 1.301\mu$, while in all subsequent observations the value of a is much less for the λ_{max} at 1.2 to 1.3μ. (See Fig. 10.)

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[25] Quoted in Liebenthal's Praktische Photometrie, pp. 54 and 340. The original paper by Lummer (Ziel der Leuchttechnik) does not mention the experimental work upon which the evidence of selectivity is based.

A series of measurements was then made on a 110-volt, 32-cp, 60-watt (General Electric) lamp having a fine filament. The values of *a* vary considerably at the lowest energy consumption, but they are all high—about $a = 7.8$. At about normal operation the value dropped to $a = 6.5$. Another 110-volt lamp of the same make was examined on 60 volts (see Fig. 3 for its energy curve). Two computations gave concordant values of $a = 7.14$ and $a = 7.18$, respectively, as shown at *a* in Fig. 10, the E_{max} being at 1.472μ.

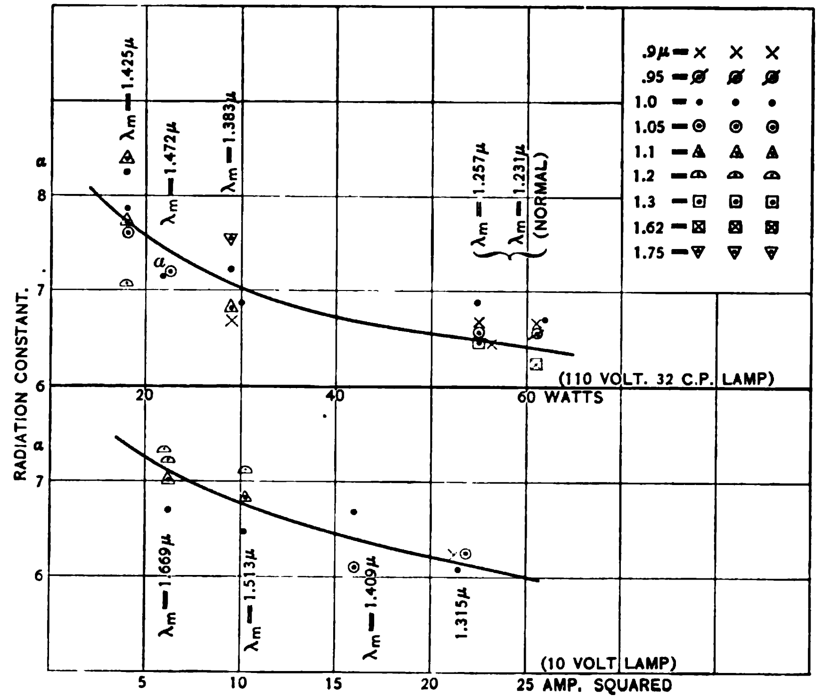

Fig. 10.—*Tungsten.*

A further study of *a* was then made, using for the purpose a General Electric Company 10-volt single-loop tungsten lamp, the filament being at least 0.2 mm in diameter. The thick filament gave much larger galvanometer deflections. For the highest energy consumption (given in amperes, "squared," Fig. 10) the brightness was somewhat below normal. The values of *a* are somewhat lower for the same λ_{max} as compared with the thin filaments, decreasing from $a = 7.1$ to 6.2. The values of a found from these four filaments differ considerably from each other, especially for the thick filament, but the important fact remains that all the values are much higher than those of platinum, when the energy curves have the E_{max} at the same wave length. From this it would appear that the high efficiency of tungsten is due not only to the high operating temperature, as concluded by Grau,[36] but also to the high value of a.

7. TANTALUM.

The measurements were made on a 110-volt, 16-cp, lamp. In the first series, the radiation entering the spectrometer slit formed an image which was about 4 mm wide, and evidently came from two filaments. The values of a are platted (I) in Fig. 11. The lamp was then readjusted, giving a sharper, narrower image upon the prism face. The values of a thus obtained are somewhat higher (II) in Fig. 11, but in both cases there is a tendency for a to decrease with rise in temperature.

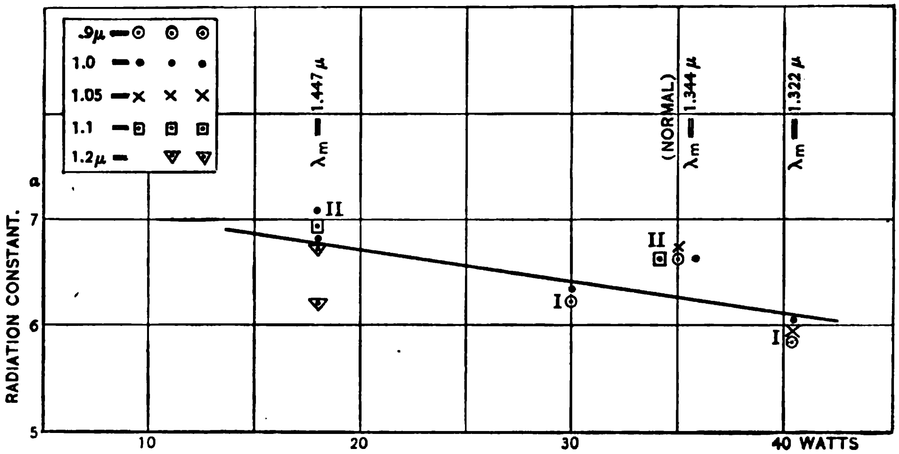

Fig. 11.—*Tantalum.*

A series of observations was made on a filament about 6 cm long placed in a lamp, as shown in Fig. 4. The filament seemed to be affected by the residual gas in the bulb, and its behavior was so unsatisfactory that the results were discarded entirely. The results, on the whole, seem to indicate that the "constant" *a* of tantalum is somewhat higher than for platinum.

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[36] Grau, Illuminating Eng., **1**, p. 186; 1908.

On "normal operation" the *a* of tungsten and of tantalum are about the same value, but for tungsten the $\lambda_{max} = 1.23\mu$ while for tantalum the $\lambda_{max} = 1.34\mu$, from which it would appear that the tungsten filament is at a higher temperature.

In Table V are given approximate values of the "constant" *a* of the aforesaid metals, as used in the 110-volt lamps investigated, when under "normal operation," which seems to admit of rather wide variations. For example, the 55-volt osmium lamp has a λ_{max} which is at variance with the preceding lamps in which the *a* is smaller. Although the osmium lamp was operated on 50-volts (only 10 per cent below normal) the λ_{max} is larger than for tungsten. In Fig. 3 the comparison is a fairer one and the position of the λ_{max} is consistent with the value of *a*. The "normal" operating temperatures, computed on the assumption that the constants are the same as for platinum, are given in the third column of this table. The values, of course, are only approximations, and are all much lower (300 to 500° less) than found by Waidner and Burgess,[37] using an optical pyrometer. This is of course due in part to the assumption made that these metals have the same constants as platinum, which is contrary to observation.

TABLE V.

Constants of 110-Volt Lamps Burning Normally.

	λ_{max}	$T_{Pt} = \frac{2620}{\lambda_{max}} - 273^\circ$	"Radiation Constant" *a*
Metallized Carbon	1.423	1570°C	6.1
Tantalum	1.344	1670	6.3
Tungsten	1.257	1810	6.6
Osmium (55-volt lamp on 50 volts)	1.401	1600	6.9

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[37] Waidner and Burgess, this Bulletin, **2**, p. 327; 1907.

8. THE ACETYLENE FLAME.

It is hardly legitimate to apply the foregoing method of computation of a for the acetylene flame, and much less for the sun, but it seemed of interest to make computations.

That the emissivity of the acetylene flame must depart far from a complete radiator is evident from the very intense emission band of CO_2 at 4.4μ superposed upon that of the incandescent carbon particles which have their maximum at 1.05μ. For computing a the energy curve of the acetylene flame found by Stewart[38] was used. For the wave lengths, $\lambda = 0.75$, 0.9, 0.95, 1.2, and 1.3μ the values of $a = 10.3$, 11.1, 13.4, 10.7, and 9.5, respectively. This gives a mean value of $a = 11$, and is an extreme example, showing that a and not temperature may cause a high luminous efficiency; for the mean temperature of the acetylene flame is about 1900° C, while the $\lambda_{max} = 1.05\mu$. The temperature of a complete radiator would have to be 2500° C in order to have its maximum emission at $\lambda_{max} = 1.05\mu$.

Spectrophotometric comparisons by Nichols[39] and by Blaker[40] of the spectra of carbon filaments with acetylene shows a marked selective emission at 0.6 to 0.7μ in the carbon filament. It is difficult to conceive how a radiator like carbon (of the type lying between insulators and electrical metallic conductors) of the thickness of the filament used, can emit such an unsaturated radiation. As mentioned on a previous page, the selective emission is due to the unsaturated radiation from the acetylene flame. The maximum of the selective emission of acetylene lies in the region of the apparent minimum found by Nichols and by Blaker at 0.7 to 0.72μ in the solid carbon. It will be interesting to investigate the change in selectivity with increase in thickness of the flame. Whether the selective emission at 0.72μ is due to a true emission band at this point, or due to a more transparent region in the infra-red, 0.76 to 0.8μ, remains to be determined. The latter seems more probable. Stewart's energy curves of the visible spectrum do not rise as rapidly as one would expect, in order to become a maximum at 1.05μ, which would indicate a discontinuity at 0.7 to 0.8μ. That the transparency of the acetylene flame must increase rapidly in passing from the visible into the infra-red is evident from the smallness of the radiometer deflections at the maximum of the energy curve, at 1.05μ, as compared with the large deflections in the visible spectrum. For a Nernst glower, which on normal operation emits somewhat like a complete radiator (due to its low reflecting power and its great thickness), the deflections at 1.05μ would have been 40 to 50 times as great as those observed by Stewart. Further evidence of the very rapid increase in transparency of extremely thin films of carbon (lampblack) is to be found in a paper by Ångström,[41] and also in an investigation by Ladenburg[42] of the transmission of a flat acetylene flame and of a Hefner flame. The latter was much more opaque than acetylene, at 0.7μ, while beyond 2.5μ they were of about the same opacity.

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[38] Stewart, Phys. Rev., **16**, p. 123; 1903.

[39] E. L. Nichols, Phys. Rev., **18**, p. 65; 1901.

[40] Blaker, Phys. Rev., **18**, p. 345; 1901.

9. THE SUN.

For computing the value of a for the sun, Abbot's[43] data was employed. First of all, it was found that no consistent value of λ_{max} could be found (eq. 5), the values varying from $\lambda_{max}=0.49\mu$ to $\lambda_{max}=0.52\mu$, due, no doubt, to the lack of data in the ultraviolet, assuming that the composite radiation is similar to the Nernst glower. The difficulty in computing the value of a lies in the lack of knowledge of the value of E_{max}, of which different values were used. Taking $\lambda_{max}=0.46\mu$, and $\lambda_1=0.4$, 0.5, 0.6, 0.7, and 1.2μ, respectively, the values of $a=21.9$, 15.6, 11.0, 7.8, and 5.4, respectively. The values of a vary uniformly over so great a range that it seems evident that we can not apply the radiation laws of a complete radiator, as has been done heretofore, in discussing the emissivity of the sun.

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[41] Ångström, Ann. der Phys., (3) **36**, p. 717; 1889.

[42] Ladenburg, Phys. Zs., **7**, p. 697; 1906.

[43] Abbot, Smithsonian Miscell. Coll., **45**, p. 74; 1903. Also: Annals Astrophys. Obs., **2**; 1908.

VI. SUMMARY.

The object of this investigation was to obtain some estimate of the emissivity of incandescent metal filaments, e. g., tantalum, tungsten, and osmium, the assumption being made that their spectral energy distribution follows the same general law as that of platinum and of a complete radiator. With substances whose energy spectra undergo no change in contour with change in temperature it does not seem unreasonable to apply our knowledge gained from the behavior of platinum under similar conditions, especially since the filaments are metals—electrical conductors—which, theoretically, should have similar emissive properties. That the method is open to criticism is admitted, but there is nothing novel about it, and until a better one is suggested the present method is the only one available without a knowledge of the temperature of the radiator. The results obtained, considered quantitatively, are not very satisfactory, and perhaps this should hardly be expected. The filaments are generally made from discrete particles of metal, mixed with a "binder," and squirted into a thin thread, after which the binder is removed, leaving the metallic particles welded together in various degrees of homogeneity and purity. The results, therefore, can not be as conclusive, neither can the accuracy be as great as will be possible when wide strips are obtainable, made from these metals, melted into a homogeneous mass, and rolled and hammered into shape, such as obtains for platinum. The nature of the material now obtainable did not appear to warrant a further continuance of the present investigation.

The results obtained show that the so-called radiation constant, a, of all the metals examined is higher than that of platinum at the same temperature. Osmium seems to have the highest value, $a=7$.

Although the individual computations of the "constant, a," vary from 3 to 5 per cent, the results thus far obtained are concordant in being uniformly high or uniformly low for a given energy consumption and as compared with platinum.

Contrary to the results of the previous investigations, the present work shows that the so-called "radiation constant, a," decreases uniformly with rise in temperature, i. e., the a has a temperature coefficient probably similar to that of electrical resistance. Theoretically, this decrease must occur, but whether the change takes place abruptly or whether it is a gradual change, as indicated in the present results, remains undetermined.

It is, therefore, an open question whether or not there is a "radiation constant, a," for metals (total radiation, $S=\sigma T^{a-1}$). This is of vital importance in the application of optical pyrometers, although the "optical constants" of the metals show no appreciable temperature coefficient in the visible spectrum. The present observations, and some of those of previous experimenters, indicate a variation in this so-called constant with temperature.

The investigation will, therefore, have to be undertaken anew, and it is proposed to examine the total radiation of platinum (and other metal strips, if obtainable), using the temperature scale to as high a point as its accuracy will admit. Further spectro-bolometric work will also be necessary, although the latter is subject to greater instrumental errors. An investigation of the infra-red reflecting power of platinum, with change in temperature, will also aid in proving or disproving this variation in emissivity constants of metals.

WASHINGTON, August 22, 1908.

DEPENDENCE OF MAGNETIC HYSTERESIS UPON WAVE FORM.

By Morton G. Lloyd.

When the electromotive force applied to the magnetization of iron is simple harmonic, the wave of magnetic flux which it produces is also simple harmonic. When the curve representing the emf. is distorted from the sinusoidal shape, the magnetic flux will also be distorted, but in a different manner. In considering the energy losses, or transformations into heat, which take place in the iron in different cases, we may proceed from two different standpoints. If we consider the effective value of the electromotive force maintained constant while the form of wave is changed, we approximate the conditions of the practical use of transformers. Under these conditions the iron is magnetized to different maximum inductions with the different forms of wave, and large differences in the iron losses may result. This subject has been treated both theoretically and experimentally in a previous paper.[1]

If, on the other hand, the maximum magnetic induction is made the same with different wave forms, then the effective electromotive force is different and the eddy currents induced in the iron are different. Here, again, we shall have the energy loss in the iron different.

Our present problem is to separate this variable loss due to eddy currents from the loss due to magnetic hysteresis and determine whether the latter varies or is constant when the maximum value of the magnetic flux is kept constant while the form of the wave is varied.

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[1] M. G. Lloyd, this Bulletin, 4, p. 477; 1908.

The formula given by Steinmetz assumes that the hysteresis loss is determined by the maximum magnetic induction. As modified by Rössler[2] this formula reads

$$W = \eta n B^{1.6} + \zeta n^2 f^2 B^2$$

where

B = maximum magnetic flux density.
n = frequency.
f = form factor of induced electromotive force.
W = power expended per unit volume.
η and ζ are constants of the iron.

The first term in the second member of the equation represents the hysteresis loss and the second term the loss due to eddy currents. The latter is assumed proportional to the square of the effective value of the induced emf. Now the maximum flux density is proportional to the average value of the induced emf., and since form factor is by definition the ratio of the effective to the average value, we have eddy current loss proportional to $(fB)^2$. The introduction of the form factor is the modification made by Rössler in order to apply the formula to waves which are not sinusoidal.

It is well known that this formula is not accurately true. Even with sine waves, the hysteresis is not strictly proportional to $B^{1.6}$, but within practical operating ranges it is approximately so. Similarly the eddy current loss may not be strictly proportional to B^2. We are concerned here, however, not with variations of B, but with variations of the loss when B is kept constant.

To determine the constancy of the hysteresis loss it is necessary to separate the total iron loss into its two components, and determine the variation of the eddy current loss independently. With a constant form factor, this separation can be approximately accomplished by measurements at two frequencies and the same flux density. If the form factor is not the same at the two frequencies (and ordinarily it is not), it must be measured and used in the separation, as follows:

$$W_1 = \eta n_1 B^x + \zeta n_1^2 f_1^2 B^2$$

$$W_2 = \eta n_2 B^x + \zeta n_2^2 f_2^2 B^2$$

$$\frac{W_1}{n_1} = \eta B^x + \zeta n_1 f_1^2 B^2$$

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[2] G. Rössler, Electrician, **36**, p. 124; 1895.

$$\frac{W_2}{n_2} = \eta B^x + \zeta n_2 f_2{}^2 B^2$$

from which

$$\zeta B^2 = \frac{\frac{W_1}{n_1} - \frac{W_2}{n_2}}{n_1 f_1{}^2 - n_2 f_2{}^2}$$

Having determined ζB^2, the value of $\zeta n f^2 B^2$ for each case, and the value of ηB^x are easily obtained. These represent the energy per cycle expended in eddy currents and in hysteresis respectively. Knowing these values for one wave form, the wave form may then be altered, the new form factor measured, and the new eddy current loss computed. The remainder of the measured loss is due to hysteresis.

If the original equation were an exact representation of the facts, this procedure would give an easy solution of the problem; but the equation is only an approximation. It assumes, first of all, that the hysteresis *is* independent of wave form in separating the two components of loss. This objection is avoided, however, if the separation is made with runs on the same form of wave, say a sine wave. Or if not exact sine waves, if the waves approach it closely, an approximately true separation would result by considering the form factors as indicated above. The test could then be made upon waves with widely varying form factors.

There are other assumptions in the equation, however, which may not correspond with fact. The question as to whether the hysteresis loss per cycle varies with the rapidity of performing the cycle is still an open question and one closely related to the problem in hand. The results obtained by different observers are so utterly diverse as to present no final answer to that question, but some of the results which have been published are so obviously spurious that they can not be considered as evidence. These researches will be discussed later. We may safely assume for present purposes that if there be a variation with frequency it is small, and within such limits as would here be used for the separation of the losses (say 30 and 60 cycles) the error introduced would also be very small. So much for the term representing hysteresis loss.

The term representing the loss by eddy currents is known to be only an approximation. If a series of measurements is taken at different frequencies, using a sine wave, it is found that the loss per cycle does not increase according to a linear law, as the equation indicates, but that at higher frequencies the observed values are smaller than would correspond to the observed values at the lower frequencies. This is illustrated in Fig. 1, which shows the variation of total loss per cycle with the frequency in some sheets of transformer iron which were tested with a sine wave. The dotted line is a straight line through the points for 30 and 60 cycles. The solid line connecting the observed values sags at the higher frequencies. This has been attributed by some writers to the inductance of the eddy current circuits, which makes itself appreciable at the higher frequencies, so that the proper expression for the energy is

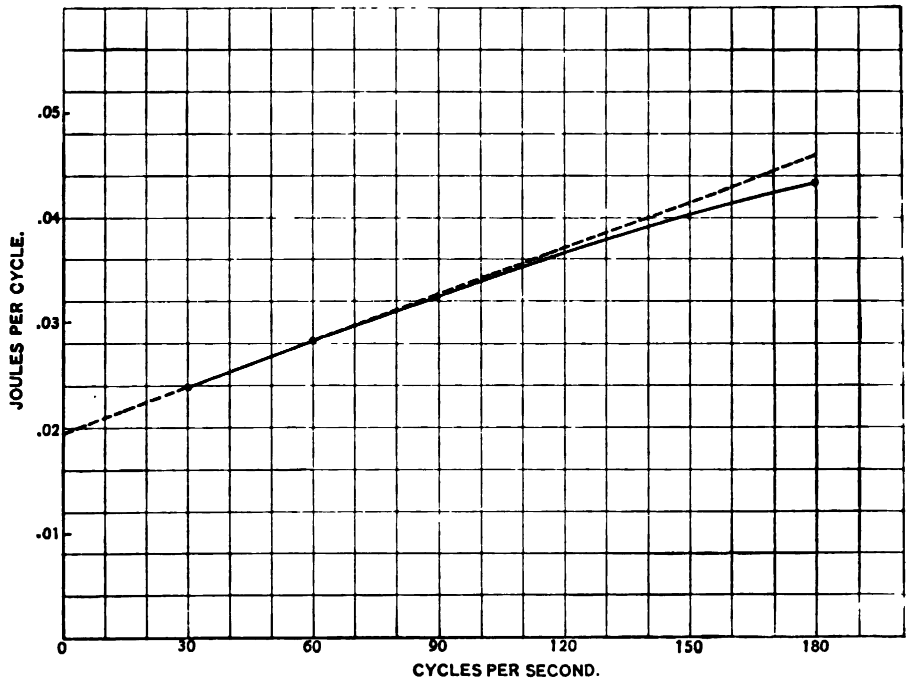

Fig. 1.—*Transformer Sheet No. 27 Gage at 5000 gausses.*

$$\frac{E^2R}{R^2+(2\pi nL)^2}, \text{ instead of } \frac{E^2}{R}.$$

This expression decreases with increasing *n*. I have not, however, been able to fit such an expression to my experimental results, and I believe the cause is to be found in another action of the eddy currents. As the frequency increases the eddy currents are increased in magnitude and exert a greater magnetizing force. This is greater near the center of the cross-section, and has the well-known effect of concentrating the flux near the outer walls of the conductor. The distribution of flux consequently changes with the frequency, the amount of flux threading the shorter paths of eddy currents, as *a*, Fig. 2, being decreased, while the flux through the outer paths, as *c*, is unchanged. The result is that the mean length of circuit of the eddy currents is increased and the mean resistance increased, thus reducing the energy below its value for a uniform distribution. The hysteresis loss is also affected by the change in distribution.

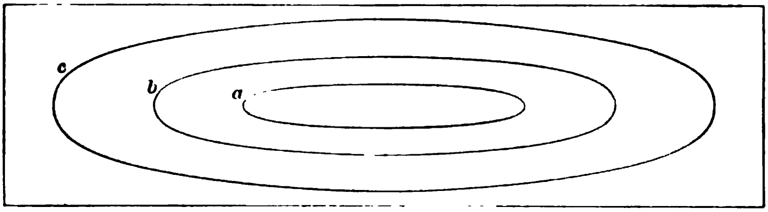

Fig. 2.

From this discussion we may conclude that an *exact* separation of the hysteresis and eddy current components of the energy loss can not be made upon the basis of the Steinmetz equation, but that an *approximate* separation can be made; and the accuracy of the separation will be greater the more nearly the following conditions are realized:

1. The two frequencies used should be low and as close together as the accuracy of observation will permit.
2. The form factor of secondary emf. should be the same at both frequencies.
3. The eddy current loss should be very small compared to the hysteresis loss.

The fulfillment of the first condition is limited by the fact that a given error of observation has a greater effect upon the results when the two frequencies are close together. In the present work 30 and 60 cycles are used exclusively.

The fulfillment of the second condition is limited by the fact that the wave is distorted by the ohmic drop of potential in the magnetizing circuit, and this effect is different at different frequencies. The magnetizing current changes but little with the frequency, while the induced emf. is proportional to the frequency. The distortion is therefore greater at lower frequencies. The best conditions are obtained when the resistance of the magnetizing circuit is low, when the iron is worked at maximum permeability, and when the cross-section of iron is large.

The third condition is best fulfilled by using wire of small diameter or sheets which are thin, by using steels of low electrical conductivity with large hysteresis, and by operating at low frequency. In the present work, transformer sheet was used for the first experiments, and later rings of steel wire. Finally, some wire of 0.002 inch diameter was obtained, in which the eddy currents were negligible.

THE APPARATUS.

The specimens experimented upon were mostly in ring form and consisted of sheet iron, both annealed and unannealed, and iron and steel wire of different diameters. A few measurements were made upon strips of transformer sheet built into a magnetic circuit after the manner of Epstein, but with less magnetic leakage. The specimen was wound with a magnetizing coil and with one or two secondary coils. The power was measured by a dynamometer wattmeter whose deflections were read by telescope and scale. The fixed coils of this instrument were traversed by the magnetizing current. The moving coil was connected to one of the secondary windings through a variable resistance or multiplier, which enabled a wide range of powers to be measured with equal accuracy—always better than 0.1 per cent. This wattmeter was calibrated with direct current. If n_1 is the number of turns in the magnetizing coil, and n_2 the number in the secondary coil, then $\frac{n_1}{n_2}$ times the wattmeter reading gives the power expended in the iron and in the secondary circuits. The effective voltage in the secondary circuits is measured by means of a dynamometer voltmeter. The square of this effective voltage divided by the resistance of any secondary circuit will give the power expended in that circuit, and this is applied as a correction to the measured watts. These corrections are never more than a few per cent.

The measurement of the maximum value of the magnetic flux, where the wave form is unknown, is not easy, and required the use of special apparatus, a description of which has already been published.[3] The value of the magnetic flux at any instant is proportional to the algebraic average of the induced electromotive force during the succeeding half cycle, providing the positive and negative half waves are alike, as in the present case. The maximum value of the flux is given by the average value of the induced emf. during the half cycle beginning with the zero value. The average value of the emf. is measured by connecting the secondary winding, through a rotating commutator, to a Weston D. C. voltmeter. The brushes of this commutator can be shifted, and the reading of the Weston instrument is proportional to the flux at the instant of commutation. By shifting the brushes until the Weston reading is a

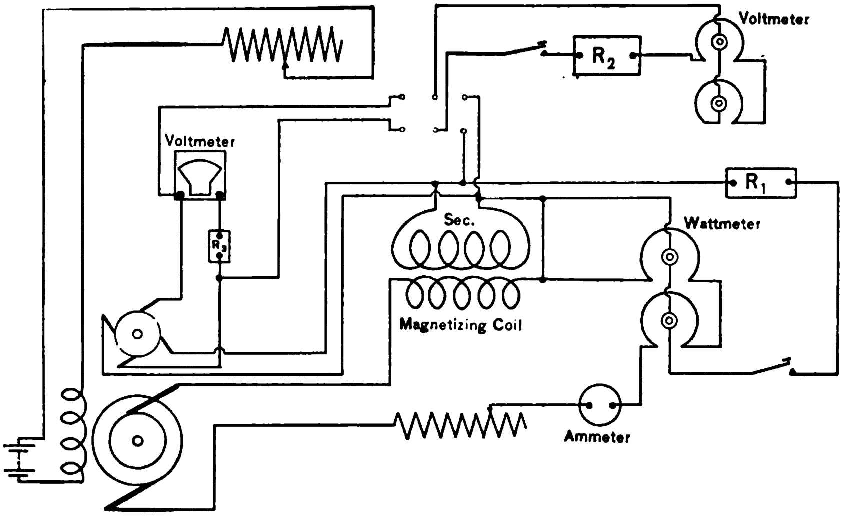

Fig. 3.—*Diagram of Connections.*

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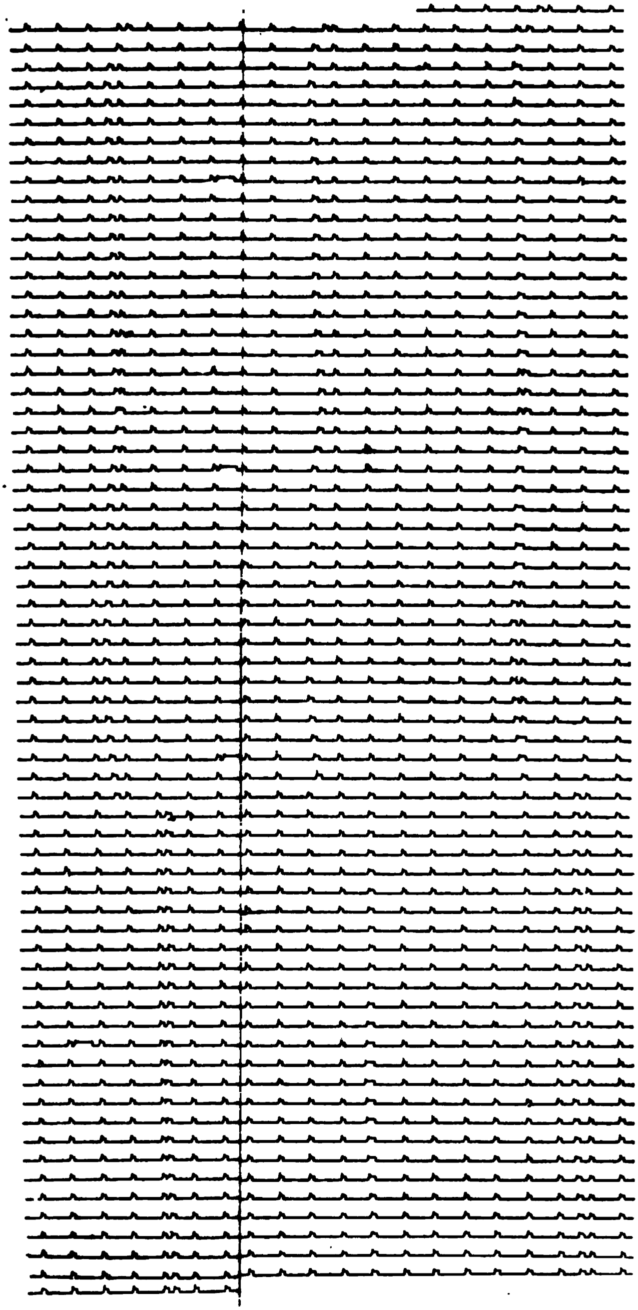
[3] Lloyd and Fisher, this Bulletin, 4, p. 467; 1908. The rotating commutator was constructed in the instrument shop of the Bureau by Mr. F. E. Mann.

maximum, the maximum value of the flux is measured. If at the same time the reading is taken upon a dynamometer voltmeter, the form factor of the emf. wave is thereby determined. By shifting the brushes until the Weston reading is a maximum, the maximum value of the flux is measured. If at the same time the reading is taken upon a dynamometer voltmeter, the form factor of the emf. wave is thereby determined. By shifting the brushes around, the successive values of the magnetic flux throughout the cycle can be read off and the wave form plotted. By connecting the commutator to the secondary of a mutual inductance whose primary is in the magnetizing circuit, the wave form of the magnetizing current may be similarly determined.

The rotating commutator is connected directly to the shaft of the generator, so that their phase relation is invariable. The generator is a four-pole, two-phase machine with surface-wound armature, and gives a sine wave. It is directly connected to a D. C. motor, driven by current from a storage battery. A second generator, with 12 poles, is coupled to the same shaft and gives a frequency three times that of the first. The motor field circuit includes a rheostat in the laboratory, by means of which the speed can be adjusted. The speed is automatically recorded upon a chronograph by means of an electrical contact which is made once every hundred revolutions through the medium of a worm gear. This chronograph was made by the International Instrument Company, and is controlled by a standard clock so that its drum makes exactly one revolution per minute. The seconds are recorded upon the drum by the same fountain pen which registers the speed of the generator, the second marks falling in rows parallel to the axis of the drum. When the generator is making 900 revolutions per minute, corresponding to a frequency of 30 cycles per second, the speed marks are 6⅔ seconds apart, giving 9 marks to the minute. If the speed has been exactly adjusted, these marks also form rows parallel to the rows of second marks, and a glance at the drum is sufficient to show whether the speed is correct. Fig. 4 shows a sample record upon the chronograph. In all cases the speed was adjusted to give 30 or 60 cycles within 0.1 per cent before readings were taken, and consequently no corrections to the wattmeter or voltmeter readings are necessary on account of frequency.



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The generator field circuit also includes a rheostat in the laboratory, and this is used in adjusting the voltage to give the desired flux density. This makes it unnecessary to have an adjustable rheostat in the magnetizing circuit to control the current. A rheostat is always included in this circuit, however, in order to prevent a big rush of current when the circuit is first closed. After closing the circuit this resistance is reduced to zero, when it is desired to use a wave as nearly sinusoidal as possible. In other experiments a large resistance was left in for the purpose of distorting the wave, which is conveniently done in this way. Distortion was also secured by connecting the second generator in series with the first and exciting it to a low voltage. Either phase could be used, and the terminals could be reversed, thus giving a wide range of wave forms, as will be seen from the illustrations and the tabulated form factors.

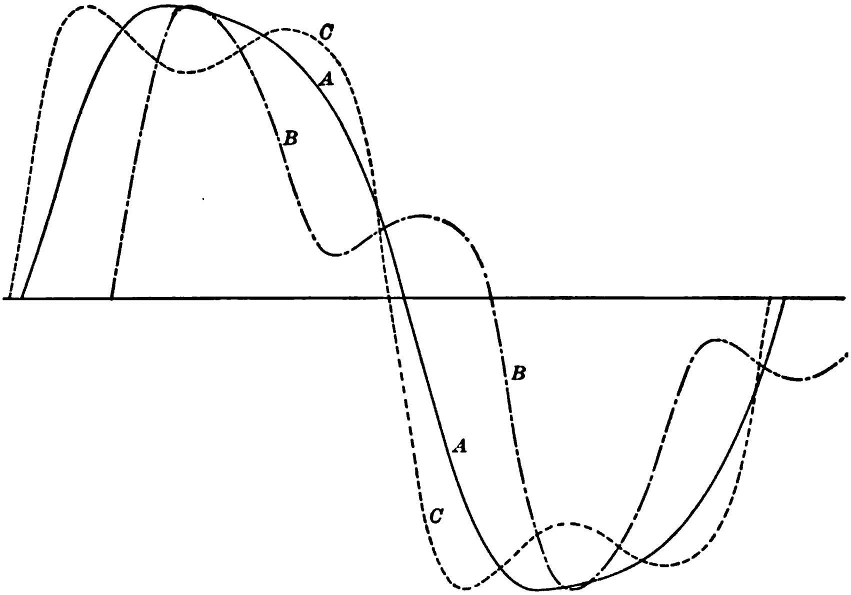

Fig. 5.—*Flux Waves.*

A. Undistorted wave for Experiments 8 and 9.
B. Distorted wave for Experiment 8.
C. Distorted wave for Experiment 9.

The Weston voltmeter used is an especially sensitive instrument, giving a full scale deflection for a current of about 0.0004 ampere. This permitted the use of a high resistance in its circuit, making negligible any error due to its inductance, or to the variable resistance of the moving contacts on the commutator. The constants of the various specimens used in the experiments are as follows:

TABLE I.

Designation	Form	Material	Weight	Outer Diameter	Inner Diameter	Magnetizing Turns	Secondary Turns	Cross Section	Eddy Current Loss at 30 cycles 10000 gausses	Average Volts for 10000 gausses 30 cycles
			grams	cm	cm			sq. cm	Per cent	
No. 28	Ring	Transformer sheet, 0.033 cm thick, not annealed.	199.5	8.9	6.9	102	320	1.048	14	4.023
No. 30	"	Same	390.4	8.9	6.9	100	315	1.984	13	7.500
No. 31	"	Same, annealed	394.9	8.9	6.9	100	330	2.074	19	8.213
No. 36	"	Iron wire, 0.02 cm diam.	185.5	11.0	8.3	260	414	0.795	1.5	3.950
No. 38	"	Steel wire, 0.005 cm diam.	157.9	10.15	7.9	377	541	0.716	0.0	4.650
No. 41	"	Transformer sheet annealed, 0.035 cm.	203.7	22.85	20.3	158	520	0.3865	19	2.412
No. 76	"	Transformer sheet, 0.059 cm thick, not annealed.	316.8	22.85	20.3	352	500	0.601	20	3.606
G	Epstein Square	Transformer sheet, 0.033 cm thick, not annealed.	1542	......	......	1000	180	1.950	7	4.218
W	"	Transformer sheet, 0.032 cm thick, annealed.	765.6	......	......	700	700	0.493	28	4.140
S	"	Common sheet iron, 0.06 cm.	657.5	......	......	700	700	0.421	18	3.537

OBSERVATIONS.

The first observations were taken in May, 1907, soon after the rotating commutator was constructed. The method followed has already been indicated. Runs were taken at 30 and 60 cycles with as little distortion of wave as possible, the voltage of the generator being adjusted to give a definite flux density. The form factor of secondary electromotive force was determined in each case, and a separation of eddy current and hysteresis losses effected. The generator of triple frequency was then connected in series with the first, and runs taken at 30 cycles with distorted wave forms, the form factor being determined in each case. Usually the form of the flux wave was also plotted by taking readings on the Weston instrument while shifting the commutator brushes from point to point. If the flux wave should have a dimpled form, the experiment must be excluded from consideration, for this indicates that the iron has not been put through a simple hysteretic cycle, but a subsidiary loop has been included in the cycle, as shown in Fig. 10, dot and dash curve.

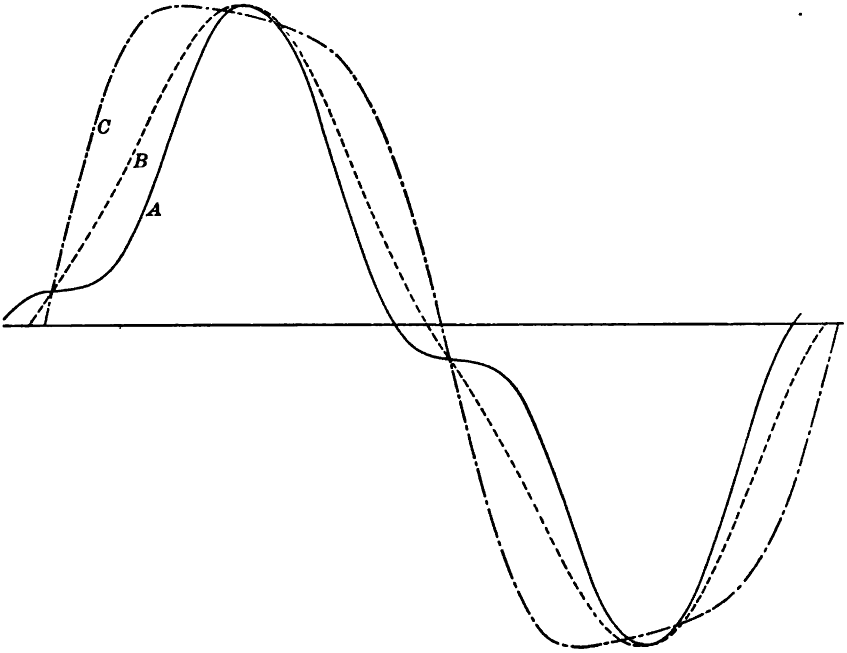

Fig. 6.—*Flux Waves with Ring No. 31.* B_{max}*=10000 gausses. Experiments Nos. 19, 20, 21.*

In such cases the energy is always greater than in the case of a single loop. Some cases of this kind are included in Table II and illustrated in the flux waves of Fig. 5. These results are bracketed, and are not to be considered as bearing upon the main problem. Flux waves were always platted when there seemed a possibility of

TABLE II. (30 Cycles.)

Experiment No.	Specimen	Maximum Flux Density (Gausses)	Form Factor with Undistorted Wave	Form Factor with Distorted Wave	Increase in Hysteresis (Per cent)	Cause of Distortion	Type of Flux Wave
1	W	5000	1.175	1.29	+ 0.1	3rd Harmonic 11 %	
2	W	10000	1.26	1.22	− 0.9	"	
3	W	10000	1.26	1.34	+ 1.2	"	
4	W	12000	1.32	1.27	− 1.1	"	
5	W	12000	1.32	1.40	0.0	"	
6	S	5000	1.28	1.21	− 0.2	3rd Harmonic	
7	S	5000	1.28	1.39	+ 0.2	"	
8	S	5000	1.29	[1.52	+ 2.2]	"	B Fig. 4
9	S	5000	1.29	[1.74	+ 5.7]	"	C Fig. 4
10	No. 28	10000	1.54	1.17	− 1.3	"	
11	No. 28	10000	1.54	1.82	+ 0.6	3rd Harmonic	
12	No. 30	10000	1.19	1.12	+ 1.2	" 21 %	
13	No. 30	10000	1.19	1.30	− 0.1	" 21 %	
14	No. 30	10000	1.19	1.21	+ 1.2	" 63 %	
15	No. 30	10000	1.19	1.48	+ 1.0	" 63 %	
16	No. 31	10000	1.17	1.09	+ 0.7	3rd Harmonic 27 %	
17	"	10000	1.17	1.35	− 2.0	" 22 %	
18	"	7500	1.16	1.38	+ 1.0	" 30 %	
19	"	10000	1.165	1.20	− 0.5	" 83 %	A Fig. 5
20	"	10000	1.165	1.075	− 1.2	" 30 %	B Fig. 5
21	No. 31	10000	1.165	1.34	0.0	3rd Harmonic 26 %	C Fig. 5
22	No. 41	10000	1.415	1.615	− 1.2	" 24 %	A Fig. 11
23	No. 41	10000	1.415	[1.83	+ 1.2]	" 29 %	Dimpled
24	No. 76	10000	1.18	1.19	− 0.4	" 29 %	A Fig. 6
25	No. 76	10000	1.18	1.38	+ 1.1	" 31 %	A Fig. 11
26	G	10000	1.11	1.057	− 0.3	3rd Harmonic 23 %	C Fig. 6
27	"	10000	1.11	1.292	+ 0.3	" 27 %	D Fig. 7
28	"	10000	1.11	1.408	+ 0.8	" 46 %	B Fig. 6
29	"	10000	1.11	1.069	0.0	" 44 %	C Fig. 6
30	"	5000	1.11	1.067	− 0.2	" 44 %	C Fig. 6
31	"	5000	1.11	1.405	+ 0.3	" 46 %	B Fig. 6

TABLE III.

Ring No. 31.

This Ring has two secondaries, one of 330 turns connected to the voltmeter and one of 100 turns connected to the wattmeter. $\left(\frac{100}{330}\right)^2 = .092$.

Experiments 19, 20, 21.

Cycles	Total Watts	Form Factor	E Effective Secondary Volts	$\frac{E^2}{R_v}$	$.092\frac{E^2}{R_v}$	Instrument Losses	Iron Losses		Loss per Cycle		Change in Loss per Cycle		
							Watts	Joules per Cycle	Hysteresis	Eddy Currents	Total	Eddy Currents	Hysteresis
60	1.9072	1.157	19.02	.0054	.0166	.0220	1.8852	.03142	.02182	.00960	....	....	....
30	0.8118	1.165	9.57	.0027	.0084	.0111	0.8007	.02669	.02182	.00487	....	....	....
30	0.8170	1.198	9.85	.0029	.0089	.0118	0.8052	.02684	....	.00513	+.00015	+.00026	−.00011
30	0.7828	1.075	8.83	.0023	.0072	.0095	0.7733	.02569	....	.00414	−.00100	−.00073	−.00027
30	0.8624	1.341	11.00	.0036	.0106	.0142	0.8482	.02827	....	.00644	+.00158	+.00157	+.00001

the wave being dimpled, and many later runs have been excluded from the tables on this account. When the distortion of wave is produced by resistance in the magnetizing circuit the wave is never dimpled, but has a characteristic shape, more or less pronounced, according to the magnitude of the resistance. Table II gives a summary of the results obtained at various times from May to December, 1907. Some of the wave forms are illustrated in Figs. 6 and 7.

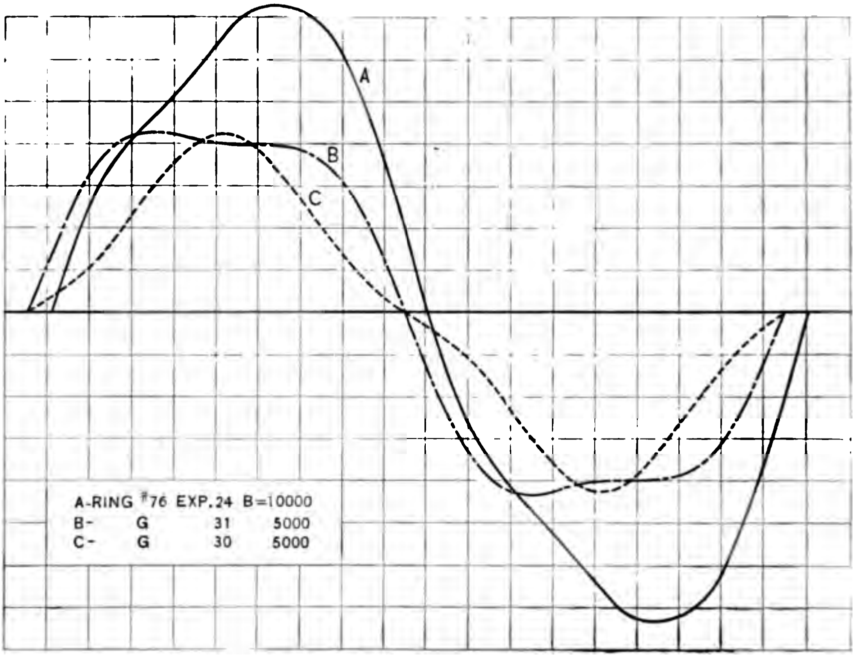

Fig. 7.—*Flux Waves.*

Table III is a sample of the observed values, with the method of computing results.

Next a ring was prepared from No. 36 iron wire with the idea of decreasing the eddy currents and thus making a more accurate separation of the two components of loss. The results obtained with this ring are given in Table IV and some of the flux curves illustrated in Fig. 8. The outstanding change in energy is here never more than 1 per cent, with a single exception. That the separation

TABLE IV.—Ring No. 36.

AT 30 CYCLES.

Experiment No.	Maximum Kilogausses	Form Factor with Undistorted Wave	Form Factor with Distorted Wave	Increase in Hysteresis	Cause of Distortion	Type of Flux Wave
				Per cent.		
32	5	1.149	1.119	0.0	3rd Harmonic 46½ %	B Fig. 7
33	5	1.149	1.389	+0.4	" "	A Fig. 7
34	5	1.161	1.249	0.0	10 Ohms	B Fig. 8
35	5	1.161	1.309	+0.1	10 "	B Fig. 8
37	10	1.193	1.379	0.0	10 Ohms	B Fig. 8
38	10	1.193	1.482	0.0	20 "	B Fig. 8
39	5	1.148	1.076	+0.3	3rd Harmonic 28 %	C Fig. 7
40	5	1.148	1.248	—0.2	" 26 %	D Fig. 7
41	5	1.148	1.426	—0.6	75 Ohms	B Fig. 8
42	5	1.148	1.212	—0.2	10 Ohms	B Fig. 8
43	5	"	1.249	—0.3	20 "	"
44	5	"	1.346	—0.5	30 "	"
45	5	"	1.406	—1.0	70 "	"
46	5	"	1.441	—1.0	100 "	"
47	5	"	1.455	—0.9	140 "	"
48	10	1.196	1.322	0.0	10 Ohms	B Fig. 8
49	10	"	1.420	+0.1	20 "	"
50	10	"	1.522	+0.5	30 "	"
51	10	"	1.566	+1.0	50 "	"
56	10	1.204	1.324	0.0	10 Ohms	B Fig. 8
57	10	"	1.477	—0.3	25 "	"
58	10	"	1.559	—0.2	35 "	"
59	10	"	1.585	0.0	50 "	"
60	10	"	1.663	—0.2	65 "	"

AT 60 CYCLES.

36	5	1.145	1.234	+0.3	20 Ohms	B Fig. 8
52	10	1.165	1.289	+0.1	20 "	"
53	10	1.165	1.415	+0.3	40 "	"
54	10	1.165	1.566	+1.0	75 "	"
55	10	1.165	1.599	+1.2	100 "	"

of losses was not all that could be desired was shown in the following way. By trial, resistances were found whose introduction in the magnetizing circuit gave the same form factor for runs at 30 and 60 cycles. Separations were thus made at three form factors between 1.15 and 1.30, with a discrepancy in the results of 1 per cent.

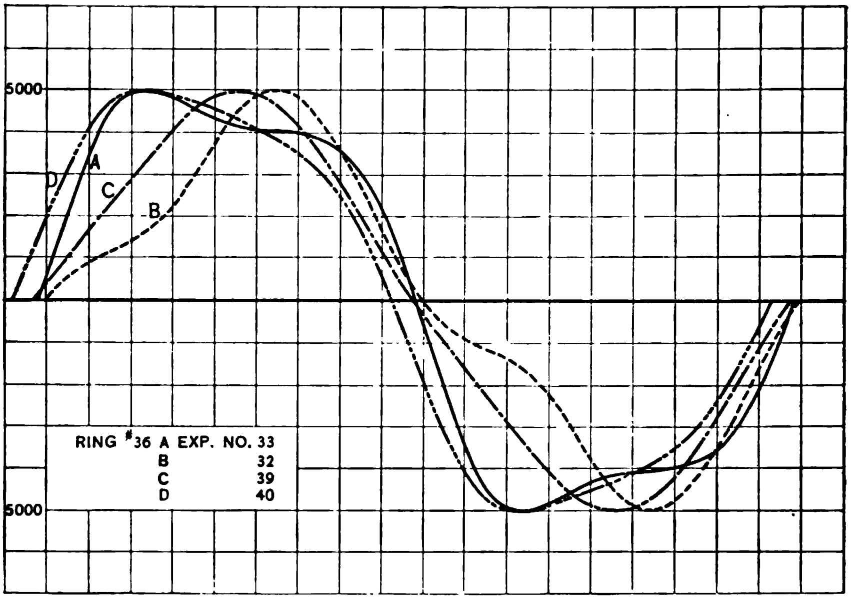

Fig. 8.—*Flux Waves.*

The next step was to obtain some soft steel wire 0.002 inch in diameter. This was soaked in shellac while being wound upon a brass form, and afterwards baked. It was then removed from the form and wound with magnetizing and secondary coils. The eddy current loss in this ring (No. 38) was less than 0.1 per cent of the whole, and changes in the eddy current loss were negligible. A separation of the losses was therefore unnecessary. There was still a slight change in the instrument losses with form factor. This was small and easily computed, and if the change in wattmeter reading differed from this computed change it indicated a change in the hysteresis. These experiments are therefore the most conclusive of any. In each case the distorted wave was applied several times, intermediate readings being taken with undistorted wave, and unless the readings could be repeated the results have been discarded. It was usually necessary to have the current flowing for some time before the observations were taken, in order to reach an approximate equilibrium of temperature. In making changes it was endeavored to keep the flux density *below* the maximum used in setting. When through inadvertence it rose higher, the specimen was demagnetized by gradually reducing the magnetizing current. In making the final adjustments before reading, the maximum flux would necessarily vary slightly upon both sides of the desired setting. The results with this ring are given in Table V and some of the flux curves in Fig. 9. Table VI gives the details of experiment No. 67.

TABLE V

Ring No. 38, at 30 Cycles.

Experiment No.	Maximum Gausses	Form Factor with Undistorted Wave	Form Factor with Distorted Wave	Increase in Hysteresis (Per cent)	Cause of Distortion	Type of Flux Wave
61	5000	1.145	1.320	0.0	3rd Harmonic 25 %	A Fig. 8
62	5000	1.168	1.407	+0.35	50 Ohms	B Fig. 8
63	5000	1.153	1.365	0.0	35 "	B Fig. 8
64	5000	1.157	1.328	0.0	3rd Harmonic 25 %	C Fig. 8
65	2500	1.124	1.254	+0.1	55 Ohms	B Fig. 8
66	5000	1.162	1.363	+0.85	39 "	B Fig. 8
67	5000	1.155	1.442	+0.9	80 "	B Fig. 8
68	10000	1.187	1.653	+0.4	54 "	B Fig. 8
69	10000	1.190	1.422	+0.3	3rd Harmonic 34 %	A Fig. 7

Energy Loops.—By taking readings of the instantaneous values of both magnetizing current and flux through a cycle, it is possible to plat these values so as to form a closed loop whose area is a measure of the energy expended by the current. This energy is due to hysteresis, to eddy currents, and to the copper loss in the secondaries. The latter can be made very small by breaking the potential circuit of the wattmeter, leaving only the high resistance Weston voltmeter as a secondary load. As indicated in Table VI, this would ordinarily be not more than one-fourth per cent of the total energy. The total iron loss can be obtained in this way, but there is no means of separating the two components.

Such loops have been platted for Rings 36, 38, and 41. Fig. 10 shows several half loops for No. 41, only two of which have been drawn in full, the important parts of the others alone being indicated. Fig. 11 shows similar results for No. 36, and Fig. 12 gives the forms of the waves of current and flux. In each case, for the loop at 60 cycles, the readings near the maximum values have been taken at closer intervals and platted to a much larger scale in the upper left-hand corner. These show distinctly that the current reaches a maximum before the flux. The elapsed time between the maxima is, in the case of No. 41, equal to 0.00028 sec. or 1.7 per cent (1/60) of the time of a complete cycle; for No. 36 it is 0.00014 sec. or 0.8 per cent of the time of a cycle. A similar condition

TABLE VI.

Details of Experiment No. 67.

Ring No. 38, at 30 cycles and 5000 gausses.

Extra Resistance	Brush Setting	Wattmeter Deflection	Effective Volts	Form Factor
Ohms	Degrees	cm		
80	70	20.91	3.354	1.442
0	89	20.59	2.687	1.155
80	69	20.86	3.351	1.441
0	89	20.59	2.687	1.155

Change=0.29 cm.

Weston reading for average volts=2.325.
Weston resistance=7885 ohms.
Wattmeter resistance=2000 ohms.

Instrument losses $=(2.687)^2\left(\frac{1}{7885}+\frac{1}{2000}\right)=0.0045$ watts.
$=0.174$ cm.

$\left(\frac{1.442}{1.155}\right)^2 0.174=0.27$ cm. Change=0.10 cm.

Increase in hysteresis=0.19 cm=0.9 %.

NOTE.—The dynamometer voltmeter is disconnected while wattmeter is read.

could not be established with certainty for No. 38, although a slight difference in phase between the two maxima was indicated.

Hysteresis loops for these same rings have been determined by using the ballistic method with direct currents, and these results also are platted in the figures. The area of these ballistic loops has been determined, and the energy compared with the wattmeter measurements.

TABLE VII.

Comparison of Hysteresis Measurements by Wattmeter and Ballistic Methods.

Ring	Maximum Flux Density B	Hysteresis			Exponent of B (Ballistic)
		Ergs per cm³ per cycle		Difference	
		Wattmeter	Ballistic		
	Gausses			Per cent	
41	5000	1318	1310	0.6	1.78
41	10000	4722	4500	4.7	
36	5000	4750	4540	4.3	1.56
36	10000	14100	13370	5.2	
38	2500	4547	4548	0.0	1.56
38	5000	13380	13380	0.0	1.34
38	10000	....	33910	....	1.35
38	15000	....	58720	....	

With Ring No. 38 the loops obtained (for maximum of 10000 gausses) by the two methods agreed within the error of observation. The hysteresis loss, as determined by the wattmeter, was the same for this ring as when determined by the ballistic method. Table VII gives a comparison of the results by the two methods. The power of B to which the loss is proportional is also indicated, and it is apparent that this may be quite different in different specimens, as well as varying with the flux.

I am indebted to Dr. C. W. Burrows for the measurements with the ballistic galvanometer, and to Mr. J. V. S. Fisher for assistance in the work with alternating currents.

DISCUSSION OF THE RESULTS.

The first results obtained appeared to indicate a definite dependence of hysteresis upon the wave form. The differences obtained were much greater than could be accounted for by errors in taking the readings. The settings on the Weston voltmeter could be made to within 0.1 per cent, and the other instruments could be read more accurately than this. Indeed, in magnetic work, it seems useless to strive for great precision in reading, since conditions can not be duplicated as closely as instruments can be read. It is well known that the present magnetic condition depends upon certain features of its past treatment, and it would seem to depend upon some others that are not so well known. After a specimen has been highly magnetized, the hysteresis loss at a lower induction will depend upon whether it has been subsequently demagnetized. This is well recognized in ballistic work, but it seems to be disregarded in magnetization with alternating current, where its influence, nevertheless, is just as potent. A transformer which has been tested at double voltage may continue for years to consume more energy than would be necessary if it were afterwards demagnetized. Even when demagnetization is resorted to, and the temperature maintained within narrow limits, the iron losses do not seem to return consistently to a previously determined value, and measurements made under condi-

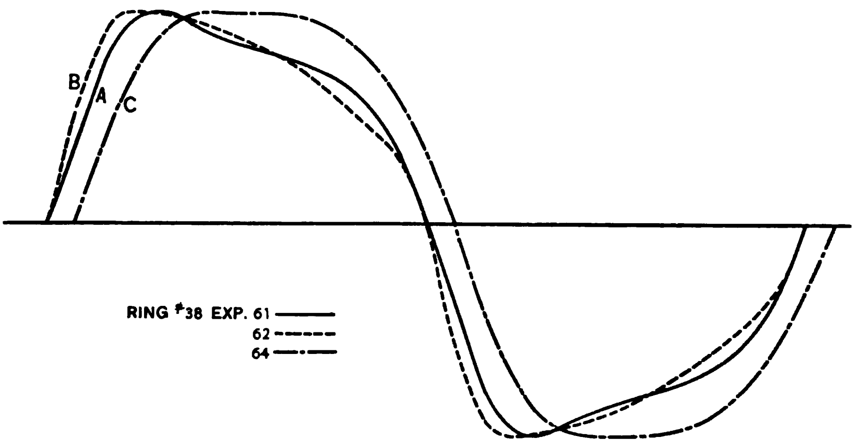

Fig. 9.—*Flux Waves.* $B_{max}=5000$ *gausses.*

tions supposedly identical do not yield results in as close agreement as 0.1 per cent. The accuracy of the instruments used in this research therefore seems more than sufficient. In the final experiments care was taken to avoid disturbing influences, and unless readings could be repeated within one-fourth per cent when changing from one wave to another and back, they were discarded. Table VI illustrates that this could be done. In the earlier experiments, however, this repetition was not required, and many of the results given are from single observations. Temperature effects, however, were avoided by changing quickly from one wave form to another.

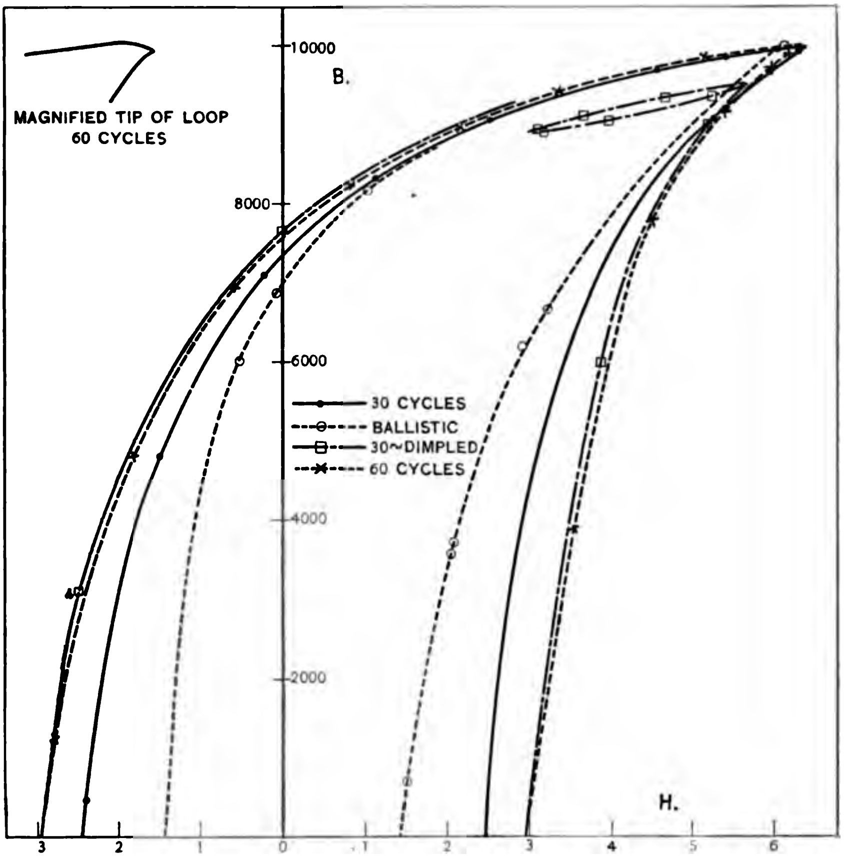

Fig. 10.—*One half of Energy Loops with direct current and with different wave forms of alternating current. Ring No. 41.* $B_{max}=10000$ *gausses.*

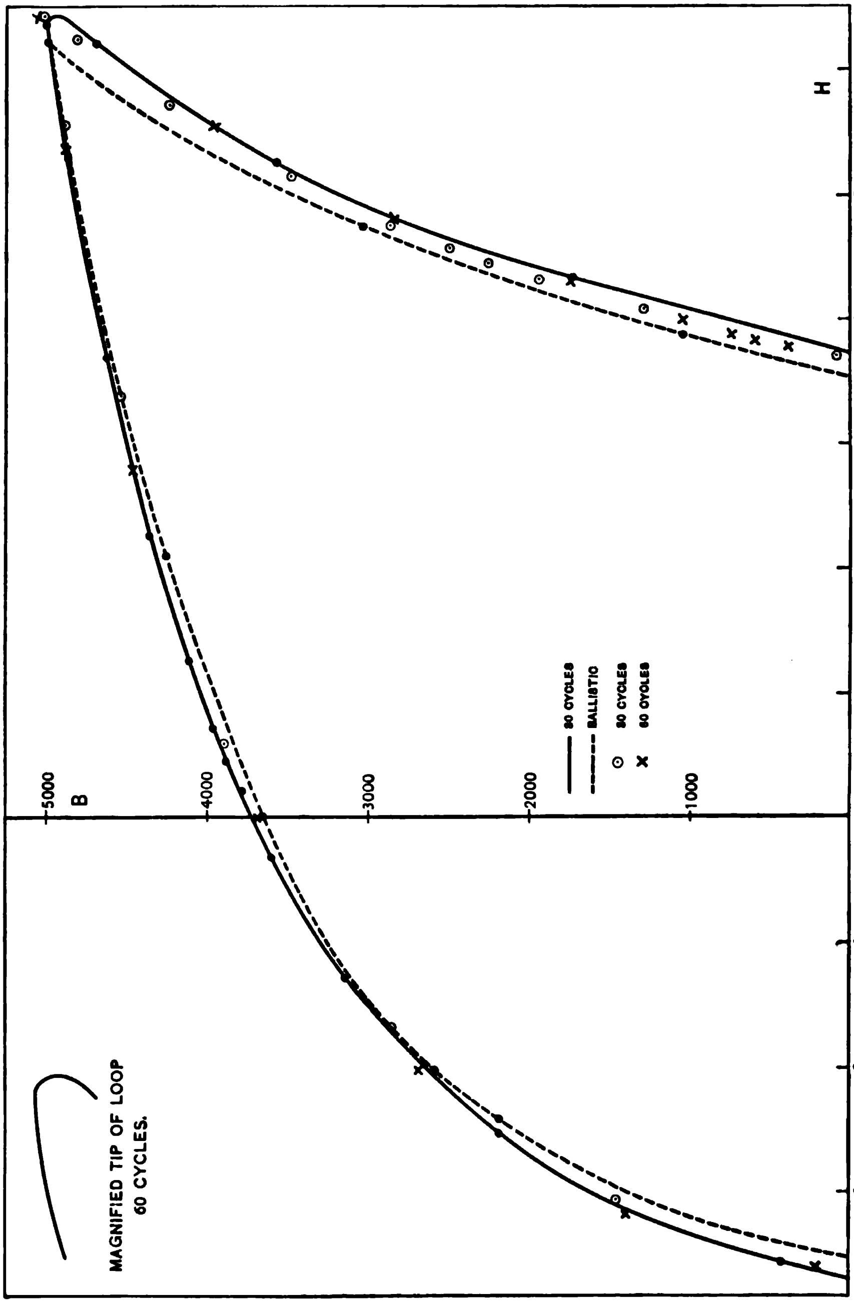

Fig. 11.—*Energy Loops with Ring No. 36.*

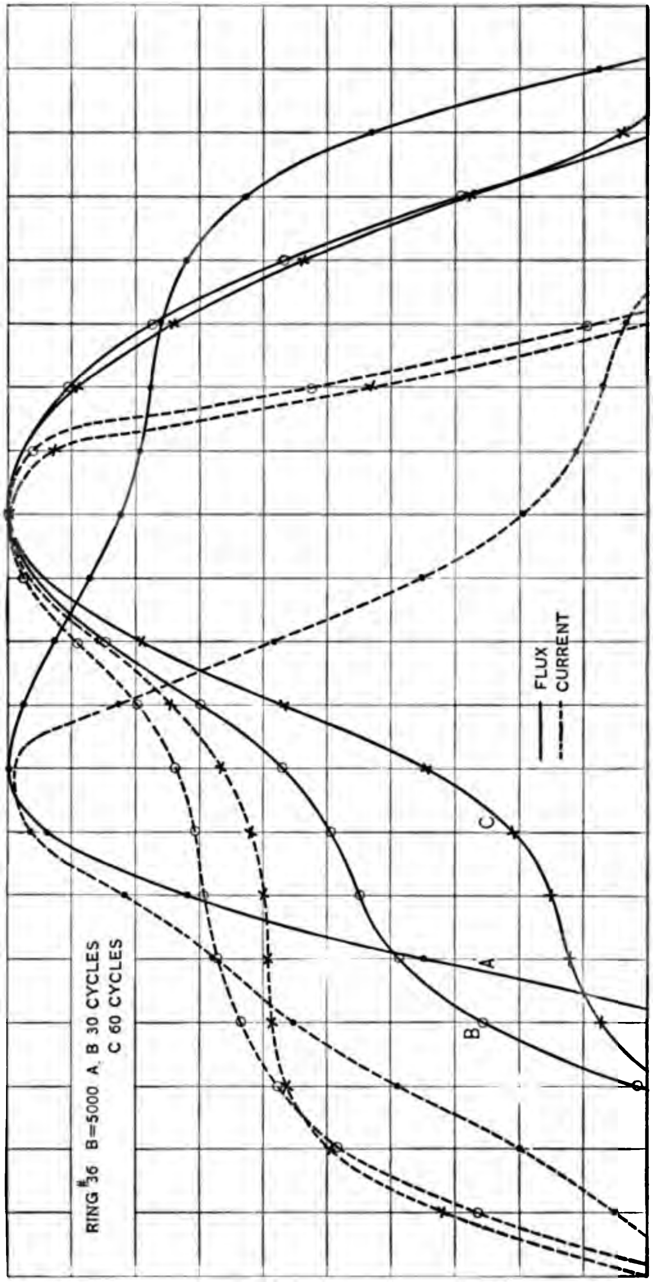

Fig. 12.—*Curves of Magnetizing Current and of the Flux produced by it, corresponding to the Energy Loops of Fig. 11.*

When the distorted wave gave a secondary electromotive force with form factor lower than that used for the separation of the losses, the hysteresis appeared smaller; when the new form factor was higher, the hysteresis appeared larger. The eddy current loss formed a goodly portion of the total in all of these early specimens, however, and there was the uncertainty of the separation to be considered, especially since the computed changes in hysteresis were all under 2 per cent. Moreover, the direction of change above noted was not maintained in all cases with the ring specimens.

With Ring No. 36 the eddy current loss was a very small part of the whole, and the indicated changes at once were confined to a limit of 1 per cent (with a single exception at 1.2 per cent). In experiments Nos. 48 to 51 at 30 cycles and Nos. 52 to 55 at 60 cycles, there seemed to be a progressive change evident with increasing distortion; yet these results are in contradiction to those of experiments Nos. 56 to 60, and in the opposite sense to those of Nos. 42 to 47 at a different induction. The variations in hysteresis appear, then, rather accidental than systematic.

Turning to Table V, the results with Ring No. 38 can not be attributed to eddy currents. The whole question hinged here upon whether the observed change in wattmeter reading was accounted for by the change in instrument losses. In three cases it was so accounted for, but in others it was unmistakably not so accounted for, and the change in hysteresis seems positive. The change is in the direction first noted; that is, the hysteresis increases with the form factor of induced electromotive force. In other words, a flat flux wave causes a higher value for the hysteresis loss than a sinusoidal or peaked wave. No relation between the quantitative value of the effect and the amount of distortion of the wave is evident. The effect may be explained on the basis of a magnetic viscosity. If the flux values lag behind the corresponding values of magnetizing force, then the hysteresis loop will be broader for a flat flux wave, since the change through the zero values is more rapid. This explanation would require also a higher value of the hysteresis with higher frequency.

The comparison of the ballistic and the wattmeter measurements shows that the latter sometimes gives a larger value for the hysteresis. In Ring No. 38, however, where all uncertainty due to eddy currents has been eliminated, there is no difference; and even with Ring No. 41, where the eddy currents are of considerable magnitude, the discrepancy does not exceed 5 per cent. This is in contradiction to the results of Lyle, Niethammer, and others, but confirms the work of Gumlich and Rose.[4]

In Figs. 10 and 11 it is noticeable that where the plotted points for any curve lie close together, indicating that at this part of the cycle the changes were proceeding slowly, the curve is here also closer to the ballistic curve, since the influence of eddy currents is minimized. In Fig. 10 the dot-and-dash line represents a cycle with a dimpled flux wave, and it consequently forms a subsidiary loop, the completion of this small loop occupying about half the time of a half cycle. Consequently the remaining part of the half cycle must also be completed in half the time, and the consequence is that the values for this 30-cycle loop are changing at about the same rate as for a 60-cycle loop. Inspection of the figure shows that the points fall very close to the curve obtained at 60 cycles, and both of these are much farther from the ballistic loop than the normal curve for 30 cycles (solid line).

In the case of both Rings Nos. 36 and 41 the maximum magnetizing force (externally applied) with alternating current is considerably larger than the maximum used in the ballistic test to obtain the same flux density. This is due to the eddy currents, which exert a demagnetizing force, and part of the nominal magnetizing force is necessary to neutralize this field. Consequently computations of permeability based on such values would yield spurious results. In the case of Ring No. 38 such a difference between the magnetizing forces in the two cases was not observed, and the permeability was apparently as high with alternating as with direct current.

DISCUSSION OF PREVIOUS INVESTIGATIONS.

The problem of the effect of wave form upon hysteresis has been attacked by several investigators, and the closely related problem of the effect of frequency by a great many more.

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[4] E. Gumlich and P. Rose, Electrot. Zs., **26**, p. 503; 1905.

Niethammer[5] has obtained the most remarkable results with respect to wave form. He measured the power consumed with a wattmeter, and computed the maximum flux density from the average electromotive force, determined by plotting the wave of electromotive force. The two components of loss were separated by taking observations at two frequencies (37 and 59 cycles). In a ring built up of sheet iron, he observed a decrease in the hysteresis with a peaked emf. wave as compared with the value with an approximately sinusoidal wave. The form factors are not given, but the wave forms used are illustrated in another article.[6] The difference in hysteresis amounted to 16 per cent at 2300 gausses, and to *over 30 per cent at 13300 gausses.* It is to be noted that the curves taken are not of secondary emf., but of primary emf. The resistance of the magnetizing coil is not given, so that it is not possible to judge whether the ohmic drop in this winding would seriously affect the wave form or the voltage reading from which flux density is computed. Evidently no allowance is made for either of these possible sources of error, and it is assumed that the wave form is the same at different frequencies. Even so, it does not seem probable that they could account for differences as large as those observed. It is to be noticed also that he found the hysteresis with alternating current larger by about 20 per cent than the hysteresis obtained by a static method.

Benischke,[7] on the other hand, found the hysteresis increased by a peaked wave of electromotive force, and decreased by a flat wave. His results carry little weight, however, as they are vitiated by the use of an incorrect relation between the maximum flux density and the observed effective voltage. He, too, made no allowance for the ohmic drop in the magnetizing coil, and did not read the voltage at the time of observation, measuring the watts with voltmeter disconnected, and consequently with a voltage different from that recorded.

Krogh and Rikli[8] measured the total iron losses in ring specimens by the wattmeter method, and computed the eddy current loss from theoretical considerations. The remainder was assumed to be due to hysteresis, and this showed an increase with the form factor of the applied emf. and also with the frequency. The change between form factors of 1.11 and 1.16 amounted to about 2 per cent at 5000 gausses and 50 cycles per second. There are three objections to accepting their results. The method of measuring the flux density is not given. The ratio of outer to inner diameter of the rings was 1.55, so that there was an enormous variation of flux density over the cross section of the material.[9] Finally, the eddy current loss, which in the case cited amounted to a fifth of the total loss, is not measured but computed from a formula, and hence uncertain.

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[5] F. Niethammer, Wied. Ann., **66**, p. 29; 1898.

[6] Electrot. Zs., **19**, p. 669; 1898.

[7] G. Benischke, Electrot. Zs., **22**, p. 52; 1901: Discussion, pp. 111, 185, 267, 313. Ibid., **27**, p. 9; 1906: Discussion, pp. 235, 236.

[8] Krogh and Rikli, Electrot. Zs., **21**, p. 1083; 1900.

Sahulka[10] used a half-period contact ring (same principle as used by me) for determining average emf. and thereby the correct flux density, and found an increased loss when using a wave similar to that shown in Fig. 6, Curve A. He used a ring composed of insulated wires, 0.018 cm in diameter, in which he assumed that the eddy currents were negligible, regardless of the fact that the loss per cycle increased with the frequency. No numerical results are given, and without further details it does not seem fair to assume that the increased loss was not due to eddy currents. The form factor of the secondary emf. induced by the given flux wave would be greater than for a sine curve, and both eddy current loss and instrument losses would be increased by using such a wave.

The form of wave used in one of their researches was varied by Gumlich and Rose[11] so that the form factor of electromotive force had values from 1.06 to 1.19. The resulting change in iron losses was fully accounted for by the change in eddy currents within the accuracy of the observations, which was about 2 per cent.

The effect of frequency has received a great deal more attention, both by earlier and more recent investigators. The earlier researches were largely faulty in method and contradictory in results, and as they have been discussed by Wien[12] it is not necessary to consider them here. The work of Wien was undoubtedly the most painstaking and thorough which had been done up to that time, and his method was ingenious. He measured the change in effective resistance and inductance of the magnetizing circuit by a bridge method, using a sinusoidal current. Equations were derived connecting the energy, the permeability, and the flux with these, and with the magnetizing current, from which also the magnetizing force could be computed. Neither flux nor energy was measured directly. Since the magnetizing current was sinusoidal, the flux was far from sinusoidal, especially at high inductions, and corrections had to be made for the resulting harmonics in the eddy currents. The reduction in maximum flux density by the demagnetizing effect of the eddy currents was allowed for, but never amounted to more than 2.7 per cent, and was negligible in the small wires. The worst error of observation was of about the same magnitude. Small rings of wire were experimented upon and the results compared with static measurements by the ballistic method. Changes as great as 70 per cent in the hysteresis of soft iron at the same high flux density were computed for a frequency of 520 cycles. Smaller proportional effects were obtained at lower frequencies and with harder steel, but the absolute values of the changes with the latter were just as large as with the soft iron.

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[9] Cf. this Bulletin, **5**, p. 435; 1909.

[10] J. Sahulka, Electrot. Zs., **28**, p. 986; 1907.

[11] E. Gumlich and P. Rose, Electrot. Zs., **26**, p. 403; 1905: Discussion, pp. 500, 576.

[12] M. Wien, Wied. Ann., **66**, p. 859; 1898.

It is greatly to be regretted that the care and energy given to this work were not expended in the direct measurement of the quantities involved. The values of flux density used in comparison here are the values computed for the maximum of the *fundamental component* of the flux wave, and since the harmonics were very prominent this probably is quite different from the maximum value for the actual wave, which has not been determined. The eddy current corrections, due to the harmonics of the flux wave, are large and uncertain, and it would seem in consequence of these two facts that the numerical values of Wien are not reliable. It seems hardly likely, however, that the entire effect found by him could be thus explained, and his conclusion seems justified, that at high frequencies the magnetization can not follow the magnetizing force quickly enough to reach the values obtained with direct current, making the iron appear harder.

Lyle[13] has also found that the hysteresis with alternating currents is greater than that measured by a ballistic test, and that it increases with the frequency. His method consisted in platting the curves of flux and magnetizing current, analyzing these, and computing the energy from the elements. The highest frequency used was 50 cycles, and the magnitude of the effect claimed (at this frequency and 10000 gausses) is about 30 per cent of the ballistic loss. He, like Wien, aimed to secure a sinusoidal magnetizing current, with the result that the flux wave was very much distorted, and the results may be interpreted as a change in hysteresis with wave form just as properly as a change with frequency. The one condition was not held constant while the other was varied, and consequently the effects of the two can not be separated. The correction for energy expended in eddy currents varies in amount from 0.7 to 2 times the frequency effect claimed, yet this correction is based entirely upon a computed estimate and not upon any experimental determination. The formula given by Lyle as expressing the relation between total loss per cycle, frequency and magnetic induction, is not in terms of the *maximum* magnetic induction, but what he calls the "effective" induction, which is the maximum induction which, with a sine wave, would induce the same effective voltage as the actual wave. Moreover, there is no separate term for eddy current loss, this being combined with the frequency term of the hysteresis loss to give about 1.5 per cent change per cycle. The author points out himself that this formula holds only when the wave of magnetizing current is sinusoidal; it is, then, of very limited application.

In direct contradiction to Wien and Lyle are the results of Maurain,[14] of Williams and Kaufmann,[15] of Guye, Herzfeld, and Schidlof,[16] and of Batelli and Magri. Maurain determined the curve of induced electromotive force, and from this computed the flux, which was platted against the instantaneous values of current to give an energy loop. In a massive specimen he obtained large changes with the frequency of course, but in the thinnest wire, where eddy currents are excluded, there was no effect up to 60 cycles.

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[13] T. R. Lyle, Phil. Mag., **9**, p. 102; 1905.

[14] C. Maurain, Journ. d. Phys., (3) **7**, p. 461; 1898.

[15] W. Kaufmann, Verh. deuts. phys. Ges., **1**, p. 42; 1899.

[16] Guye and Herzfeld, C. R., **136**, p. 957; 1903; Guye and Schidlof, C. R., **139**, p. 517; 1904.

Williams and Kaufmann used a different apparatus for observing the relative instantaneous values of current and magnetization, which were platted similarly. No effect as great as 2 per cent (limit of accuracy) was found. Both of these methods are subject to the same criticism that applies to the energy loops platted in this article, and the former has the additional disadvantage that the values of flux are not directly determined.

Guye's method consisted in measuring the heat developed in the wire specimen by means of the bolometer. For the finest wire used, 0.0038 cm in diameter, there is no change in the loss per cycle up to 1200 cycles; in the larger wires the increasing loss is explained by eddy currents. The method of determining flux is not mentioned. The accuracy is 1 per cent.

Batelli and Magri[17] used a Braun tube for getting the energy loop; and using fine wires (0.005 cm in diam.) found the area slightly less at 5000 than at 25 cycles. In large specimens the area is increased at high frequencies, owing to eddy currents.

Gumlich and Rose[18] platted loops from instantaneous values with alternating current and from ballistic readings with direct current and found agreement at high inductions and only slight differences at low inductions. By wattmeter measurements they found a slight increase of hysteresis with frequency below 12000 gausses. Their method involved the separation of the eddy current loss by the Steinmetz formula.

Peukert[19] also claimed to find an increase of hysteresis with frequency, but it is to be noted that he assumes a sine wave and *calculates* the eddy current loss.

The latest research in this field is by Schames,[20] who reverts to a calorimetric method, and finds that above 10000 gausses the hysteresis increases with frequency, but below that flux density is constant. His specimens were straight strips; frequencies of 400 and 500 cycles per second were used. The curve of secondary emf. was platted to obtain the maximum flux density, the secondary coil being placed at a point to measure the average value of B throughout the specimen, rather than the average value of $B^{1.6}$. The flux density varied through wide limits between the middle and end of the strips, and this distribution might vary with the frequency on account of the effect of eddy currents. The eddy current loss was not measured but computed, and although it was considered negligible for the thinner specimens it amounted to half the measured loss in other cases. The author further admits that the value of this correction is very uncertain. The secondary voltage could not be read to better than 2 per cent except for inductions above 10000 gausses. In view of these facts we must conclude that the absolute values given by Schames are doubtful, and as to qualitative results they show that the effect of frequency can not be large, but must either be small or nil.

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[17] A. Batelli and L. Magri, Atti Acc. Lincei, **15**, p. 485; 1906.

[18] E. Gumlich and P. Rose, Electrot. Zs., **26**, p. 503; 1905.

[19] W. Peukert, Electrot. Zs., **20**, p. 674; 1899.

[20] L. Schames, Diss. Würzburg 1906; Ann. d. Phys., **22**, p. 448; 1907.

CONCLUSIONS.

For a definite maximum value of the flux density, the hysteresis is greater with a flat wave of flux, but the effect is small, and from the industrial standpoint negligible, even with very distorted waves.

If, however, the wave of flux is dimpled, the hysteresis may be much increased.

The hysteresis determined by the ballistic method may be smaller than that which obtains with the use of alternating current, but the differences are small.

The separation of hysteresis and eddy current losses by means of runs at two frequencies, using the Steinmetz formula, is not accurate, but is a close approximation when the sheets are thin.

WASHINGTON, October 10, 1908.

A NEW FORM OF STANDARD RESISTANCE.

By Edward B. Rosa.

1. INTRODUCTION.

In an article in this Bulletin nearly a year ago,[1] an account was given of an extended investigation of the effect of atmospheric humidity upon the resistance of coils of manganin wire prepared in the usual way by the use of shellac. It was there shown that the resistance always increases by an appreciable amount when the humidity of the atmosphere surrounding the coil increases, owing to the swelling of the shellac as the latter absorbs moisture from the atmosphere, the wire embedded in the shellac being stretched by an amount depending on the relative humidity of the air and the length of time the coil is exposed. When the atmospheric humidity decreases, the shellac gives up moisture and contracts in volume, the wire shortening at the same time, and the resistance of course decreasing. Although the actual change in volume of the shellac is very small, and the consequent change in the resistance is also small, yet in the case of resistance standards, especially those of relatively fine wire, the change is very serious, and enormous as compared with the errors allowable in measuring such standards.

It was found in this investigation that keeping the coils submerged in oil, using the pure, high-grade petroleum oil, as is usually done in precision resistance measurements, did not prevent this change, although it largely retarded it and apparently reduced the total change. Such oil in a damp atmosphere absorbs moisture and passes it on to the shellac, and in a drier atmosphere gives it up again. The oil, indeed, tends to come to a state of equilibrium with the atmosphere as regards its moisture content, and hence if the relative humidity of the atmosphere is constant the resistance of a coil of wire in the oil will tend to become constant; the time required to attain such a steady state is, however, very considerable. In practice the moisture content of the oil is varying day by day as the relative humidity of the atmosphere changes, and if in the resistance measurements the oil is vigorously stirred, to secure uniformity of temperature throughout the oil and equality of temperature between the resistance standard and the oil, the rate of absorption of moisture (or of giving up moisture when the air is drier than the average) will be accelerated. Hence, stirring the oil may greatly increase the rate of change of the resistance, and the same coils when measured in two different baths of oil, one having been stirred more than the other, may differ appreciably.

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[1] Rosa and Babcock, this Bulletin, 4, p. 121; 1907.

These variations of resistance may be of importance not only in resistance standards, but also in precision resistance boxes, Wheatstone bridges, potentiometers, and other resistance apparatus, where small changes in the resistances of the coils are serious. In potentiometers it is the relative resistance that should remain constant, so that if all coils change alike no serious harm is done. But all coils do not change alike, the higher resistances generally changing more than the lower ones. Hence the potentiometer will be subject to slight errors that will depend upon the weather or the season of the year. In a recent article in this Bulletin[2] it was shown that several precision resistance boxes which were frequently calibrated showed very marked seasonal changes of resistance, with a maximum in summer and a minimum in winter; and yet most of these resistances were kept submerged in oil. It was this seasonal change which led to the discovery of the effect of atmospheric humidity on shellacked resistances.

In a recent number of the *Zeitschrift für Instrumentenkunde*,[3] Dr. S. Lindeck describes experiments which he has made since the publication of our paper a year ago on the influence of atmospheric humidity upon electrical resistances. His results confirm our experiments, not only as to the change of resistance in the air but also in oil when the relative humidity of the atmosphere above the oil varies.

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[2] E. B. Rosa and N. E. Dorsey, this Bulletin, **3**, p. 553; 1907.

[3] "Über den Einfluss der Luftfeuchtigkeit auf elektrische Widerstände," Zeitschrift für Instrumentenkunde, August, 1908.

Our results have also been confirmed by Mr. F. E. Smith[4] at the English National Physical Laboratory. Mr. Smith subjected some of his resistance standards to extreme conditions, drying them out in an atmosphere freed from moisture by phosphorus pentoxide and then exposing them to an approximately saturated atmosphere. The change in resistance was from 14 to 64 parts in 100,000; but in some other coils leakage due to condensed moisture reduced the resistance so that the resultant change was a decrease instead of an increase. It is impossible to say at just what humidity leakage would begin to appear, and it seemed to us better not to subject coils to so high a humidity. Lindeck used a humidity of 80 per cent as his maximum, and found relatively large increases of resistance. In our experiments at the Bureau of Standards the humidity was seldom carried above 80 per cent, in order to avoid leakage and also to avoid permanent injury to the coils; and to conform as nearly as possible to actual conditions, the humidity was seldom carried below 25 per cent.

Lindeck found two methods of reducing the change produced by humidity. The first was to use a heavy paraffin oil, which absorbs and transmits moisture to a less degree than the petroleum oil commonly used. The second was to use a very thin tube on which to wind the resistances, and to cut slits in it, so that the tube yields readily to external pressure. When the moisture is absorbed and the shellac swelled, the wire is accordingly stretched less, because the yielding of the tube relieves the tension in part. The objection to the first method is that the heavy paraffin oil is less mobile and does not equalize the temperature of the bath as readily as the lighter oil, and is less convenient in use. The second method does not permit as substantial a construction as the usual form of coil, and would not seem to be a satisfactory solution, even if it prevented the change to a greater degree than it does. As, however, neither method, according to Lindeck, prevents the coils entirely from changing, some other method must be employed. The methods which we have employed at the Bureau of Standards will be described below.

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[4] Phil. Mag., September 1908, p. 450.

As set forth in our articles above referred to, the coils may be dipped in paraffin and the shellac completely protected from the atmosphere, so that whatever moisture is contained will remain constant, and the resistance is then very constant. If, however, the resistances are to be put into oil, either for the purpose of increasing their current carrying capacity or of fixing their temperatures more exactly, paraffin can not be used. Sealing them in air-tight boxes is then an effective protection from the effects of atmospheric humidity. Hence, dipping in paraffin the coils of Wheatstone bridges, potentiometers, etc., gives a satisfactory protection, while resistance standards of the highest precision and resistance boxes which are to carry more current than usual, as in accurate alternating current measurements, can better be protected by sealing in metal boxes which are filled with pure oil.

2. DESCRIPTION OF THE NEW SEALED STANDARDS.

Having found that coils which changed greatly when exposed to the air were remarkably constant when kept in an atmosphere at constant humidity, as when sealed in a test tube or preserved in any air-tight inclosure, it was evident that resistance standards ought to be so mounted as to possess this advantage. The design which I adopted more than a year ago, and which experience has since shown to be perfectly satisfactory, is shown in Figs. 1 and 2.

The coil is wound in the usual manner on a brass cylinder 30 mm in diameter and 70 mm long, and is contained within a cylinder 40 mm in diameter and 12.5 cm high. The coil is shellacked, dried, and annealed in the usual manner, as originally specified by the Reichsanstalt. The coil is supported, as shown, by a small tube (closed at the bottom) which serves as a thermometer tube. The hard-rubber top through which the leads pass is threaded, and screws into the outer brass cylinder which forms the case. When the coil is finally adjusted, the case is nearly filled with pure oil that has been freed from moisture, and the top screwed firmly into place. To make the joint perfectly tight, shellac is usually put into the threads before screwing up. Shellac is also put into the joints in

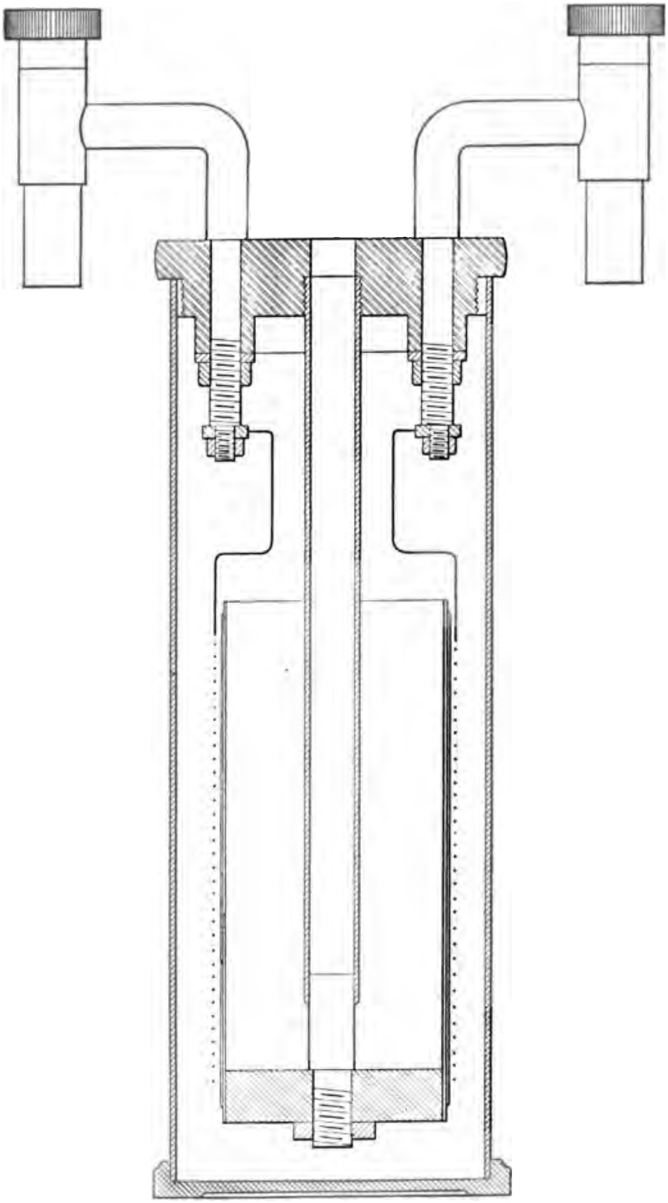


Fig. 1.—*Section of Sealed Resistance Standard*

the top, where the leads and thermometer tube pass through the ebonite.

The terminals dip into mercury cups, but the upper potential terminals may ordinarily be used as current terminals. Only in low-resistance coils (1 ohm and less) is the difference in resistance appreciable.

Sealing the standards in this way possesses several distinct advantages. First and most important, of course, is the protection against the changes in resistance due to the absorption of moisture by the shellac (or varnish or other protective covering). In the second place, it protects the coils from dirt or mechanical injury, to which they are liable if open. And, finally, if any bare spots exist, due to shellac coming off or being imperfectly applied, the oxidizing effect of moisture in the oil bath and of the atmosphere is far greater than it can be when the coil is sealed air-tight in very pure dry oil. Hence the resistances are not only protected from the serious fluctuations from day to day, and the still greater seasonal changes due to moisture, but also from slower changes due to oxidation of the manganin and the possible sudden changes due to accident.

The new form of resistance standard is much smaller than the Reichsanstalt type, so long and so favorably known throughout the world as a standard of resistance. It weighs only about 400 g filled, and measures only 7.5 cm across the terminals instead of 16 cm. For measurements up to an accuracy of .001 per cent it is measured as it stands, its current capacity being ample when using reasonably sensitive galvanometers, and the small thermometer in the central tube giving its temperature with all needed accuracy. The temperature coefficient is generally not greater than .002 per cent per degree, so that a quarter of 1 degree uncertainty in the temperature would cause an error less than that allowed.

When used as standards of the highest precision, and measured to one part in 1,000,000 or closer, it is necessary to know the temperature of the coil very accurately. The standards are then submerged in an oil bath which can be stirred and kept at constant temperature and the precaution taken not to use current enough to make the temperature of the wire uncertain. The standards when properly prepared and aged are remarkably constant in value, and in

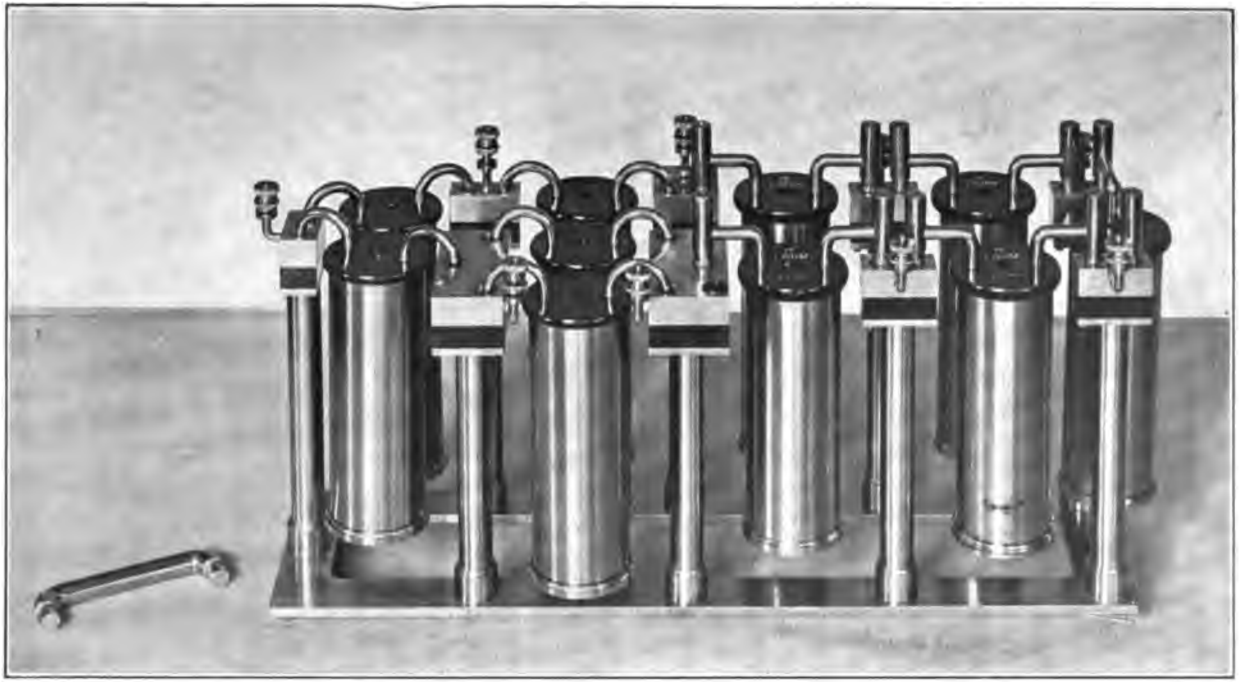


Fig. 11.—*Bridge for Intercomparing Coils of Different Denominations. Link Removed*

measuring them to test their constancy I have been obliged to take extreme precautions in order not to have their slight variations exaggerated by errors of measurement. The temperature coefficients of the coils which I have been using vary through wide limits, the average value between 15° and 30° C generally falling between 4 and 20 parts in a million per degree, but sometimes exceeding 20.

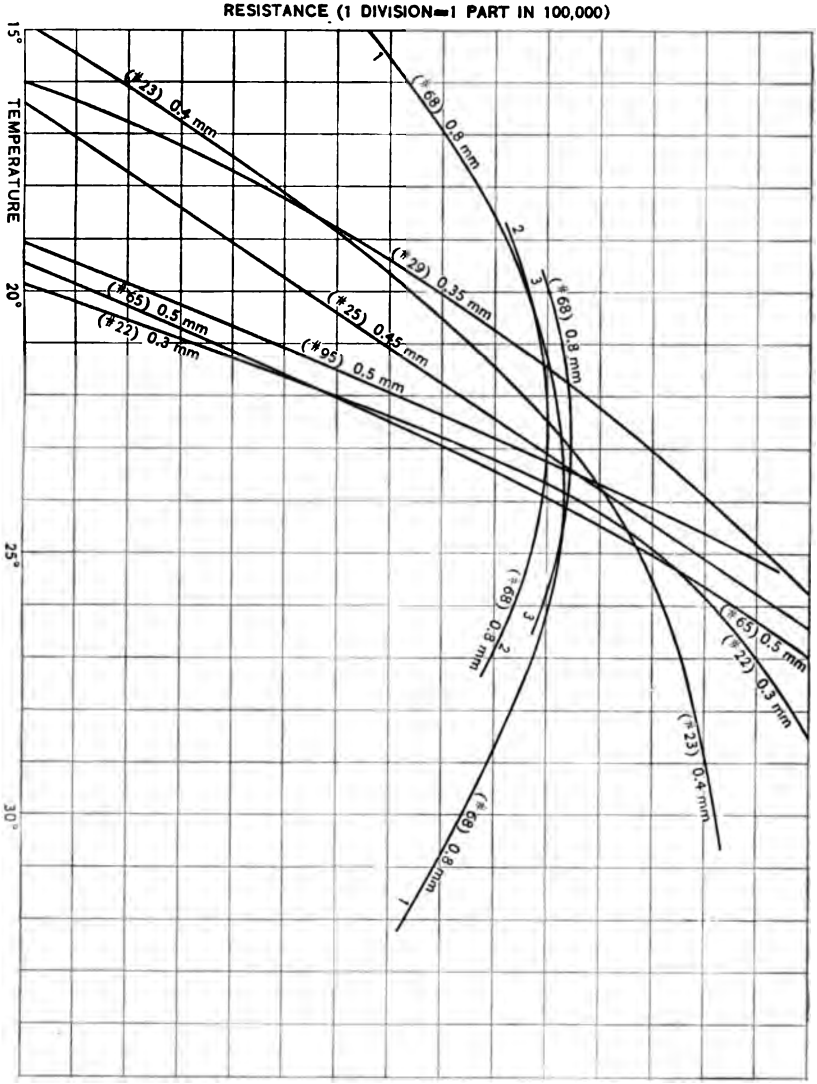


Fig. 3.

Different specimens of manganin wire vary greatly as to temperature coefficient, and hence it must be carefully selected if the coils are to have coefficients as small as these.

3. TEMPERATURE COEFFICIENTS OF MANGANIN WIRE.

A series of coils of the style just described was prepared, most of them being made in the instrument shop of the Bureau, but a few were made by the Leeds & Northrup Company. Some of the standards first made consisted merely of coils taken out of a dial rheostat by Otto Wolff and mounted and sealed in our cases. These resistances, being old, were well seasoned, and although their values were not steady in the rheostat they became steady shortly after mounting in the new way. In order to find wire suitable for making new coils of various denominations, we measured the temperature coefficients of a considerable number of specimens, and some of the curves found are shown in Figs. 3, 4, and 5, some of which were determined for me by Dr. G. W. Middlekauff.

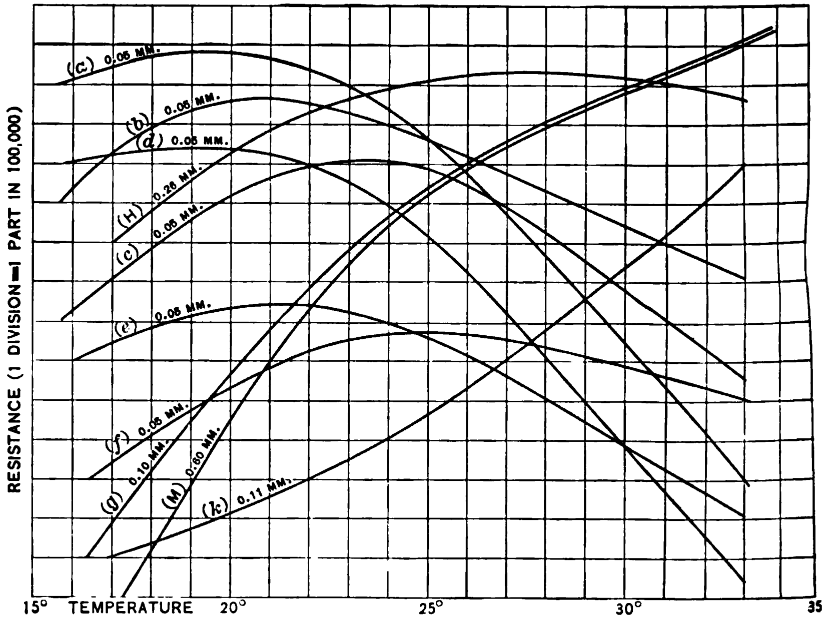


Fig. 4.

The curves are substantially parabolas, and the temperature coefficients (represented by the slopes of the curve) vary greatly through the range 15° to 25°, which is the important region for laboratory purposes.

It is not generally known that different specimens of manganin vary so widely, and these curves will perhaps serve as a warning to anyone anxious to make good resistance standards not to use manganin wire without first testing it for temperature coefficient. Some of our 1-ohm coils have been made of strips wound on cylindrical forms properly insulated by silk and covered with shellac and annealed, as the wire resistances are treated. It will be noted that the finer wires have generally the smaller temperature coefficients. These specimens of manganin all came from Otto Wolff, the Berlin agent of the German manufacturers. These curves are not intended to show that a large number of specimens of manganin wires would always vary as much as these do, but merely to show the results of our experience on a considerable number of samples. Between 15° and 25° the mean value of the coefficient varies in the different specimens from 1 to 22 parts in a million per degree. It sometimes goes as high as 40 or 50 per million, or more. In some specimens it

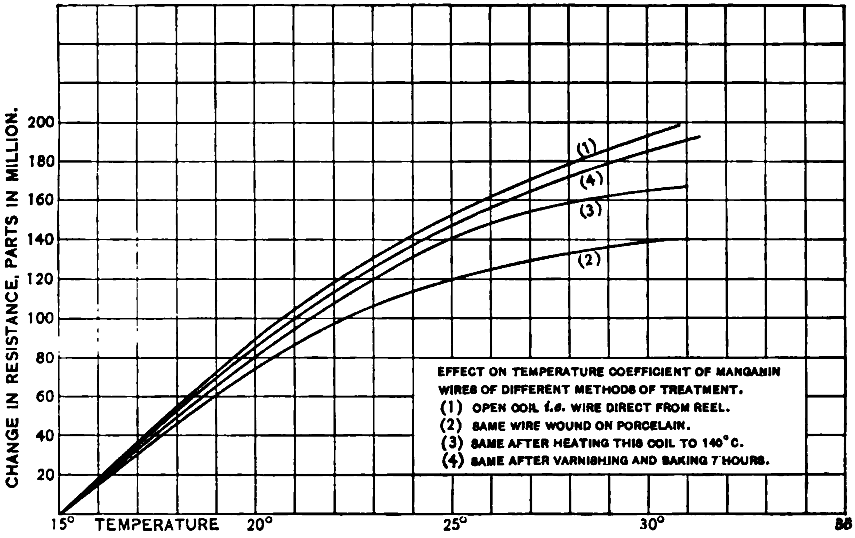


Fig. 5.

will be noted that the coefficient becomes negative at temperatures as low as 20° C. The finished coils made from these wires would have somewhat different values of the coefficients, generally larger.

It is of great importance in standards for the best service that the temperature coefficients be small. The errors in measurements due to uncertainty in temperature are then smaller, and a larger current can be used in the measurement without appreciable error due to heating. If the coefficient in the working range is below 10 parts per million per degree, an uncertainty of 0°1 is only 1 part in a million, and with care the uncertainty can be kept much below this.

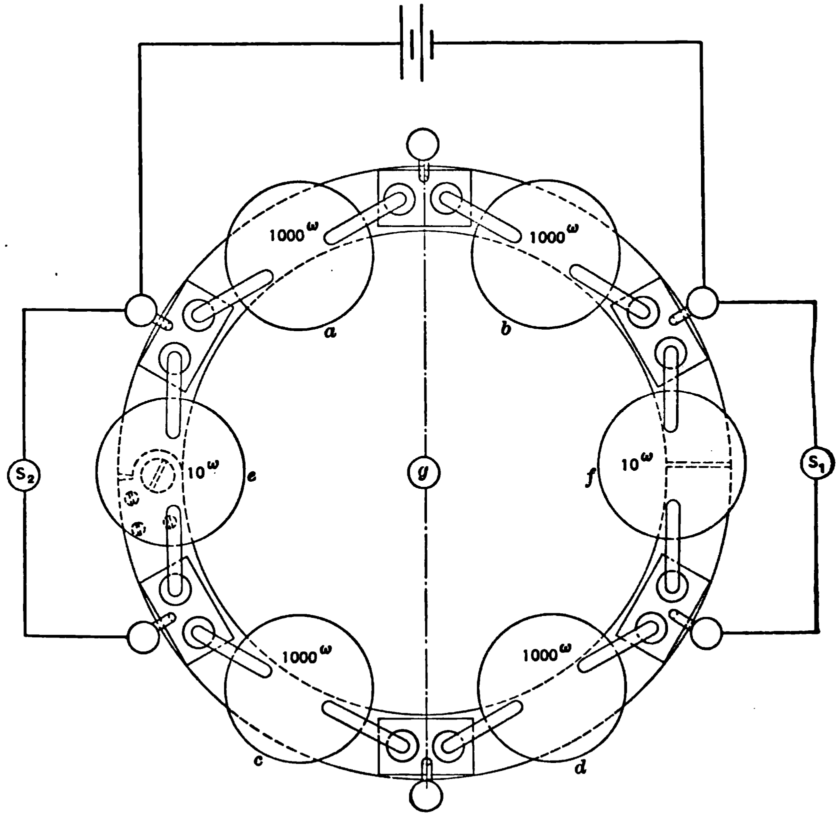


Fig. 6.

In Fig. 3 three curves are given, showing maximum values of the resistance (that is, zero temperature coefficient) at about 23° C. These three curves are for different specimens of wire of 0.8 mm

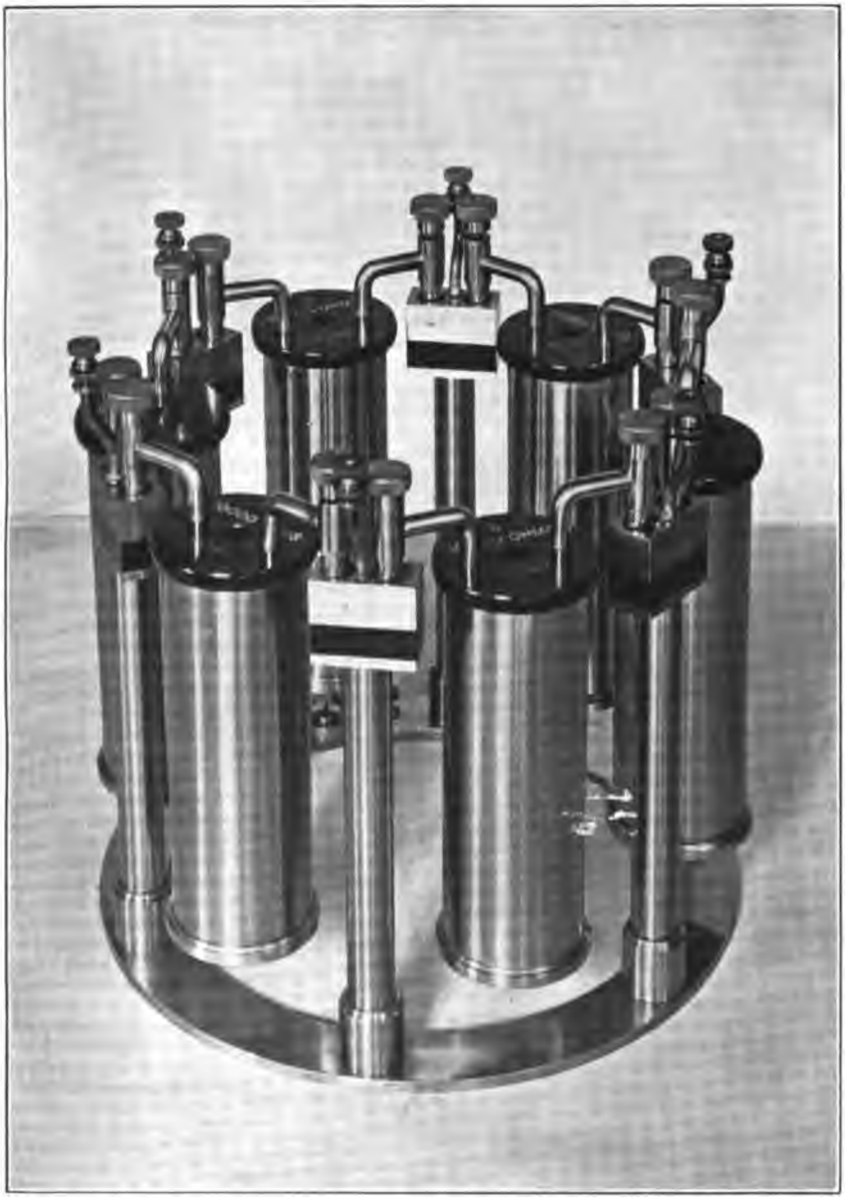


Fig 7.—*Bridge for Intercomparing Sealed Resistance Standards.*

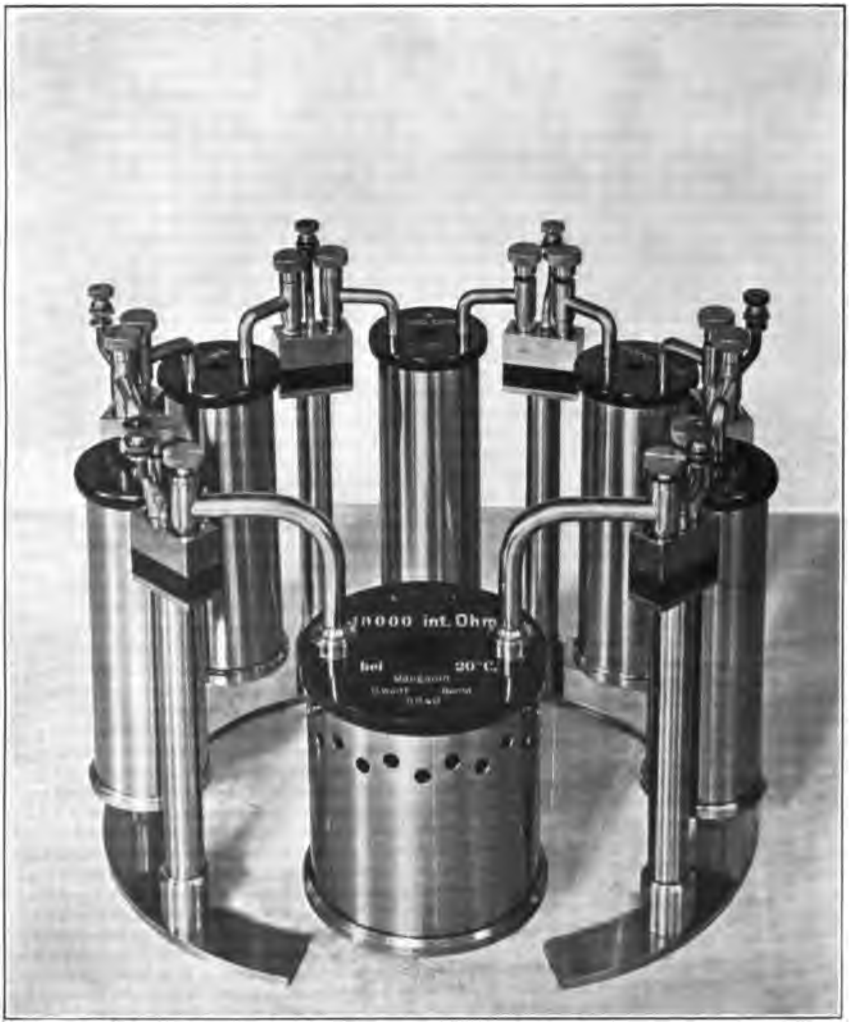


Fig. 8.—*Bridge for Intercomparing Resistance Standards Opened to Contain a Larger-sized Coil.*

diameter from the same spool, tested before making up into a coil. This is an exceptional specimen of manganin of this size, and when made up gave coils of relatively small temperature coefficient, although the maximum was shifted to a somewhat higher temperature.

4. THE APPARATUS EMPLOYED.

The apparatus employed in comparing the coils of any denomination with one another is shown in Figs. 6, 7, and 8. It is a Wheatstone bridge with the coils arranged in a circle, having two extra coils inserted, on which shunts may be applied for balancing the bridge instead of shunting directly the ratio coils. Or these two openings may be closed by heavy links and the shunts applied directly to the ratio coils. The circular frame is so hinged that it may be opened far enough to admit a larger coil, as for example, one of the Reichsanstalt form, which may thus be directly and conveniently compared with one of the new Bureau of Standards form, Fig. 8. The apparatus is very convenient, is compact, requires a relatively small oil bath, or may be used without an oil bath except in comparisons of extreme precision, and will accommodate any kind of a resistance standard that is provided with terminals for dipping into mercury cups.

For stepping up from one denomination to another I have designed the form of bridge shown in Fig. 9. This is a very compact and convenient apparatus for comparing the sum of five coils of one denomination with two coils in parallel of the next higher denomination. Thus the difference between five 1-ohm coils in series and two 10-ohm coils in parallel is obtained by the method of substitution. Thus the mean of the tens is determined in terms of the mean of the ones. Also five 10's have been thus compared with two 100's, five 100's with two 1000's, and five 1000's with two 10,000's, all by the method of substitution and with extreme precision. If the temperature coefficients of all the coils are known with precision, or the comparisons are made at the standard temperature, the values of all the coils in terms of the mean of the 1-ohm standards are thus determined with very great accuracy.

When the bridge is arranged as shown in Figs. 10 and 11, it consists of four 10-ohm coils, and the balance is obtained by means of the shunts S_1 S_2; the five 1-ohm coils are out of circuit, the two cups V_1

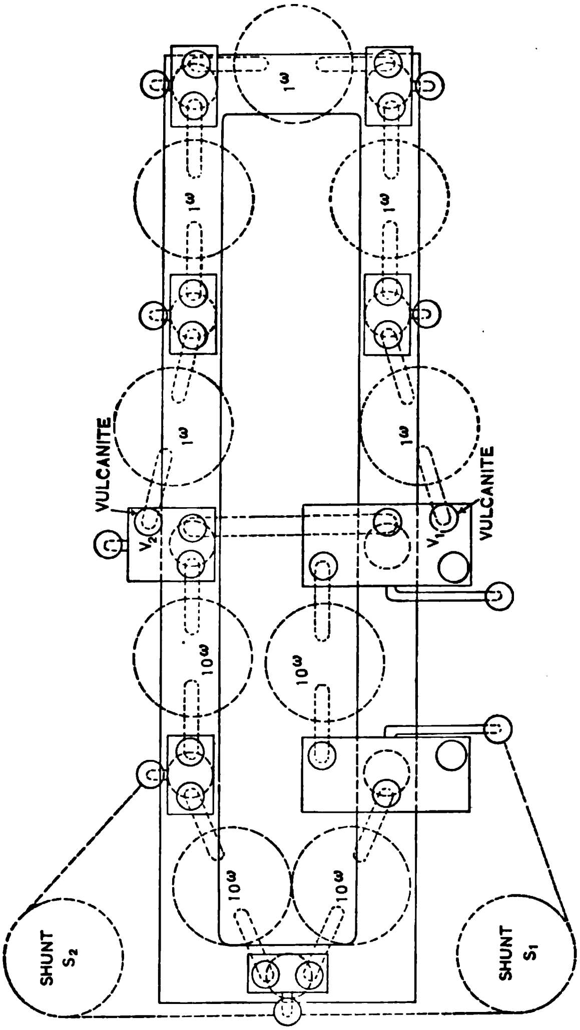
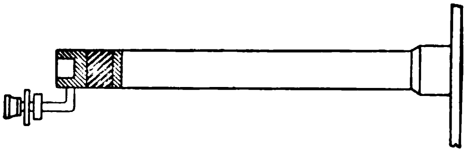


Fig. 9.—*Plan of Bridge shown in Fig. 12.*



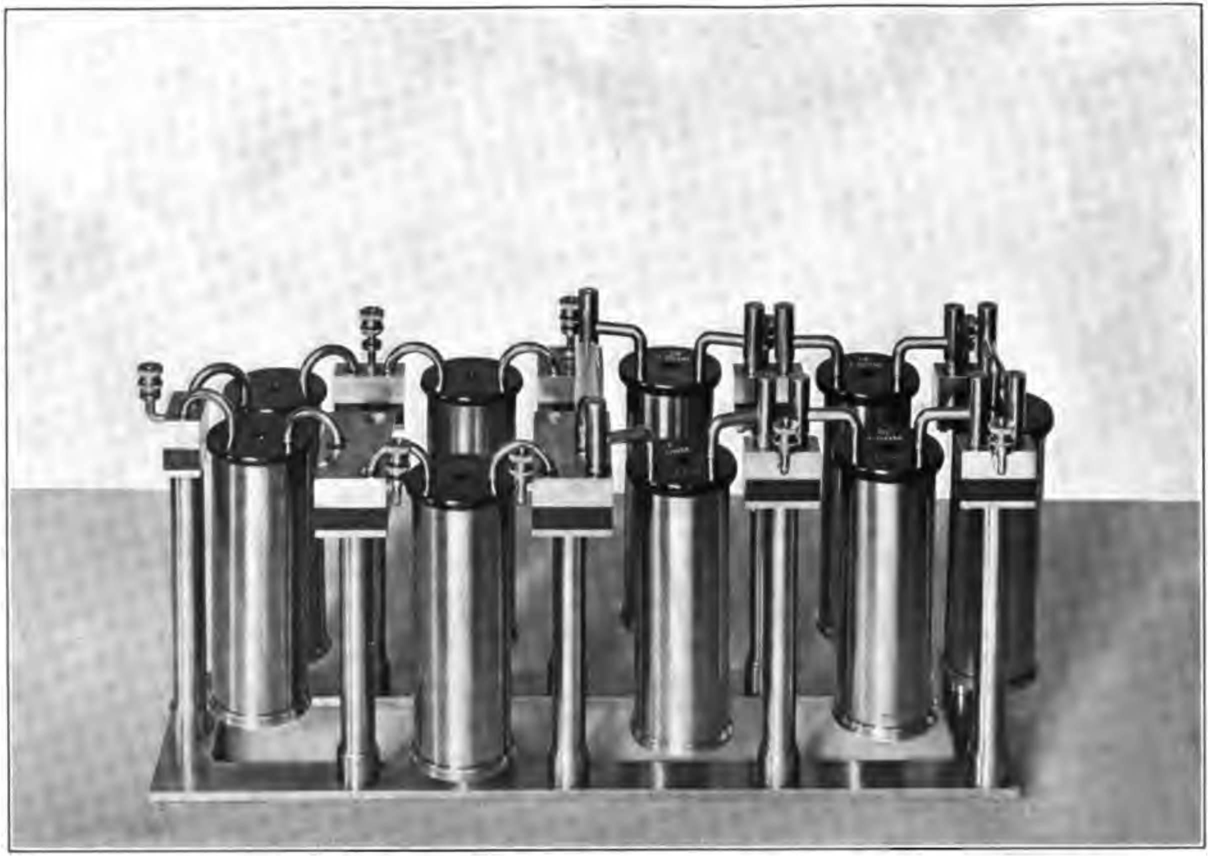


Fig. 10.—*Bridge for Intercomparing Coils of Different Denominations. Link in place.*

and V_2 being insulated. The coil 10_A may be replaced by 10_B and their difference determined directly. The link L is then removed and the coils arranged as in Fig. 11. Now the five 1-ohm coils and the two 10's in parallel together make up one arm of the bridge, and the change in the shunt gives the difference between 10_A or 10_B and $\frac{10_A+10_B}{4}$ plus the sum of the five 1-ohm coils. Allowance must, of course, be made for the resistance of the link, which amounts to only about 2 parts in a million of a 10-ohm coil. The following measurements made September 29, 1908, show how the method works out in practice:

Results of stepping up from the units to the tens, September 29, 1908

Using coil 10_1 as a standard of reference, the following relative values were measured:

$$10_2-10_1=-\ 41.3 \qquad (1)$$

$$10_3-10_1=-\ 34.0 \qquad (2)$$

$$10_4-10_1=+\ 85.0 \qquad (3)$$

$$10_5-10_1=+106.4 \qquad (4)$$

$$\left[\left.\begin{matrix}10_1\\10_2\end{matrix}\right\}\text{parallel}+(\Sigma 5+l_1)\right]-[10_1+l_2]=-34.9 \qquad (5)$$

where the results are expressed in millionths of the nominal value of the 10-ohm coils.

$\Sigma 5$ = sum of the 1-ohm coils Nos. 1, 2, 3, 4, and 5.

l_1 = resistance of the connecting blocks between these coils $=8\times 10^{-6}$ ohms.

l_2 = resistance of the link $=23\times 10^{-6}$ ohms.

$\therefore\ l_1-l_2=-15\times 10^{-6}$ ohms.

$=-1.5$ millionths of the value of a 10-ohm coil.

For $\Sigma 5$ we derive from intercomparison of the 1-ohm coils and on the assumption that their mean value is constant and 2.5 millionths less than the nominal value.

Coil	Correction to Nominal Value	
1_1	− 1.0	Sum = − 13.0
1_2	− 1.5	
1_3	0	
1_4	− 6	
1_5	− 4.5	

or $\Sigma 5 = 5 - 13.10^{-6}$ ohms

= 5 ohms − 1.3 millionths of the whole resistance when used in the 10-ohm combination.

Writing

$$\begin{aligned} 10_1 &= 10 \text{ ohms} + \alpha \\ 10_2 &= 10 \quad \text{''} \quad + \beta \\ 10_3 &= 10 \quad \text{''} \quad + \gamma \\ 10_5 &= 10 \quad \text{''} \quad + \epsilon \end{aligned}$$

we have

$$\left.\begin{matrix} 10_1 \\ 10_2 \end{matrix}\right\} \text{in parallel} = 5 \text{ ohms} + \frac{\alpha+\beta}{4}$$

and from equations (1) and (5)

$$\begin{aligned} \frac{\alpha+\beta}{4} - 1.5 - 1.3 - \alpha &= -34.9 \\ \beta - \alpha &= -41.3 \\ \alpha &= +43.6 \\ \beta &= +\ 2.3 \end{aligned}$$

From these follow immediately the corrections (in parts of a million) of the nominal values of the 10-ohm coils.

Coils	By Equations (2)-(5)
10_1	+ 43.6
10_2	+ 2.3
10_3	+ 9.6
10_4	+128.6
10_5	+150.0

Fig. 13 shows a form of bridge used in measuring temperature coefficients where the two parts, connected by long links, are in baths at different temperatures, one constant and the other variable.

5. CONSTANCY OF THE RESISTANCES.

The relative values of a number of 100- and 1000-ohm coils are shown by the curves of Fig. 12, which represent observations extending over a period of six months. The values as plotted are on the assumption that the mean of the 1000-ohm coils numbered 1, 2, 3, 4 is a constant. The first six coils, both of the 100's and 1000's, are

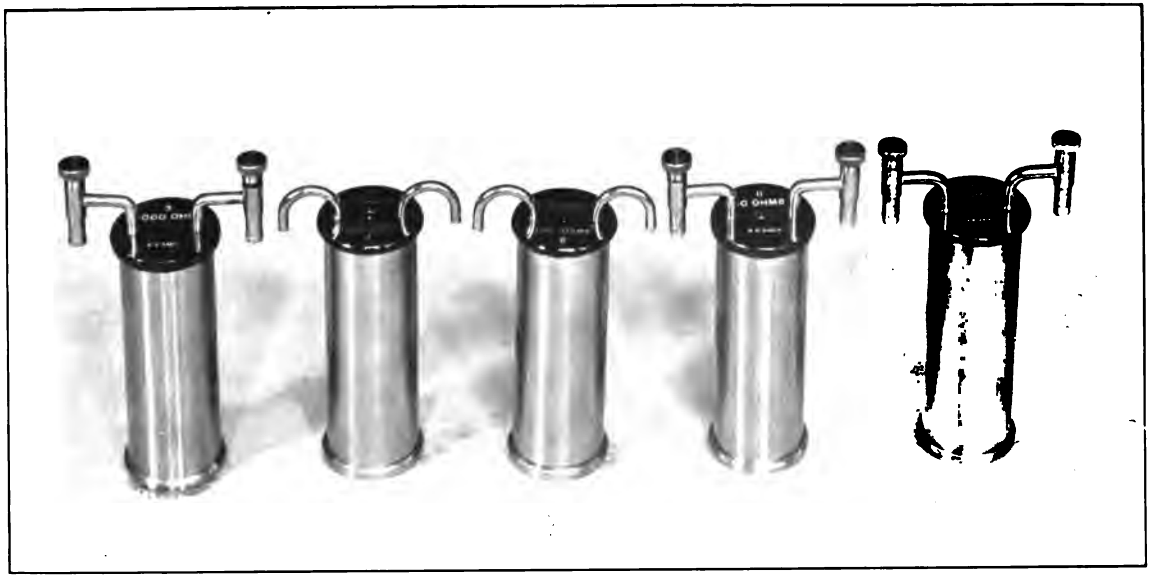


Fig. 2.—*Group of Sealed Resistance Standards.*

Wolff coils taken from one of his precision rheostats and mounted as sealed standards. Coils 7 and 8, both of the 100's and 1000's, were new, and not at first as constant as the older ones. The two coils, 100-ohm No. 9 and 1000-ohm No. 9, were mounted in precisely the same way, but with several holes in the cases, as shown in the photograph, Fig. 14. Their curves, given in the middle of Fig. 12, show how greatly they varied as compared with the sealed coils. The results of measurements on three Wolff coils of the Reichsanstalt form are also given in Fig. 12. Two of these coils (one of 1000 ohms and one of 10,000 ohms) were mounted as usual, but the third had its case replaced by a case without holes, which was sealed with varnish at the junction of case and top, so as to give the coil an air-tight inclosure. The curves show that the two coils in their own cases varied considerably, the 1000-ohm coil 14 parts in 100,000 and the 10,000-ohm coil 16 parts in 100,000 during this period. Up to November 21 the measurements were made rather infrequently, and the curves therefore appear smooth; but between November 21 and January 8 the open coils were measured eleven times, and their changes up and down as shown in the curves corresponded to varying conditions of the weather. The sudden considerable change between November 21 and November 22 was produced by an unusually damp atmosphere on the evening of November 21, when the windows of the laboratory were open a few hours. The sealed Wolff coil, No. 3057, remained remarkably constant for several months, and the slight upward turn of the curve is possibly due to a slight leak of air.

These and other sealed coils of 10, 100, 1000, and 10,000 ohms have been frequently measured during the past few months, and show a remarkable constancy. Six 10-ohm coils, eight 100-ohm coils, eight 1000-ohm coils, and six 10,000-ohm coils have been under observation. While they are not all constant, there is *only one coil of the above twenty-eight that has changed as much as 2 parts in 100,000 during the past twelve months.* This one coil I regard as defective, but I do not know the reason for its change. Of the other twenty-seven coils, more than half have changed less than 1 in 100,000. The measurements have not been systematic enough to say just how many parts in a million the changes have been, except for a part of the coils.

Fig. 15 shows the results of careful measurements on thirteen 1-ohm coils during the past three months. The dots which represent the individual measurements are plotted on such a scale that one small square is equal to 1 part in a million. For example, the two measurements of July 18 and 20 on coil No. 1 differed by 2 parts in a million. The horizontal lines represent the values of the coils, assuming them all to be constant. The average deviation of the separate measurements from the mean is given for each coil at the right of the figure. In seven coils this is less than 1 part in a million, *and for the entire thirteen the mean of these average deviations is exactly 1 part in a million.* These comparisons were made at temperatures ranging between 15° and 28° C, some of them thus involving considerable temperature corrections. *These small variations are no more than the probable errors of the measurements, including the uncertainty in the temperature corrections, so that there is no evidence that any one of the thirteen 1-ohm standards has suffered any change whatever during the past three months,* unless possibly the two coils 13 and 14 have increased *very slightly.* On the other hand, open 1-ohm coils exposed to atmospheric influences sometimes change appreciably in a single day,[5] or even in a few hours; and although the average values of the 1-ohm standards at the Reichsanstalt have undoubtedly remained very constant for some years, Dr. Lindeck has recently found[6] a difference of about 13 parts in a million in his 1-ohm coil (A) between April and October. His coils of larger denomination—of 10, 100, 1000, and 10,000 ohms—changed in the same time from 28 to 75 parts in a million.

Fig. 16 is taken from Lindeck's recent paper (p. 234), and shows the changes in resistance of 14 resistance standards with reference to his 1 ohm (A) which was assumed to be constant. The observations extended over a year, and show that the resistances of all the 10, 100, 1000, and 10,000 ohm coils rise to a maximum about the 1st of October and subside to a minimum somewhere between January and April, the observations not being sufficiently numerous to fix the maximum and minimum points closely; and these would, of course, vary from year to year. The 0.1 ohms

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[5] F. E. Smith. Phil. Mag. p. 450; Sept., 1908.

[6] Über den Einfluss, etc., p. 234.

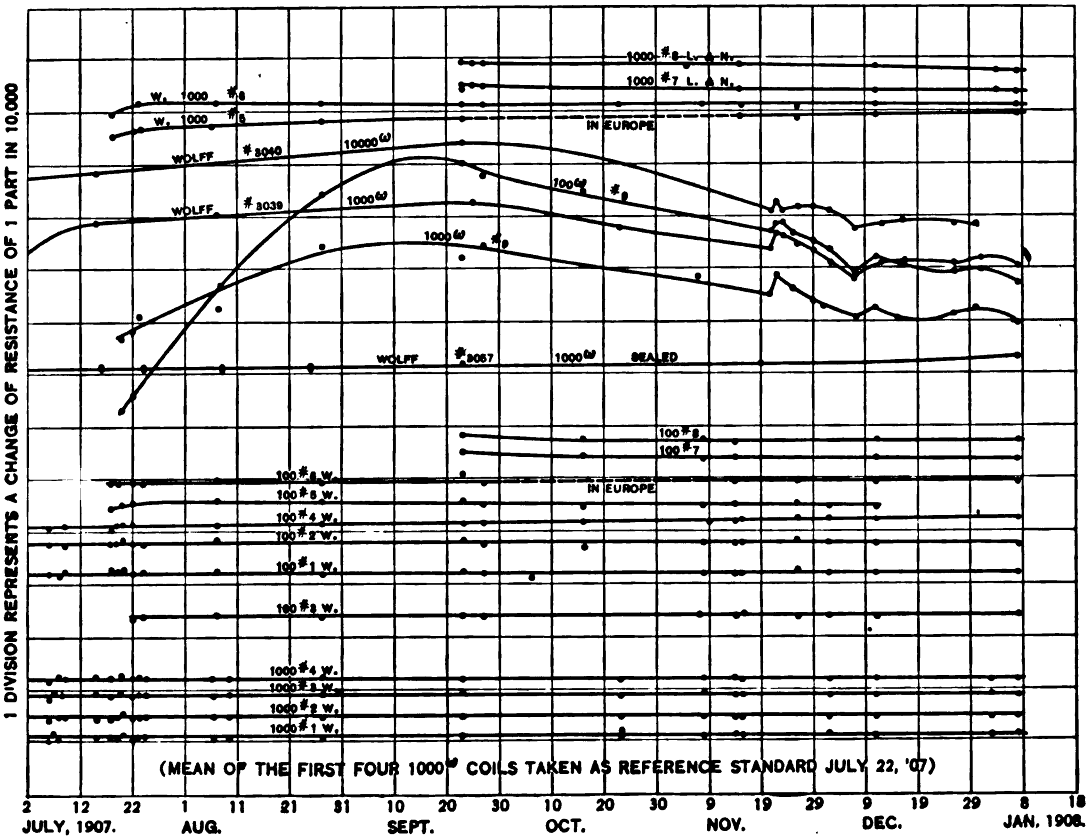
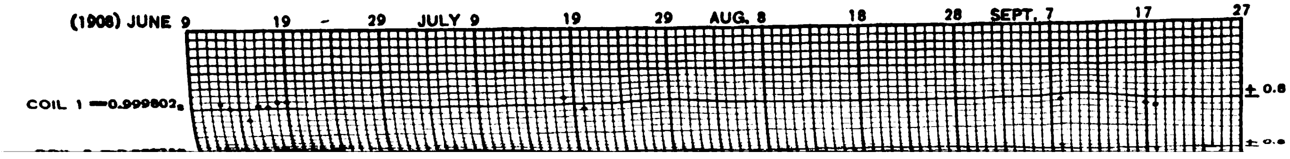
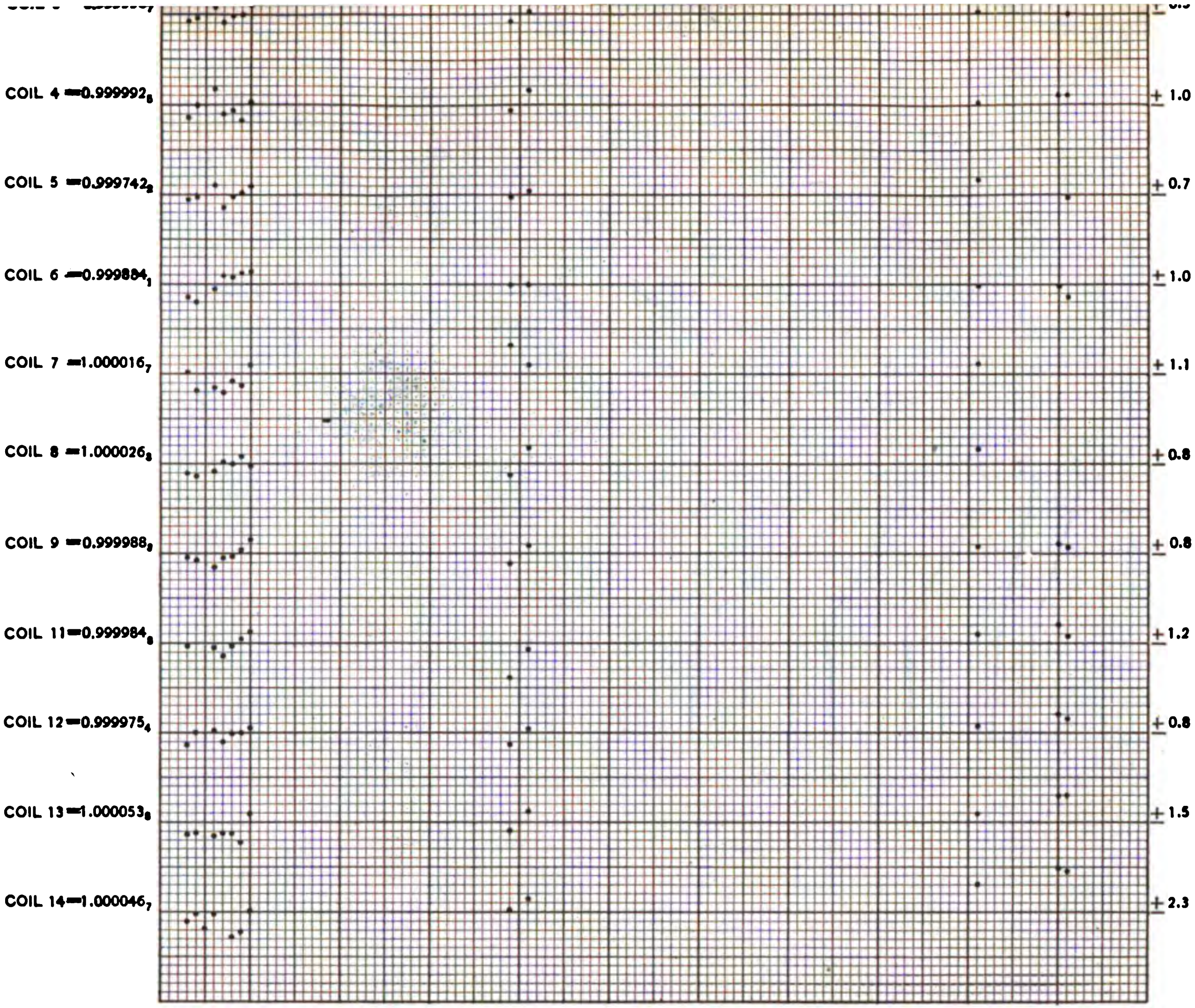


Fig. 12.—*Showing Difference in Behaviour of Open and Sealed Standards.*





The straight lines represent the average values of the coils, which are given at the left.
The numbers at the right give the average deviation of the observed values from the average values, in millionths of the nominal values of the coils.
Each small division represents 1 millionth of the nominal value of the coils.

Fig. 15.—*Thirteen One-Ohm Sealed Standards.*

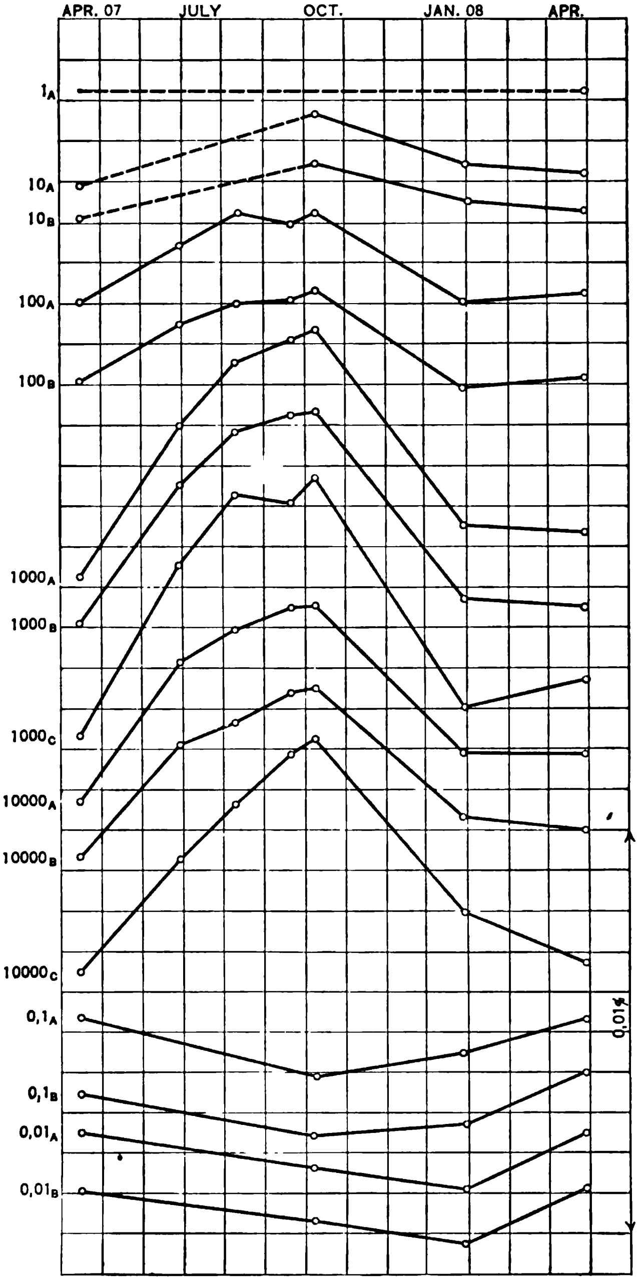


Fig. 16.—*Variation of Resistance of Open Standards as found by Lindeck. One space=1 part in 100,000.*

(wound with very coarse wire) and 0.01-ohm coils (wound with sheet manganin) seem to *decrease* in summer, all by about the same amount, namely, 13 parts in a million. This is, of course, as Dr. Lindeck points out, because the 1-ohm coil has really *increased* and the 0.1 and 0.01 ohm coils have remained constant. Hence the change in the 10-ohm coils is the sum of 15 parts in a million, as shown in the figure, and the 13 parts change of the reference coil, or 28 parts in a million altogether. The 100-ohm coils changed about 35 parts in a million, the 1000's about 73 parts on the average, and the 10,000-ohm coils about 71 parts in a million. Inasmuch as these coils can be compared to 1 part in a million, or better, and as standards ought to be known to at least 1 part in 100,000, it is evident that changes of the above magnitude, which correspond to temperature changes of from 1° to 7° C (when the coefficient is 10 parts in a million per degree), are altogether too great to be permanently satisfactory for important standards of reference. Moreover, these changes occurred in the favorable climate of Berlin, whereas in London or Washington the change could be expected to be much greater.[7]

Hence it appears that our sealed standards, most of which contain comparatively new coils, and of these most were made in our own instrument shop, have changed on an average far less than the standards of the Reichsanstalt (the latter being in the favorable climate of Berlin), while our 1-ohm sealed standards, which are of course wholly unaffected by atmospheric humidity and will have the same values in any climate, have not changed enough to discover during the past summer, and some of them have been constant as compared with one another for nearly a year.

It ought to be stated that the reference standards of the Reichsanstalt are now being kept in an atmosphere of constant humidity in order to avoid these changes, and at the National Physical Laboratory the reference standards are about to be sealed for the same reason.

A further study of these resistances is, of course, necessary, but I think it not improbable that if three or four national laboratories

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[7] Lindeck, Zs. für Instrumentenkunde, August, 1908, p. 236.

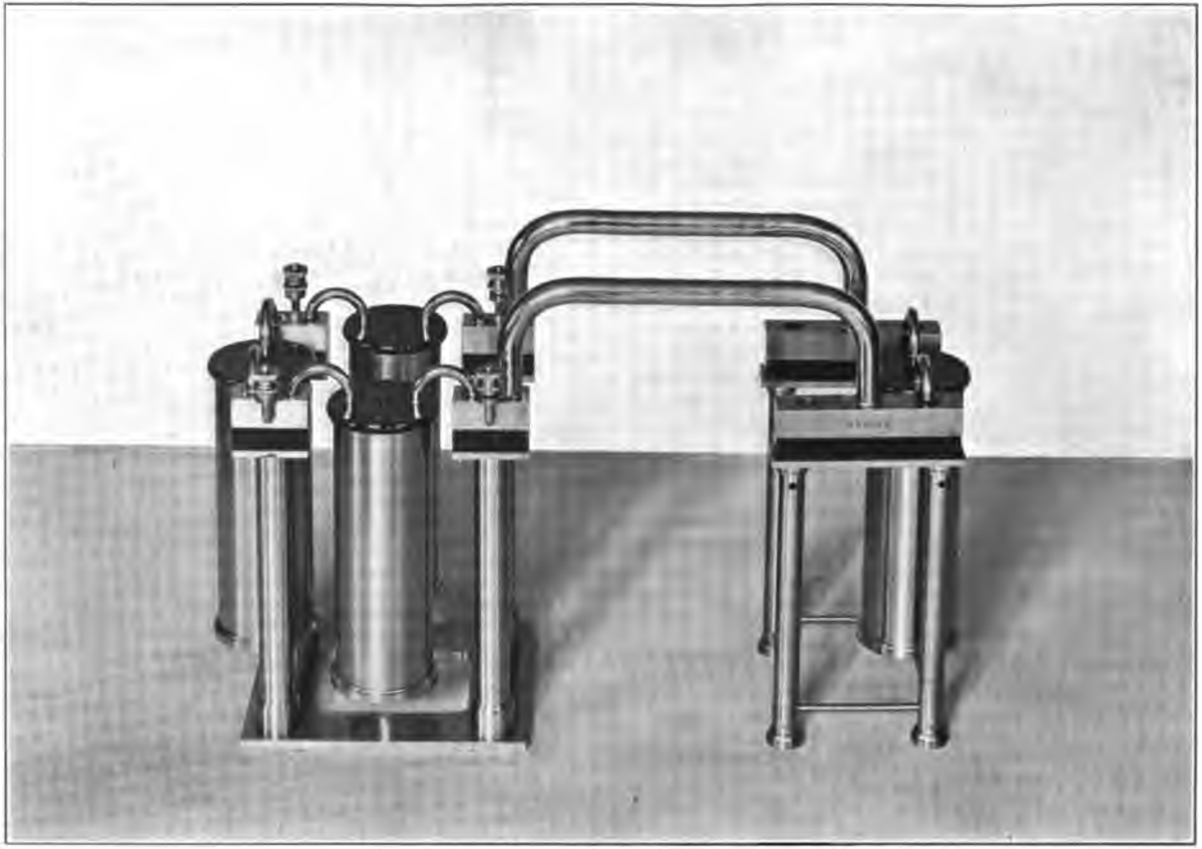

Fig. 13.—*Bridge for Measuring Temperature Coefficients.*

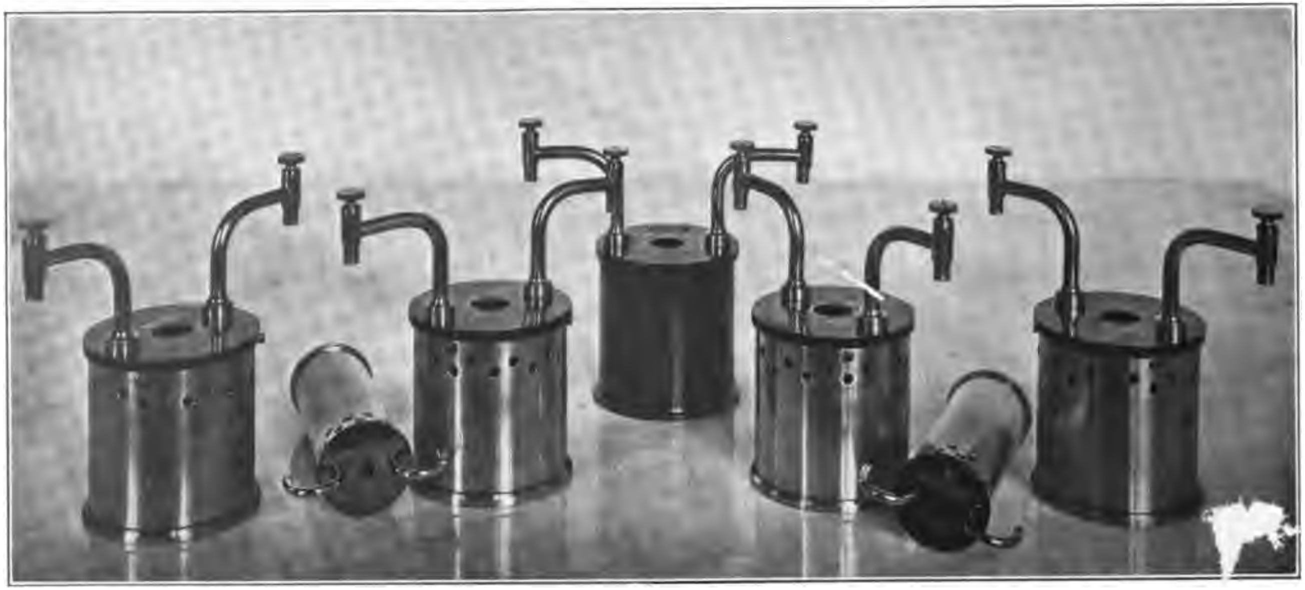

Fig. 14.—*Resistance Standards of plate 12.*

were each supplied with a dozen sealed standards of the Bureau of Standards type, and they were occasionally intercompared by means of a number of traveling coils of the same type, that a mean international ohm could thus be maintained constant to within 1 or 2 parts in a million, and a reference to primary mercury ohms would need to be made only at relatively infrequent intervals.

6. COMPARISONS OF RESISTANCES IN LONDON AND BERLIN.

In February, 1908, I took a number of these resistances to Europe and had them compared at the National Physical Laboratory, London, and the Physikalisch-Technische Reichsanstalt, Charlottenburg, partly to obtain a check on the value of the resistance unit of the Bureau and partly to obtain a comparison between the resistance standards of the above-named institutions. Such comparisons had been made between London and Berlin, but probably because of changes in the comparison coils the results were not entirely satisfactory. The recent comparisons show that the values assigned to the coils in Washington differed from the values found at the Reichsanstalt by about 1 part in 100,000 on the average for the 1-ohm and 100-ohm coils. This shows excellent agreement in the stepping up at the two institutions.

TABLE I.

Results of Measurements on Resistance Coils made in Berlin, London, and Washington.

Nominal Value	Coil No.	P. T. R.			N. P. L.	N. B. S.		Differences in millionths	
		Oct. 28, 1907	Mar. 12, 1908	Mar. 20/21, 1908	Feb. 18, 1908	Sept. 30, 1907	Feb. 3, 1908	P.T.R.-N. B. S.	N.P.L.-N. B. S.
1 ohm..........	1	....	....	0.99981_6	0.99982_9	....	0.99980_1	+15	+28
	2	....	....	$.99975_3$	$.99977_6$	....	$.99975_0$	+ 9	+26
	3	....	1.00001_1	1.00000_9	1.00002_3	....	1.00000_0	+10	+23
	12	....	0.99997_5	0.99997_5	0.99999_7	....	$.99998_2$	− 7	+15
100 ohms.......	1	....	99.991_6	99.991_6	....	....	99.990_3	+13	+13
	2	....	99.987_3	99.987^1	99.987_2	....	99.985_8	+14	+14
	3	....	....	....	....	....	99.994_1	....	....
	6	99.971_8	....	....	....	99.971_0	....	+ 8	....
	9	....	100.004_7	100.000_8	(open coil)		....	....	....
1000 ohms......	1	....	....	....	999.92_0	....	999.90_2	....	+18
	2	....	....	....	1000.03_4	....	1000.01_9	....	+15
	3	....	....	....	....	....	999.90_2	....	....
	5	999.78_9	....	....	....	999.78_0	....	+ 9	....

The measurements made on these coils are given in Table I, the table being an extension of that prepared for me by Dr. Lindeck at the time of the measurements in Berlin in March. It shows that the difference between the Bureau of Standards' values and those of the Reichsanstalt for the 1-ohm coils was nearly the same for coils 1, 2, and 3, the difference for coil 12 differing appreciably from the others. This was due to the fact that the first three coils were seasoned and constant, having been made and mounted in August, 1907. Coil 12, on the other hand, was a new one, having been adjusted and sealed in January, 1908, just before being taken abroad.

The following table of differences in *millionths of ohms* shows how constant the relative values of 1, 2, and 3 have been.

	Washington, Jan., 1908	**London, Feb. 18, 1908**	**Berlin, Mar. 20, 1908**	**Washington, Sept., 1908**
1-2	51	53	57	53
3-2	250	249	250	249
3-12	18	26	34	24

The differences for the 3-12 would seem to indicate that 12 was decreasing at first, but that later it increased a little. This is not impossible, as the seasoning involves changes both in the shellac covering and the manganin itself. The mean difference *PTR–NBS* for the first three 1-ohm coils is 11 millionths, and *NPL–NBS* is 26 millionths for the same coils, giving a difference of +15 millionths for *PTR–NPL*. That is

PTR–NBS= +11 millionths
NPL–NBS= +26 millionths
NPL–PTR= +15 millionths.

For the 2-100-ohm coils the differences are as follows:

PTR–NBS= $+13_5$ millionths
NPL–NBS= $+13_5$ millionths
NPL–PTR= 0 millionths.

The measurements at the National Physical Laboratory on the 1000-ohm coils gave nearly the same difference *NPL–NBS* as the hundreds, i. e., 17_5 millionths.

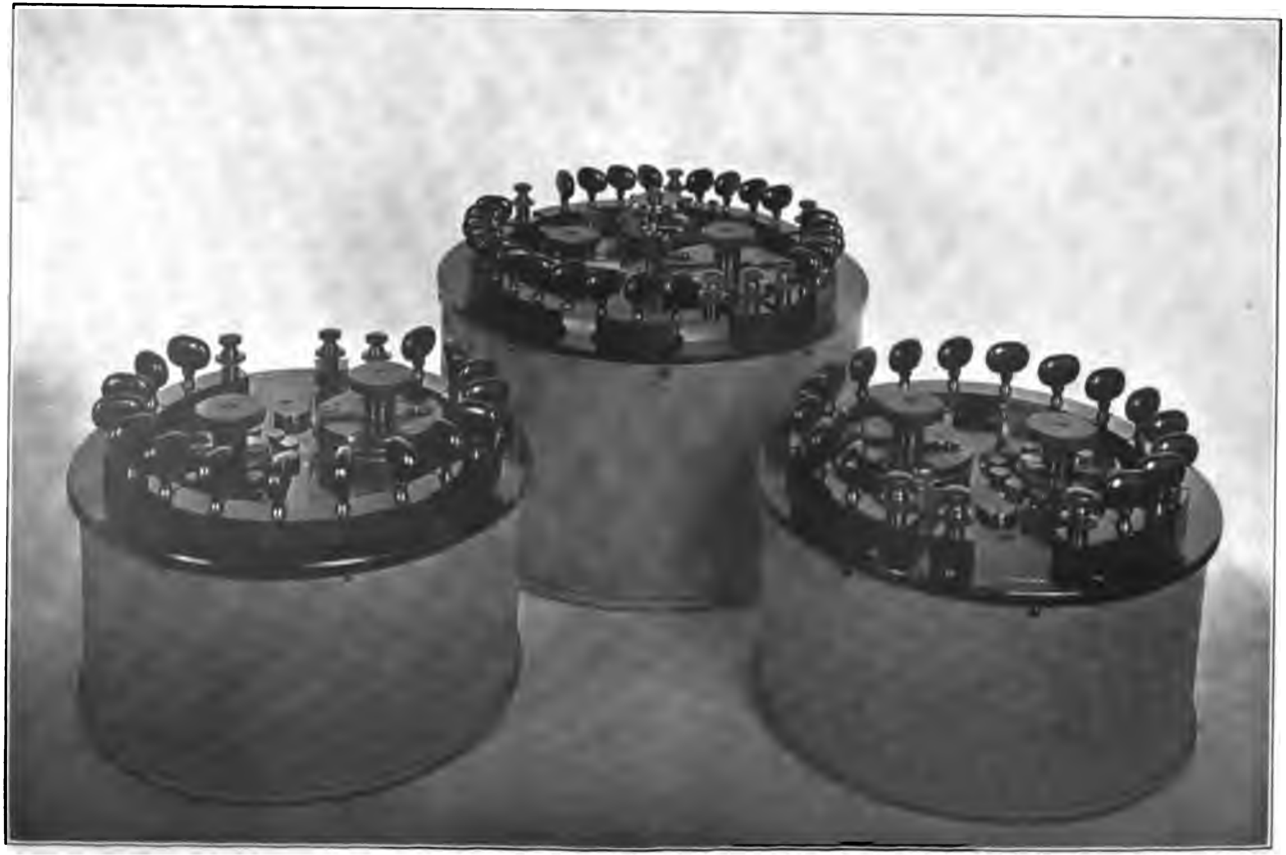

Fig. 17.—*Sealed Resistance Boxes for Alternating Current Measurements.*

It is only fair to say that the measurements at the National Physical Laboratory were not made under as favorable circumstances as could be wished, the bridge being adapted to a different style of coil. Hence, Mr. Smith regarded the 1-ohm comparisons as less reliable than the others.

Such intercomparisons of resistances between the several standardizing institutions should be made from time to time, and a mean value of the international ohm derived therefrom, which would be used by all countries. At present the values of resistances in England are in terms of their mercury ohm of the National Physical Laboratory, while in Germany they are in terms of the standards of the Reichsanstalt. The values used in America have been obtained from the Reichsanstalt, while in France they have been based on their own mercury ohms, which are different by an unknown amount from those of London and Berlin. Now that resistance standards are available that can be depended upon to within a few parts in a million, probably for long intervals of time a concrete international ohm can be maintained that can be used in all countries, and the value of a resistance standard will then be the same from whatever source it is derived. It only remains to have established some permanent official body as an international electrical commission which may cause the necessary intercomparisons of the several mercury standards to be made, and deduce therefrom the value of the common international ohm. In the same way intercomparisons of other concrete standards, as standard cells, inductance coils, etc., will fix the values concretely of the various international units and so promote uniformity in standards and advance the precision of electrical measurements.

7. USE OF SEALED RHEOSTATS AND BRIDGES.

In Fig. 17 is shown a special form of Wheatstone bridge which we have called an Anderson bridge, employed for measuring inductances,[8] and two special precision rheostats. The bridge is used with alternating currents, employing tuned galvanometers, and to increase its current carrying capacity, in order to give a high sensibility, the bridge has a metal case which is filled with petroleum. The resistances were sufficiently variable to render it necessary to make rather frequent calibrations until something more than a year ago, when the case was sealed practically air-tight. Since then nine coils have been measured frequently in terms of our sealed standards, and the values carefully reduced to 20° C. The record shows that during the past fourteen months the three 50-ohm coils, three 100-ohm coils, and the three 200-ohm coils have remained constant to within about one part in 100,000 on the average, there being no evident difference between the resistance in summer and winter. This is so much better than such a bridge has ever done before in this laboratory that it deserves notice. It is far better than the best open *standards* of the same denominations do. We have also other precision apparatus protected similarly by sealing, and also many boxes and potentiometers protected by paraffining the coils.

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[8] Rosa and Grover, this Bulletin, 1, p. 291; 1908.

As stated above, and as might be expected, the sealed resistance standards are not perfectly constant; but the variations of a number of coils from the mean of the group are so slight that it requires extremely accurate measurements to detect them, and on the average in coils that have been properly prepared amount to a very few parts in a million in a year. Exceptional coils may change more, and for that reason they should be given an ageing test before being trusted as standards of highest precision. The work that has been done during the past year has clearly shown the great advantage which these sealed standards possess, but there is further work to be done in getting standards of smaller temperature coefficients and in studying the performance of these over a longer period. They do not, of course, displace primary mercury standards, but they will make it unnecessary to refer so often to them, and as they will travel in any climate they make it possible to get accurate intercomparisons of the various primary standards that may be set up in different parts of the world. A further study of the standards will be carried on by Dr. F. A. Wolff of this Bureau.

In conclusion, I wish to express my obligation to several members of the Bureau for assistance in the work. The earlier measurements of the standards were made chiefly by Mr. H. D. Babcock. The recent measurements have been made chiefly by Dr. C. A. Pierce and Mr. G. E. Post. The coils and the special bridges were constructed by Mr. Joseph Ludewig, of the instrument shop of the Bureau.

WASHINGTON, October 1, 1908.

ERRORS IN MAGNETIC TESTING WITH RING SPECIMENS.

By Morton G. Lloyd.

The torus has been much in favor as a form for a specimen to be subjected to magnetic measurements, as it gives a continuous magnetic circuit of the material under test without a joint of any kind. Moreover, its shape makes it necessary to know only two of its linear dimensions in order to calculate the constants involved. The use of this form ordinarily requires the laborious winding of magnetizing and other coils on each specimen by hand, but apparatus for commercial testing is now in use which obviates this tedious process.[1]

If the coils be wound uniformly over the entire ring, the leakage of magnetic flux will be negligible, and it may be safely assumed that the flux crossing every section of the ring is the same. The distribution of this flux over the cross section is, however, not uniform, but the flux density is greater near the inner surface of the ring. In making measurements it is customary to determine the total flux by a measurement of the electromotive force developed in one of the coils, or by the throw of a ballistic galvanometer. The average value of the flux density is then found by dividing by the area of cross section; let it be B_0. The average value of the magnetizing force can be computed from the ampere-turns and dimensions; let it be H_0. The ratio $\frac{B_0}{H_0}$ is taken to be the permeability corresponding to the field H_0. This is only an approximation, as the average permeability is not in general given by the ratio $\frac{B_0}{H_0}$, and even if it were, we have no assurance that it is the permeability of that part of the specimen which is subjected to the average magnetizing force.[2] Indeed, we must know how the permeability varies with the radius of the ring, or with H, in order to determine the discrepancy involved; and unless we have an algebraic expression connecting these quantities, it is almost impossible to compute the quantities concerned even after a preliminary measurement has given the approximate values of B corresponding to different magnetizing forces. However, if the radius a of the section is kept sufficiently small in comparison to the radius R of the ring, the variation of permeability will also be small, and may be neglected. With given dimensions of the ring, this variation will depend upon the value of the magnetic induction, as the permeability is at first an increasing function and later a decreasing function of the induction. From the known general properties of this function the variation will be greatest for inductions near the steepest part of the magnetization curve.

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[1] J. A. Möllinger, Electrot. Zs., **22**, p. 379; 1901.

J. W. Esterline, Proc. Am. Soc. Testing Materials, **3**, p. 288; 1903.

If the ratio $\frac{a}{R}$ is made sufficiently small, the value of the magnetizing force at the center of the section may be taken as the average value H_0, and this is an additional simplification. This again is only an approximation, since the magnetizing force at any point is $\frac{2NI}{x}$ where x is the distance from the axis of the ring, and NI is the product of current and turns. Since the average value of $\frac{1}{x}$ is not $\frac{1}{R}$ (the reciprocal of the average value of x), H_0 is not the value of H at the mean radius R. The true value of H_0 can be computed for a ring whose cross section is any simple geometrical figure. The formula for a circular section was derived by Kirchhoff[3] and has been used by Rowland[4] and others, and the rectangular section was early used by Stoletow. For sheet iron, where ring stampings are piled up, the rectangular section is the only one available.

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[2] Cf. A. Stoletow, Phil. Mag., **45**, p. 40; 1873.
G. vom Hofe, Wied. Ann., **37**, p. 482; 1889.

[3] G. Kirchhoff, Pogg. Ann. Ergbd., **5**, p. 1; 1870; Ges. Abhandlungen, p. 229.

[4] H. A. Rowland, Phil. Mag., **46**, p. 140; 1873.

The values of H_0 for circular and rectangular sections have been computed for various values of $\frac{a}{R}$ in terms of H_R, and are given in Table I, and platted in the curves of Fig. 4.

The values for a rectangular section have been computed and published by Edler,[5] but his results are not always correct. The hyperbolic logarithms used are taken to only three significant figures, with the result that the final values are *not reliable to tenths of one per cent*, but may err as much as one-fourth of one per cent.

In determining the hysteresis of a ring specimen, two methods are available. The one is to determine corresponding values of H_0 and B_0, plat a curve between them, and measure the area of this curve. This method involves the same errors as the determination of permeability. The other method is to measure the electrical energy supplied to the ring and at the same time B_0. Alternating currents are usually used for this purpose, and this requires the determination of the form factor of the induced electromotive force in order to know the maximum value attained by B_0.[6] Secondly, eddy currents are induced in the specimen, but if not large can be approximately determined and separated by measurements at two frequencies. Moreover, on account of the nonuniform distribution of the flux in the section of the ring, the power expended is not the same as it would be if the distribution were uniform. This is owing to the fact that the energy per cycle is not proportional to the flux density B, but to some power of it, approximately B^2 for eddy currents and $B^{1.6}$ for hysteresis. Here, again, the average value of $B^{1.6}$ is not the same as the 1.6th power of the average B.

It can easily be shown that where equal volumes of material are traversed by the fluxes of different density, as in a straight bar, the loss due to nonuniform distribution must be greater than the loss with uniform distribution.[7] In the case of a ring, however, the denser flux follows a shorter path, and it has been shown by Richter[8] that if the energy is proportional to a power of the magnetic induction between 1 and 2, the loss is less with the actual distribution than it would be with a uniform distribution. This is due to the fact that the denser flux is established near the inner surface of the ring and traverses a smaller volume than the flux near the outer circumference, and the greater loss per unit volume where the flux is dense is more than counterbalanced by the reduced volume subjected to this loss.

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[5] R. Edler, Mitteil. Techn. Gewerbe-Museums, Vienna, **16**, p. 67; 1906. Science Abstracts, 9 B, p. 158; 1906.

[6] Cf. Lloyd and Fisher, this Bulletin, **4**, p. 469; 1908: or Robinson and Holz, Gen. Electric Rev., **10**, p. 236; 1908.

[7] Cf. B. Soschinski, Electrot. Zs. **24**, p. 292; 1903.

[8] R. Richter, Electrot. Zs. **24**, p. 710; 1903.

Consequently, in most cases of closed magnetic circuits, as in the transformer, the iron loss is less than it would be with uniform distribution of the same total flux. The dimensions are usually such as to involve very great variations in the flux density, and since the distribution of flux varies with the amount of flux (owing to varying permeability) no valid deductions as to the properties of the iron can be made from experiments in which the voltage applied to such an apparatus is varied. Apparatus of this kind may be used to determine the relative quality of different specimens of iron under the same conditions, but no reliance can be placed upon absolute values obtained under such conditions, nor upon variation of loss with B.

If we were dealing with a medium of constant permeability we could calculate the effect of nonuniformity of distribution. With iron, however, the distribution changes with each change in B_0, so that a general solution of the problem is not practicable, but particular cases may be worked out where the variation of permeability with induction is known.[9] It is well to remember that for low inductions the nonuniformity is greater, while with high values of the induction it is less than for constant permeability.

This is illustrated in Fig. 1, where the actual variation of B across the section of a certain ring is given for three different values of the magnetizing force. This ring had inner and outer diameters of 6.9 and 8.9 cm, respectively. Consequently $\frac{a}{R} = .1282$. When the value of B at the mean diameter is 2000 gausses the extreme values are 1350 and 2900, a range of 73 per cent of the mean. With 7800 gausses in the middle, the extreme values are 6700 and 8700, a range of 26 per cent. With 15,000 in the middle, the extremes are 14,500 and 15,350, a range of less than 6 per cent.

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[9] See Richter, loc. cit.

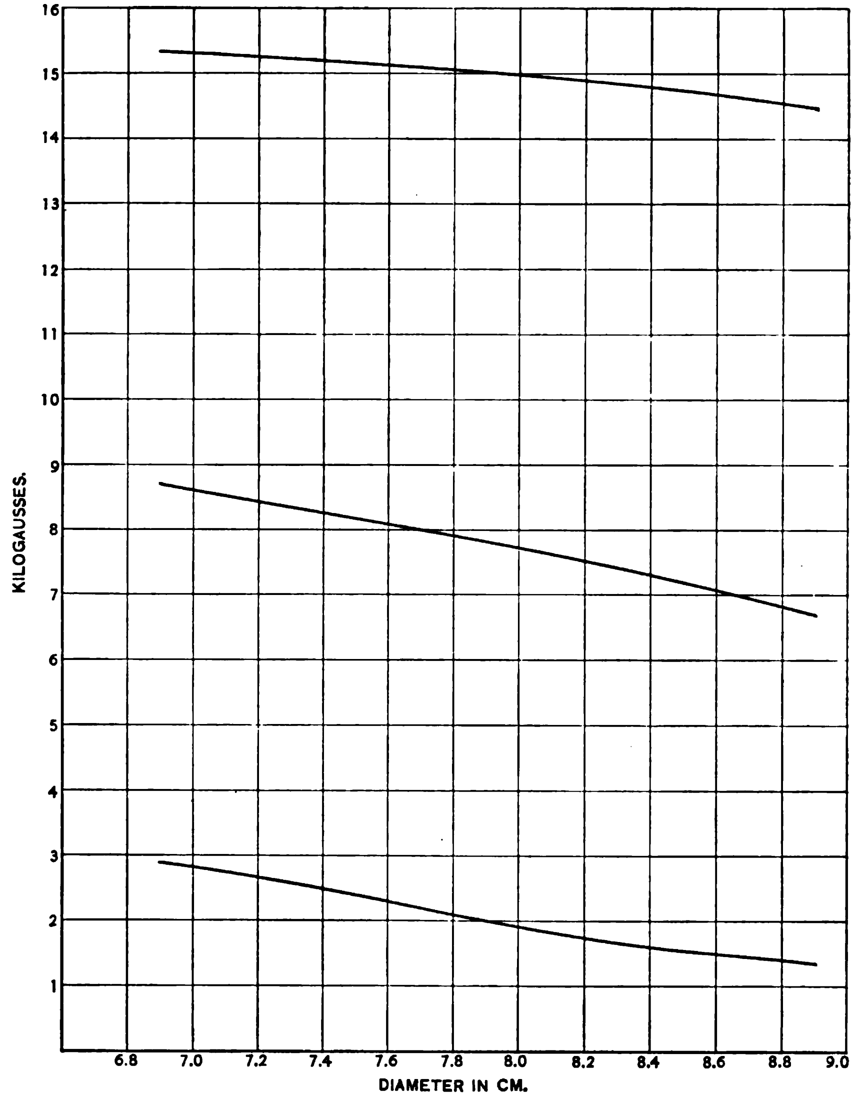

Fig. 1.—*Showing Variation in Flux Density with Distance from Axis of Ring, for Three Different Values of Magnetizing Current.*

In what follows the permeability is assumed constant, and for given dimensions of the ring the ratio of the theoretical loss for uniform distribution to the actual loss is computed, the total flux remaining constant. The solution for rings of rectangular section has been given by Richter in the paper already referred to, but is given here for the sake of completeness. The solution for a ring of circular section is believed to be new.

MAGNETIZING FORCE.

Ring of Rectangular Section.—Fig. 2 shows a section through the axis of the ring. Let NI = total current-turns of magnetizing coil uniformly distributed.

$p=\frac{a}{R}$ = ratio of radial width to mean diameter of ring.

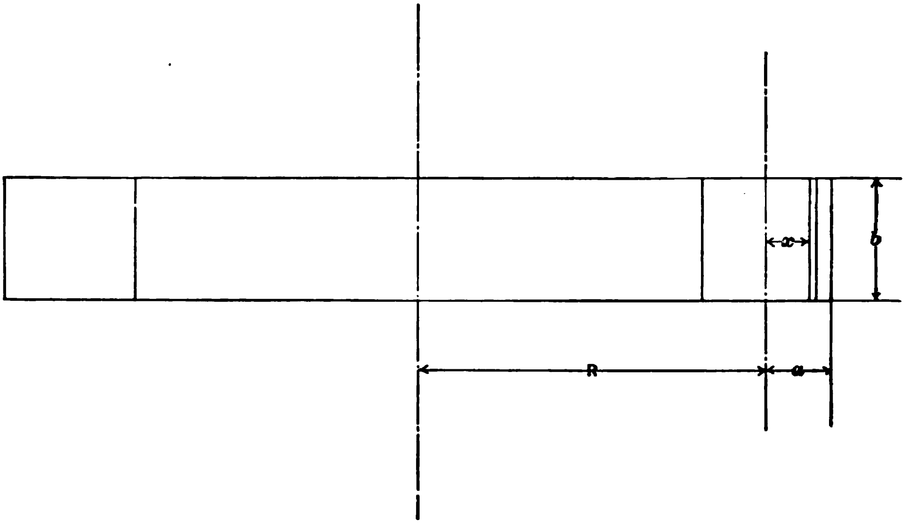

Fig. 2.

The flux in an elementary ring of thickness dx and of unit permeability is

$$d\Phi=\frac{4\pi NI}{\frac{2\pi(R+x)}{bdx}}=\frac{2bNIdx}{R+x}$$

$$\Phi=2bNI\int_{-a}^{a}\frac{dx}{R+x}=2bNI\log\frac{R+a}{R-a}$$

$$=2NIb\log\frac{1+p}{1-p}$$

The average magnetizing force is

$$H_0=\frac{\Phi}{2ab}=\frac{NI}{a}\log\frac{1+p}{1-p}$$

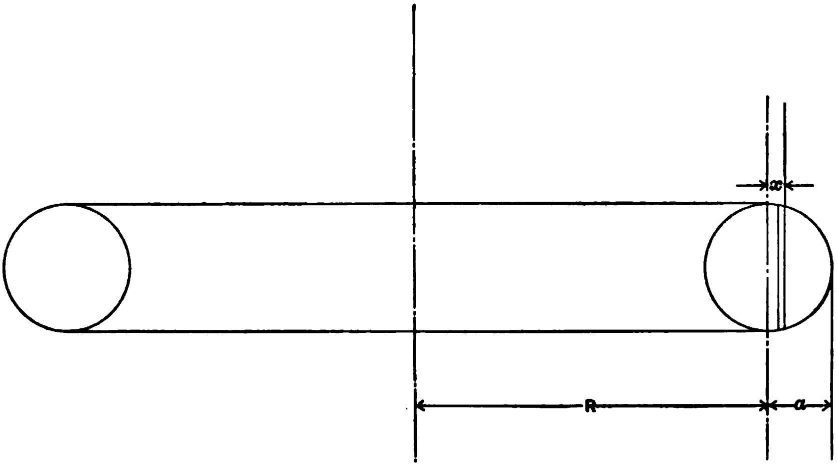

Fig. 3.

The magnetizing force at the center of the section is

$$H_R=\frac{4\pi NI}{2\pi R}=\frac{2NI}{R}$$

and

$$\frac{H_0}{H_R}=\frac{1}{2p}\log\frac{1+p}{1-p}$$

The values of $\frac{H_0}{H_R}$ are given in Table I and platted in Fig. 4 for values of p from $\frac{1}{2}$ to $\frac{1}{19}$.

Ring of Circular Section.—Fig. 3 shows section through axis of ring.

$$d\Phi=\dfrac{\frac{4\pi NI}{2\pi(R+x)}}{2y\,dx}=\frac{4NI}{R+x}\sqrt{a^2-x^2}\,dx$$

$$\Phi=4NI\int_{-a}^{a}\frac{\sqrt{a^2-x^2}}{R+x}dx=4\pi NI\left[R-\sqrt{R^2-a^2}\right]$$

$$H_0=\frac{\Phi}{\pi a^2}$$

$$H_R=\frac{4\pi NI}{2\pi R}=\frac{2NI}{R}$$

$$\frac{H_0}{H_R}=\frac{2R}{a^2}\left[R-\sqrt{R^2-a^2}\right]=\frac{2}{p^2}\left[1-\sqrt{1-p^2}\right]$$

$$=1+\frac{1}{4}p^2+\frac{1}{8}p^4+\frac{5}{64}p^6+\frac{7}{128}p^8+\ \ldots\ldots$$

TABLE I.

Ratio of the Average Value of *H* to the Value at the Mean Radius in Rings of Rectangular and Circular Section.

p	$\frac{H_0}{H_R}$	
	Rectangular	**Circular**
$\frac{1}{2}$	1.0986	1.0718
$\frac{1}{3}$	1.0397	1.0294
$\frac{1}{4}$	1.0216	1.01625
$\frac{1}{5}$	1.0137	1.0102
$\frac{1}{6}$	1.0094	1.0070
$\frac{1}{7}$	1.0069	1.00515
$\frac{1}{8}$	1.0052	1.0040
$\frac{1}{10}$	1.0033	1.0025
$\frac{1}{19}$	1.0009	1.0007

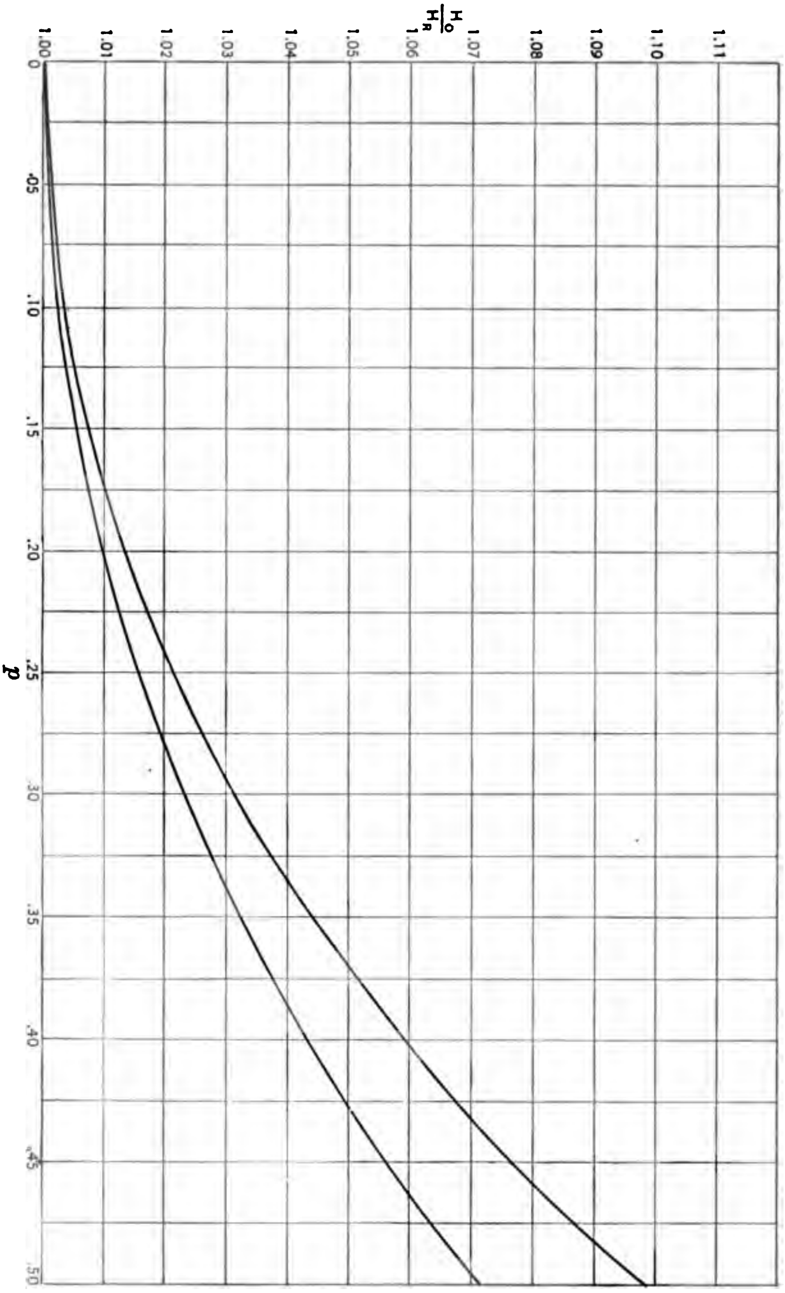

Fig. 4.—*Showing Ratio of Mean Magnetizing Force to the Value at the Mean Radius, for Rings of Different Dimensions with Rectangular (upper curve) and Circular (lower curve) Cross-Sections.*

TABLE II.

Limiting Values of p and $\frac{1}{p}$ for Given Allowable Error in Using H at Mean Radius for Mean H.

Per Cent Error	Rectangular		Circular	
	p	$\frac{1}{p}$	p	$\frac{1}{p}$
0.1	0.055	18.2	0.067	14.9
0.2	.077	13.0	.090	11.1
0.5	.120	8.3	.138	7.2
1.0	.172	5.8	.200	5.0
2.0	.240	4.2	.277	3.6

These values are given in Table I and plotted in Fig. 4. It is to be noticed that the values of H_0 differ less from H_R when the section is circular than when it is rectangular, for the same values of p. With a given area of section, however, it is easy to make p small for the rectangle by increasing the height b and decreasing a. Table II shows the limiting values of p in order to keep the error within assigned limits in using H_R in place of H_0.

ENERGY LOSSES.

Rectangular Section.—We assume constant permeability μ, and energy proportional to the mth power of the flux density. Let η be the energy per unit volume for unit flux density.

$$H_x=\frac{4\pi NI}{2\pi(R+x)}$$

The energy expended in an elementary ring per cycle is

$$dW=\eta\left[\frac{4\pi NI\mu}{2\pi(R+x)}\right]^m 2\pi(R+x)bdx$$

and in the entire ring it is

$$W=\eta(2\mu NI)^m 2\pi b\int_{-a}^{a}\frac{dx}{(R+x)^{m-1}}$$

$$=2\pi b\eta(2\mu NI)^m\frac{(R+a)^{2-m}-(R-a)^{2-m}}{2-m}$$

unless $m=2$. If $m=2$

$$W=2\pi b\eta(2\mu NI)^2\log\frac{R+a}{R-a}$$

For uniform distribution the flux density is everywhere

$$\mu H_0=\mu\frac{NI}{a}\log\frac{1+p}{1-p}$$

and

$$W_0=\eta\left(\mu\frac{NI}{a}\log\frac{1+p}{1-p}\right)^m\int_{-a}^{a}2\pi(R+x)b\,dx$$

$$=\eta\left(\mu\frac{NI}{a}\log\frac{1+p}{1-p}\right)^m 2\pi b\;2aR$$

$$\frac{W_0}{W}=(2p)^{1-m}\left(\log\frac{1+p}{1-p}\right)^m\frac{2-m}{(1+p)^{2-m}-(1-p)^{2-m}}$$

If $m=2$

$$\frac{W_0}{W}=\frac{1}{2p}\log\frac{1+p}{1-p}$$

The latter value is also obtained if $m=1$ and is the same as $\frac{H_0}{H_R}$.

For intermediate values, $\frac{W_0}{W}$ is greater, reaching a maximum for a value of m equal to $\frac{3}{2}$, as is shown in Fig. 5, where the values of $\frac{W_0}{W}$ for $p=0.1282$ have been platted for varying values of m.

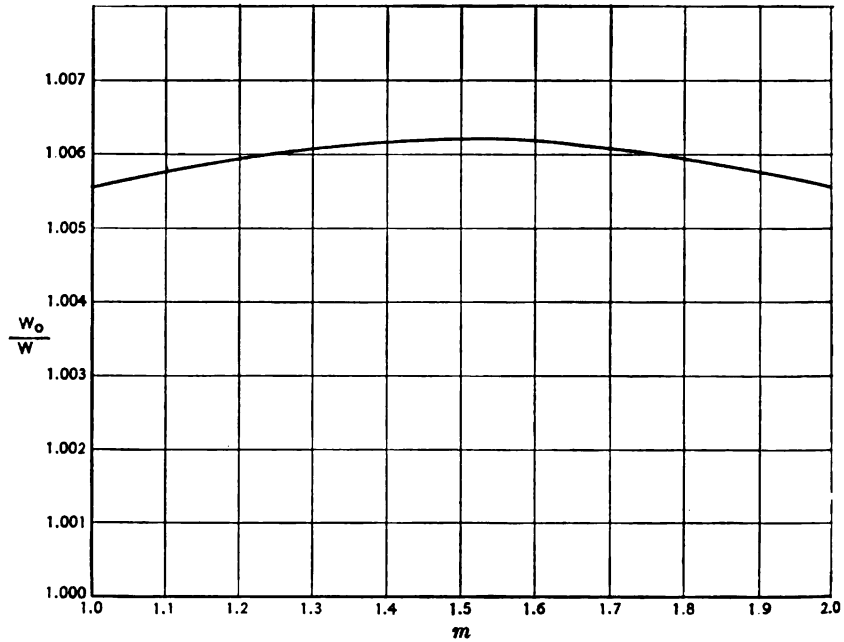

Fig. 5.—*Showing Variation of* $\frac{W_0}{W}$ *with m when p=0.1282.*

The case of particular interest for hysteresis is where $m=1.6$ and the corresponding values of $\frac{W_0}{W}$ have been worked out in Table III and platted in Fig. 6 for $p=\frac{1}{2}$ to $p=\frac{1}{19}$.

Circular Section.—The energy per cycle in an elementary ring is in this case

$$dW=\eta\left(\frac{4\pi\mu NI}{2\pi(R+x)}\right)^m 2\pi(R+x)2\sqrt{a^2-x^2}dx$$

and the total energy is

$$W=\eta\, 4\pi(2\mu NI)^m \int_{-a}^{a}\frac{\sqrt{a^2-x^2}}{(R+x)^{m-1}}dx$$

TABLE III.

Ratio of Hysteresis Loss with Uniform Distribution of Flux to the Actual Loss, Assuming Constant Permeability, in Rings of Rectangular and Circular Section.

p	$\frac{W_o}{W}$	
	Rectangular m=1.6	Circular m=1.5
$\frac{1}{2}$	1.1117	1.0841
$\frac{1}{3}$	1.0447	1.0327
$\frac{1}{4}$	1.0244	1.0183
$\frac{1}{5}$	1.0153	1.0114
$\frac{1}{6}$	1.0105	1.0077
$\frac{1}{7}$	1.0076	1.0058
$\frac{1}{8}$	1.0059	1.0041
$\frac{1}{10}$	1.0035	1.0023
$\frac{1}{15}$	1.0012	1.0008

I have not been able to work out a general solution for this integral, but it can be evaluated for particular values of m, such as 1, 2, $\frac{3}{2}, \frac{5}{2}$, etc.

If $m=1$

$$W=4\pi\eta\ 2\mu NI\frac{\pi a^2}{2}$$
$$=4\pi^2\eta\mu NIa^2$$

If $m=2$

$$W=4\pi\eta(2\mu NI)^2\pi R(1-\sqrt{1-p^2})$$

If $m=\frac{3}{2}$

$$W=4\pi\eta(2\mu NI)^{3/2}\int_{-a}^{a}\frac{\sqrt{a^2-x^2}}{\sqrt{R+x}}dx.$$

This integral can be put in the form of known elliptic integrals and thus evaluated.

Let $x=a\cos 2\theta$

Then

$$\int_{-a}^{a}\frac{\sqrt{a^2-x^2}}{\sqrt{R+x}}dx=\int_{0}^{\frac{\pi}{2}}\frac{2a^2\sin^2 2\theta d\theta}{\sqrt{R+a(1-2\sin^2\theta)}}$$

$$=\frac{8a^2}{\sqrt{R+a}}\int_{0}^{\frac{\pi}{2}}\frac{(\sin^2\theta-\sin^4\theta)d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

where $k^2=\frac{2a}{R+a}$. From the tables of Bierens de Haan, the values for the two integrals are

$$\int_{0}^{\frac{\pi}{2}}\frac{\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta=\frac{1}{k^2}F(k)-\frac{1}{k^2}E(k)$$

$$\int_{0}^{\frac{\pi}{2}}\frac{\sin^4\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta=\frac{1}{3k^4}\left[(2+k^2)F(k)-2(1+k^2)E(k)\right]$$

where $F(k)$ is the complete elliptic integral of the first kind and $E(k)$ is the complete elliptic integral of the second kind. We have, then,

$$\int_{-a}^{a}\frac{\sqrt{a^2-x^2}}{\sqrt{R+x}}dx=\frac{8a^2}{\sqrt{R+a}}\left[\frac{2(k^2-1)}{3k^4}F(k)+\frac{2-k^2}{3k^4}E(k)\right]=\frac{8a^2}{\sqrt{R+a}}C$$

Now

$$(R+a)^{\frac{1}{2}}=R^{\frac{1}{2}}(1+p)^{\frac{1}{2}}$$

$$=R^{\frac{1}{2}}\left[1+\frac{1}{2}p-\frac{1}{8}p^2+\frac{1}{16}p^3-\frac{5}{128}p^4+\frac{7}{256}p^5-\ldots\ldots\right]$$

$$=R^{\frac{1}{2}}A$$

We have, then, for $m=\frac{3}{2}$

$$W=4\pi\eta(2\mu NI)^{\frac{3}{2}}\frac{8a^2}{\sqrt{R}}\frac{C}{A}$$

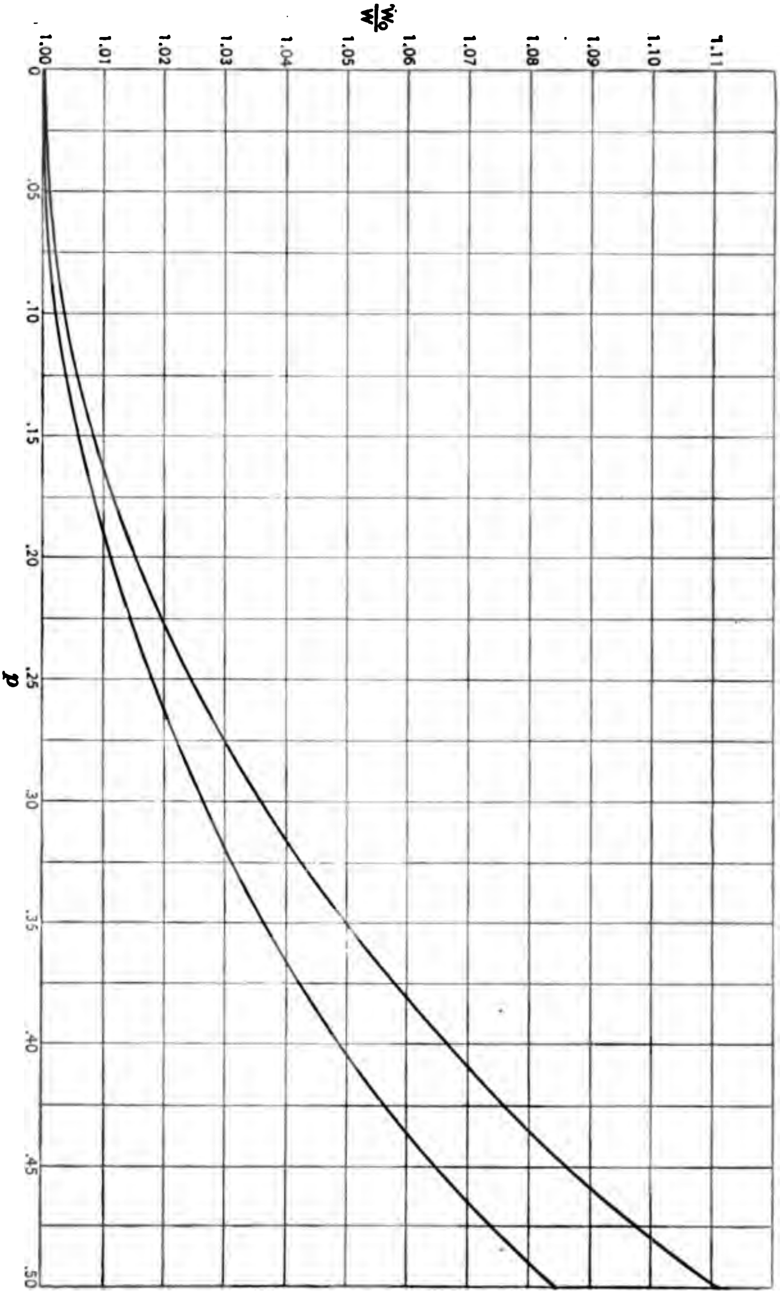

Fig. 6.—*Ratio of Hysteresis with Uniform Distribution to Actual Hysteresis (if Permeability constant) for Rings of Rectangular (upper curve) and Circular (lower curve) Cross-Sections.*

For uniform distribution

$$H_0=\frac{4\pi NI}{\pi a^2}R(1-\sqrt{1-p^2})$$

$$W_0=\eta\left[\frac{4\pi\mu NIR}{\pi a^2}(1-\sqrt{1-p^2})\right]^m\int_{-a}^{a}2\pi(R+x)2\sqrt{a^2-x^2}dx$$

$$=\eta\left[\frac{4\pi\mu NIR}{\pi a^2}(1-\sqrt{1-p^2})\right]^m 2\pi^2a^2R$$

If $m=1$

$$W_0=\eta\left[\frac{4\pi\mu NIR}{\pi a^2}(1-\sqrt{1-p^2})\right]2\pi^2a^2R$$

and

$$\frac{W_0}{W}=\frac{2}{p^2}(1-\sqrt{1-p^2})$$

If $m=2$

$$W_0=\eta(4\mu NI)^2(1-\sqrt{1-p^2})^2\frac{2\pi^2R^3}{a^2}$$

and

$$\frac{W_0}{W}=\frac{2}{p^2}(1-\sqrt{1-p^2})$$

If $m=\frac{3}{2}$

$$W_0=4\sqrt{2}\pi^2\eta(2\mu NI)^{\frac{3}{2}}\frac{R^{\frac{5}{2}}}{a}(1-\sqrt{1-p^2})^{\frac{3}{2}}$$

and $$\frac{W_0}{W}=\frac{\sqrt{2}\,\pi}{8p^3}\,\frac{A}{C}(1-\sqrt{1-p^2})^{\frac{3}{2}}$$

Now

$$1-\sqrt{1-p^2}=\frac{p^2}{2}\left(1+\frac{1}{4}p^2+\frac{1}{8}p^4+\frac{5}{64}p^6+\frac{7}{128}p^8+\ \dots\ \right)$$

$$=\frac{p^2}{2}B$$

$$(1-\sqrt{1-p^2})^{\frac{3}{2}}=\frac{p^3}{2\sqrt{2}}B^{\frac{3}{2}}$$

$$\frac{W_o}{W}=\frac{\pi}{16}\frac{A}{C}B^{\frac{3}{2}}.$$

TABLE IV.

Calculation for $m=\frac{3}{2}$ with Ring of Circular Section.

p	k^2	$\sin^{-1}k$	log E	log F	E	F	C	A	B	$\frac{\pi}{16}\frac{A}{C}B^{\frac{3}{2}}$
0.50000	0.66667	54.736	0.1007778	0.3072753	1.261182	2.028969	0.24670	1.22478	1.07178	1.0841
.33333	.50000	45.000	.1305409	.2681272	1.350644	1.854074	.22919	1.15470	1.02942	1.0327
.25000	.40000	39.232	.1459400	.2498144	1.399394	1.777520	.22085	1.11804	1.01625	1.0183
.20000	.33333	35.264	.1554326	.2390274	1.430318	1.733914	.21594	1.09545	1.01020	1.0114
.16667	.28572	32.112	.1618880	.2318830	1.451737	1.705623	.21269	1.08012	1.00704	1.0077
.14286	.25000	30.000	.1665669	.2267933	1.467462	1.685750	.21031	1.06905	1.00515	1.0058
.12500	.22222	28.125	.1701186	.2229767	1.479512	1.671000	.20864	1.06066	1.00396	1.0041
.10000	.18182	25.239	.1751520	.2176336	1.496761	1.650569	.20623	1.04881	1.00251	1.0023
.05263	.10000	18.435	.1849064	.2074841	1.530758	1.612442	.20149	1.02598	1.00069	1.0008

The values of $\frac{W_o}{W}$ for $m=1$ and $m=2$ are again equal to $\frac{H_o}{H_R}$ and may be found in Table I. For $m=\frac{3}{2}$ the values of C have been determined by the use of Legendre's Tables, while A and B are rapidly converging series and can be carried out to the necessary accuracy. The values given in Table III and platted in Fig. 6 are correct to 0.01 per cent. As the value of $\frac{W_o}{W}$ changes very slowly with m in the neighborhood of $m=\frac{3}{2}$, these values are very nearly the same as would be obtained for $m=1.6$ and represent very closely the conditions for hysteresis. The steps in the computation are shown in Table IV.

TABLE V.

Limiting Values of p and $\frac{1}{p}$ for Given Allowable Error in Hysteresis Measurements in Rings, Assuming Constant Permeability.

Per cent Error	Rectangular m = 1.6		Circular m = 1.5	
	p	$\frac{1}{p}$	p	$\frac{1}{p}$
0.1	0.046	22.	0.065	15.4
0.2	.070	14.3	.090	11.1
0.5	.117	8.6	.135	7.4
1.0	.162	6.2	.187	5.35
2.0	.227	4.4	.262	3.8

These results show the ratio of the loss with uniform distribution to the loss with the actual distribution, if the permeability were constant. With iron specimens the ratio may be greater or less, for with low inductions the distribution will be less uniform, while with high values of the induction it will be more uniform than with constant permeability. For the case of constant permeability, Table V gives the limiting value of p for the error to be within the assigned limit.

WASHINGTON, August 19, 1908.

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THE TESTING OF TRANSFORMER STEEL.

By M. G. Lloyd and J. V. S. Fisher.

Many methods have been employed for the testing of sheet iron and steel for energy losses when subjected to alternating magnetization, but all that have so far been employed have been lacking in some desirable qualifications. The plotting of a hysteresis loop from readings of magnetizing force and the resulting magnetic induction, and the measurement of the area of this loop, is a slow and tedious process and gives no indication of the eddy current losses. Methods depending upon the relative motion of the specimen and a magnet, necessitate an air-gap in the magnetic circuit, with a resulting induction in the specimen which is far from uniform. Moreover, if a permanent magnet be used, as in the Ewing and the Blondel apparatus, one is restricted in the values of the flux density which may be used, and the apparatus is only suitable for comparative measurements at one flux density.

Consequently, methods of testing with alternating currents have come to be regarded as the only satisfactory way of making these measurements, and present efforts are directed to securing the best conditions for this form of test. In the wattmeter method, the electrical energy which is supplied to maintain the alternating magnetization is measured with a wattmeter, while the maximum magnetic induction produced in the specimen is determined by a voltage measurement at the terminals of the magnetizing winding or at the terminals of a secondary winding placed around the same core of test material.

The specimen may be employed in three forms. (1) It may be in the form of straight strips placed in contact with a yoke, thus forming a closed circuit of ferromagnetic material. (2) The straight strips may be used without any yoke. (3) The specimen may be arranged to form a closed magnetic circuit in itself.

The second form gives a distribution of flux which is far from uniform, and is therefore objectionable. The first form gives a more uniform flux, but it is necessary to distinguish between the energy supplied to the specimen and that supplied to the yoke. This can only be done satisfactorily by knowing the constants of the yoke, and only then by having the distribution of flux uniform, a condition difficult to secure. Consequently, for accurate measurements the third form is the most reliable, although for factory use the first or second may prove more convenient where accuracy can be sacrificed for other considerations.

Assuming then a closed magnetic circuit of the material, to be tested by the wattmeter method, the following conditions should be realized as far as possible.

1. The flux should be uniformly distributed over the cross-section of the specimen, and should be the same at every section. This requires that there should be no leakage of flux through the air.

2. A definite form of wave of magnetic flux should be used, or in other words, a definite form of wave of secondary electromotive force, since the form factor of this wave enters into the computation of maximum flux density from the observed effective voltage.

3. The material used should be cut in a form such that only a small part of it is contiguous to a cut edge, since it is well known that all methods of cutting have a hardening effect upon the material bordering upon the cut. This means that the strip, whether straight or in ring form, should not be too narrow. This condition may be dispensed with if all specimens are annealed under definite conditions after cutting to size, and prior to testing.

4. The amount of material required should not be greater than is necessary to get a fair average value.

5. The corrections to be made to the readings of the instruments to get final values should be as few and as small as possible.

Two general forms of magnetic circuit are available. The material may be stamped into rings, or the circuit may be built up from straight strips. Leakage is most effectually avoided by using rings. With this form of specimen, however, it is impossible to satisfy simultaneously conditions 1 and 3, unless rings of very great diameter are employed, and in the latter case there is a very great waste of material. The nonuniformity of flux existing in rings of small diameter, even when uniformly wound, and the errors resulting therefrom, have been discussed in a previous article.[1] The use of rings is thus restricted to cases where the material is annealed after stamping, and the radial width of the ring should be very small in comparison to its diameter. When rings are employed, the labor of winding each specimen separately with a magnetizing coil may be obviated by the use of the apparatus of Esterline [2] or Möllinger.[3]

To meet condition 2 it is sufficient to know or measure the form factor of secondary voltage.[4] By making runs at two frequencies it is then possible to separate the eddy current and hysteresis losses, and, if desired, to compute the eddy current loss for a standard wave form.[5]

It is far preferable, however, to work throughout with a sine wave when a generator is available which will fulfill this condition. There are three things which may prevent the realization of this condition. In the first place the machine may not generate a sinusoidal electromotive force. In fact, it may be stated as a general proposition that no generator gives a perfect sine wave. The only question is as to the magnitude of the harmonics present, and whether these are negligible. It can not be assumed that these are negligible simply because the machine was designed to give a sine wave, or because a rough oscillogram does not indicate definite distortion. The only way to be certain is to take an accurate curve from the machine and analyze it by measurement of the ordinates.

As an example of the necessity of this, we cite an instance occurring at the Bureau of Standards. Here the tests are usually made with a generator whose emf. wave contains a third harmonic whose amplitude is 0.6 per cent of the amplitude of the fundamental, and none of the higher harmonics are present to such an extent as 2 per cent. The form factor is almost exactly that for a sine wave. One day a specimen, which had already been tested with this generator, was tested with a second generator supposed to give a sine wave, and whose oscillogram appeared smooth and inoffensive. The losses appeared more than 4 per cent lower than by the previous test. This led to a closer examination of the wave given by the generator, which was traced by the Rosa apparatus,[6] analyzed and found to contain nearly 7 per cent of the third harmonic, sufficient to account for the observed difference in losses.[7]

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[1] M. G. Lloyd, this Bulletin, **5**, p. 435; reprint No. 108.

[2] J. W. Esterline, Proc. Am. Soc. Testing Materials, **3**, p. 288; 1903.

[3] J. A. Möllinger, Elektrot. Zs. **22**, p. 379; 1901.

[4] An apparatus for measuring form factor is described by Lloyd and Fisher this Bulletin, **4**, p. 469; 1908. Reprint No. 87.

[5] See M. G. Lloyd, this Bulletin, **5**, p. 381; 1909. Reprint No. 106.

A second cause of distorted wave form is to be found in armature reaction, which may alter the emf. of a loaded generator when the curve on no load is sinusoidal. For this reason, it is best in testing to use a machine so large that it is only slightly loaded by the test current.

A third cause of distorted wave is to be found in the drop of potential due to the ohmic resistance of the circuit. The generator emf. is made up of two parts, one of which is balanced by the emf. induced by the changing flux, while the other produces current. If the flux be sinusoidal, the emf. induced by it is also sinusoidal. But owing to the fact that the permeability of the material varies with the magnetic induction, the magnetizing current can not be sinusoidal. The component of emf. producing current has the same form of wave as the current, and it also can not be sinusoidal. The total emf. of the generator, then, to produce sinusoidal flux, is made up of one component which is sinusoidal and one which is not, and therefore is not sinusoidal. Its shape must vary with the conditions, and hence it would be useless to attempt to secure such a form of generator emf. To approximate sinusoidal flux it is necessary to have a sinusoidal emf. at the generator, and to make the component of this, which sends current (equal to the product of current and resistance) negligible in comparison with the component which balances the emf. induced in the apparatus.

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[6] See Rosa and Grover, this Bulletin, **1**, p. 138; 1905. Reprint No. 9.

[7] See M. G. Lloyd, this Bulletin, **4**, p. 484; 1908. Reprint No. 88

It is desirable then to keep both resistance and current low in the magnetizing circuit. If this ohmic drop of potential can be made negligible, then the wave form of flux will differ from a sine curve only by a negligible amount.

The resistance in the magnetizing circuit consists of the armature, leads, magnetizing coil, measuring instruments and perhaps of windings of transformers used to step up or down to the proper voltage. It should not include a regulating rheostat, but the current should be controlled through the generator field. Each of these items should be made as low as possible, and thus appears another reason for choosing a generator of capacity large in comparison to the load to be placed upon it. The magnetizing current may be kept low by having a magnetic circuit of low reluctance. Air gaps should be avoided, and any joints in the magnetic circuit made as good as possible.

With the same magnetizing current, the induced emf. is proportional to the cross-section of test material; consequently the greater the quantity of material used, the less the distortion of wave. With a definite cross-section and windings, the induced emf. is proportional to the maximum flux density, but the magnetizing current is not proportional to the flux density, owing to varying permeability. The larger the permeability, the larger the ratio of flux to magnetizing current. The distortion will consequently be less if the iron be magnetized in the region of maximum permeability, and the distortion is sure to become appreciable if the flux density be carried too high, and may become appreciable at very low flux densities, even when it is negligible through the range of working flux densities used industrially.

In the method of Epstein,[8] which has been adopted as standard in Germany,[9] these conditions are fairly well met. Ten kilograms of sheet are cut into strips 50 by 3 cm and assembled into four bundles, over which solenoids are slipped. The four bundles are arranged in the form of a square, having butt joints at the corners, where the magnetic material is separated by a sheet of thick paper. This interruption to the magnetic circuit tends to make the flux more uniform across the section of test material, but it also makes the leakage greater, and the flux less equal at different sections. Thus the flux at the center of one bundle may exceed the flux near one end by as much as 8 per cent. This air-gap also makes the reluctance of the magnetic circuit high, and consequently the magnetizing current high, and a large quantity of test material must be used. Condition 4 is here sacrificed for the better attainment of conditions 1, 2, and 3, and yet conditions 1 and 3 are not very well satisfied.

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[8] I. Epstein, Elektrot. Zs. **21**, p. 303; 1900.

[9] Elektrot. Zs. **24**, pp. 657, 684; 1903.

A modification of this method has been developed at the Bureau of Standards which differs from the above principally in the arrangement of the test material. A smaller quantity of material in wider strips may be used, while at the same time a greater uniformity of flux is secured. The amount of material used is from 1.5 to 2 kg (less than 4 pounds), and an accuracy of 1 per cent is attained.

DESCRIPTION OF THE APPARATUS.

The specimen to be tested is cut into strips 25.4 by 5 cm (10 by 2 inches). These are assembled into four bundles, in each of which adjacent strips are separated by strips of press board of equal width and thickness, but 2 cm shorter. Each bundle is wrapped with friction tape, and is inserted in a solenoid, and the four are then arranged in a square so that the plan view shows the edges of the strips. (See Fig. 1.) The solenoids are wound upon fiber frames which are 22.7 cm long, and have inside dimensions 5 by 1 cm. At the corners of the square, short pieces of test material are bent at right angles and interleaved between the strips of adjacent bundles, as shown in the figure. There are as many of these corner pieces as there are test pieces, and they are graduated in length so as to give a uniform lap of about 2 mm. A special clamp, shown in Fig. 2, is tightened over these laps, so as to give a good magnetic joint.

Each solenoid has in its first layer two windings of double-silk-covered No. 20 wire, each consisting of 45 turns. Over these are wound 250 turns of No. 14 copper wire, also double-silk-covered, to form a magnetizing coil. The four solenoids are connected in series, making a total of 1000 magnetizing turns and two secondaries of 180 turns each. One of these secondaries is connected to a voltmeter for determining the magnetic flux. This instrument is a deflecting mirror dynamometer, giving a sufficient deflection with 0.004 ampere. The other secondary is connected to the moving coil circuit of a watt dynamometer of the same type as the voltmeter. The magnetizing current traverses the field coils of this wattmeter, whose deflections are a measure of the power supplied to the core and the secondary coils. The copper

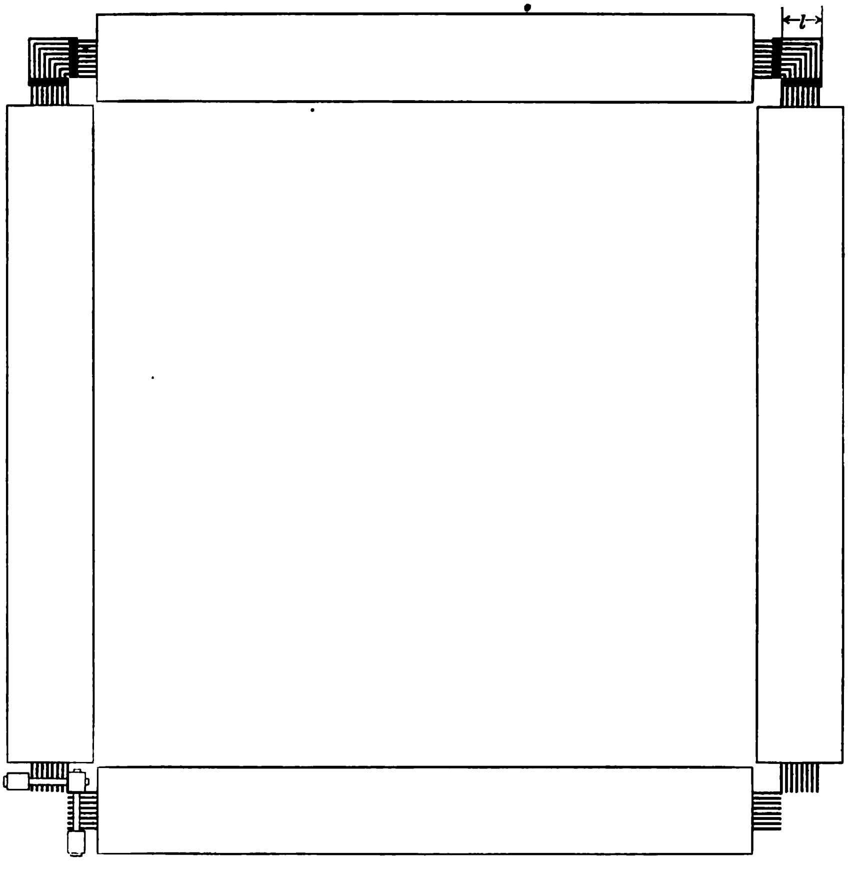

Fig. 1.—Plan of Apparatus with Test Pieces in Position. Corner pieces have been removed from two and clamps from three corners.

loss in the primary is thus eliminated from the power measurement,[10] as is evident from the following considerations.

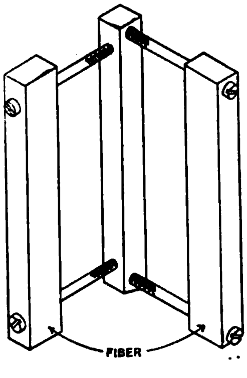

Fig. 2.

Let

N_1 = primary turns.

N_2 = secondary turns.

Φ = flux threading both primary and secondary.

i = primary or magnetizing current.

e = emf. applied to primary.

L = self-inductance of primary due to any flux not included in Φ.

Then

$$e-N_1\frac{d\Phi}{dt}-L\frac{di}{dt}=ri$$

The instantaneous power expended is

$$ei=ri^2+N_1\,i\frac{d\Phi}{dt}+Li\frac{di}{dt}$$

The integral of this expression, extended over a complete cycle, will give the net power. The term ri^2 represents the primary copper loss. The term $Li\frac{di}{dt}$ when integrated over a complete cycle, is equal to zero. The term $N_1\,i\frac{d\Phi}{dt}$ will not integrate to zero when there is either hysteresis in the core or secondary (including eddy) currents flowing, since in either case Φ is not in phase with i. This term represents the power expended in the core and in the secondary circuits. $N_2\,i\frac{d\Phi}{dt}$ is proportional to this, and its integral value represents the reading of the wattmeter when connected as in this apparatus. So that the wattmeter reading, when multiplied by $\frac{N_1}{N_2}$, gives the power expended in the core and in the secondary circuits. Error will arise only when there is flux threading the core and linked with the primary, which is not linked with the secondary. This is avoided by winding the secondary under the primary, and making the two coextensive in length. The energy in each secondary is obtained by squaring the secondary voltage and dividing by the resistance of its circuit. By using a low number of turns in the secondaries and sensitive instruments, these corrections are kept very small and are accurately known.

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[10] Due to C. P. Steinmetz, Trans. A. I. E. E. 9, p. 624; 1892.

Voltmeter and wattmeter each have variable multipliers, whose resistance is adjusted to give a suitable deflection in each case. The accuracy of reading is usually better than 0.1 per cent, and is higher than the conditions require.

The frequency is determined by a Hartmann and Braun frequency meter, which has been calibrated by the use of a chronograph, or where greatest accuracy is required the chronograph is used directly.[11]

When it is desired to measure the magnetizing current, an ammeter can be introduced into this circuit, but the magnetizing current for ordinary inductions is so low that it is difficult to secure an ammeter of sufficiently low resistance. The only type of portable instrument which has answered this purpose is the Duddell thermo-ammeter. This can be obtained with a range of 0.5 ampere and a resistance of 0.2 ohm.

The use of corner pieces bent at right angles caused at first some apprehension as to its effect upon the results. It is known that bending, like any other mechanical treatment, will change the magnetic properties of the iron. If this affects enough of the iron to seriously alter the average value, it would condemn the method. Experiments which were directed to the determination of this effect showed that it is of no importance. The corner pieces are bent sharply in a machine which distorts the material to only a very short distance from the angle. The distorted material has its constants considerably changed without doubt, but as the corner pieces constitute only about 5 per cent of the entire circuit, and as only a part, say 30 per cent, of this material is altered by a fraction, say 20 per cent, of its initial value, an error of much less than 1 per cent is made by regarding this material as unaffected by the bending, and 1 per cent is the limit of accuracy claimed for the method.

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[11] This use of the chronograph, with specimen record, is described by M. G. Lloyd, this Bulletin, **5**, p. 388; 1909. Reprint No. 106.

To examine experimentally the effect of bending, a measurement was made in the usual way; the strips were then removed and each was bent at right angles and then bent straight at a point close to the first bend, so that the length was only slightly altered. A new measurement showed that the losses had increased by 1 per cent. A single bend would of course affect them by much less than this. Another experiment consisted in making measurements upon annealed and unannealed specimens, each with corner pieces of its own material. The unannealed corner pieces were then used with annealed test pieces, and after making the proper corrections and allowances for loss in the corner pieces (as hereinafter described) the values for the test pieces were found in agreement with those previously obtained with annealed corner pieces. As the annealed material is very much more sensitive to mechanical treatment than the unannealed, the effect must have caused different results in the two cases if it were operative to more than a negligible degree.

A third method of checking this point was tried, and led to the same conclusions, but the above are considered sufficient evidence.

On account of the lapping of the corner pieces over the ends of the test pieces, the flux density is low in this part of the material, and the results must be corrected therefor. The amount of lap is determined by the relative weights of corner and test pieces, as compared with the relative lengths of the two parts of the circuit. When the corner pieces are of the same material as the test pieces, it is assumed that the flux density is halved in the portion of material which laps, and the energy loss is consequently only one-third normal.

B = nominal (or average) maximum flux density.
B_1 = maximum flux density at ends of test pieces.
M = mass of test pieces.
m = mass of corner pieces.
l = dimension shown in Fig. 1.
W = measured loss.
$\frac{m}{M}$ = proportional increase in mass of magnetic circuit, due to corner pieces.

$\frac{l}{25.4}$ = proportional increase in length of magnetic circuit, due to corner pieces.

$\frac{m}{M} - \frac{l}{25.4} = c$ = mass of corner pieces which lap, expressed in terms of mass of test pieces.

$2c$ = total material (lapped and lapping iron) in which flux density is $\frac{B_1}{2}$

$2cW\left[1 - \left(\frac{B_1}{2B}\right)^x\right] = 2cWk$ = correction to W for lap, where x expresses the law of variation of loss with B.

k varies slightly with the conditions. If $B_1 = B$ (no leakage) and $x = 1.6$, $k = 0.67$. For $x = 2.0$ this becomes 0.70. For 4 per cent leakage, $x = 1.6$, $k = 0.68$. With sufficient accuracy for the purpose k may be taken as 0.70 throughout, so that the correction becomes 1.4 cW and the loss per unit mass is $\frac{W}{M+m}(1+1.4c)$ or $\frac{W}{(M+m)(1-1.4c)}$ with sufficient accuracy. The latter form is the most useful in practice, since a number of observations at different flux densities are usually made upon a single specimen, and the correction may be made once for all to the mass. The quantity $(M+m)(1-1.4c)$ may be called the "effective mass."

At first, a set of corner pieces was made for each set of test pieces of the same material, but this has been found unnecessary. Corner pieces of approximately the same quality and thickness may be used with satisfactory and reliable results, providing the constants of the material are known. The Bureau has now accumulated a sufficient variety of corner pieces so that it is seldom necessary to make a new set, unless unusual precautions are to be taken. When using corner pieces of different material from the test pieces, it is necessary to compute the loss in the entire corner pieces, and then determine an "effective mass" resulting from the lap reducing the flux in the test pieces. Since the thickness of corner pieces may be different from that of the test pieces, it is necessary to consider this, and the flux at the lap may be considered to divide evenly between the two, or in proportion to their thickness. As the results do not differ materially, we assume that in each lapped part the flux density is half the value in the rest of the material.

Let

t = thickness of test pieces,
t_1 = thickness of corner pieces,
w = loss per unit mass in corner pieces,

and other quantities as before. We neglect leakage which is small. c must now be computed by using for M the mass M_1 of test pieces of same material as corner pieces. The loss in the corner pieces, if there were no lap, would be $wm\left(\frac{t}{t_1}\right)^x$. Considering the effect of lap it is $w\ (m-0.7\ cM_1)\left(\frac{t}{t_1}\right)^x = W'_c$. The correction to the loss in test pieces due to lap is $0.7c\ (W-W_c)$ and the loss per unit mass is $\frac{W-W_c}{M(1-0.7c)}$. If the corner pieces are of the same thickness as the test pieces, the loss in them becomes $w(m-0.7\ cM_1)$, and this expression will usually give the correction closely enough. Here again the quantity $(m-0.7\ cM_1)$ can be determined once for an entire set of measurements.

If the flux at the lap be assumed to divide between the two in proportion to thickness, we have for the loss in the corner pieces

$$w\left[(m-cM_1)\left(\frac{t}{t_1}\right)^x + cM_1\left(\frac{t}{t+t_1}\right)^x\right]$$

and the correction for lap in the test pieces is

$$c\ (W-W_c)\left[1-\left(\frac{t}{t+t_1}\right)^x\right]$$

The leakage with this arrangement of test material, i. e. the difference between flux at ends of test piece and at middle is usually not greater than 1 per cent. Since the greatest length of a line of induction is 8 per cent larger than the least, there is a probability of the same difference in flux density between the outer and inner sheets. As the average flux density is measured, this can not make an error greater than a small fraction of 1 per cent.[12] Inequality in the four arms has an equally slight effect. Upon removing 1 sheet in 12 from opposite sides, the loss per unit mass was not appreciably altered.

OBSERVATIONS AND RESULTS.

The procedure followed in making a test is as follows. The material is cut into strips of the given dimensions by the use of a sharp machine shear with nearly parallel jaws. The number of strips is determined by the thickness, and for gage No. 29 amounts to 48. The strips are then weighed, bundled and mounted in the solenoids. The effective voltage corresponding to any given flux density and frequency is computed from the following relation:

$$E = 4\, fNn\phi 10^{-8} = \frac{4.44 \times 180 \times 10^{-8}\, Bn\, M}{101.6\rho}$$

where n = frequency, ϕ = total flux, f = form factor of secondary emf., and ρ = density. In the work here reported ρ has been taken as 7.77 grams per cc, but it is proposed hereafter to determine the density for each specimen.

The dynamometer-voltmeter is calibrated for one voltage as determined above, and when taking observations for watt loss, the generator voltage is adjusted until the same deflection is obtained. For other frequencies and flux densities, the resistance in the voltmeter circuit is altered until it is proportional to the product Bn, so that the same deflection is always used. For the lower values of this resistance, the slight correction due to

TABLE I.

SPECIMEN K.

Test Pieces = 1327 Grams. Corner Pieces ≐ 80.2 Grams. c = 0.029.
Effective Mass = 1350 Grams.

Cycles	Flux Density	Wattmeter Deflection	Watts	Instrument Losses	Iron Losses	Joules per Cycle	Ergs per Gram per Cycle	Hysteresis	Eddy Currents
60	10000	21.38	4.276	0.079	4.197	0.0699	518	394	124
30	10000	18.87	1.887	.0395	1.848	.0616	456	394	62
60	5000	13.54	1.354	.0395	1.314	.0219	162	130	32
30	5000	12.20	0.610	.0197	0.590	.0197	146	130	16

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[12] See M. G. Lloyd, this Bulletin, **5**, p. 435; 1909. Reprint No. 108.

the inductance of the instrument is also made. In computing the power supplied to the voltmeter circuit, it may then be remembered that this energy is also proportional to Bn, since the same current is used throughout. A similar series of resistances is usually used in the potential circuit of the wattmeter, so that the power consumed here is also readily computed. The deflection here, however, will increase as the product Bn increases, but will usually remain within working limits of permissible deflections. These limits are so chosen that within them the deflections are proportional to the watts. The multipliers for the potential circuit are so chosen that for one of them the actual watts corresponding to a given deflection, when multiplied by $\frac{N_1}{N_2}=\frac{1000}{180}$ is numerically equal to the deflection. The wattmeter is then direct reading for this range for the values of the watts required, and for other ranges a simple factor, determined by the value of Bn, gives the desired result.

The generator used gives an emf. wave which is sufficiently close to the sinusoidal, and the form factor of the secondary emf. has been determined and found sufficiently near to that assumed through the working ranges of flux density at the frequencies used.

TABLE II.

SPECIMEN H_1.

Test Pieces = 1304 Grams. Corner Pieces of Specimen K = 80.2 Grams.
c = 0.029. Effective Mass = 1278 Grams.

Cycles	Flux Density	Wattmeter Deflection	Watts	Instrument Losses	Iron Losses	Joules per Cycle	Corner Pieces	Test Pieces	Ergs per Gram per Cycle
60	10000	21.55	4.310	0.076	4.234	0.0706	0.0028	0.0678	531
30	10000	18.95	1.895	.038	1.857	.0619	.0024	.0595	465
60	5000	13.74	1.374	.038	1.336	.0223	.0009	.0214	168
30	5000	12.34	.617	.019	0.598	.0199	.0008	.0191	150

The frequencies ordinarily used are 60 and 30 cycles, the latter being chosen because it makes the separation of hysteresis and eddy current losses easy to compute. The generator is driven by an electric motor whose field circuit contains a rheostat in the laboratory, permitting adjustment of speed for a definite frequency. The motor is supplied with power from a storage battery so that the speed may be maintained steady. The field circuit of the generator is connected through another rheostat in the laboratory, which permits adjustment of the generator voltage to give the flux density desired. No rheostat is used in the magnetizing circuit. When necessary, a transformer of ample capacity is used to step up or down to the voltage required for the test. The electrical connections are shown in Fig. 3, where for simplicity a single secondary is represented, and may indeed be used in practice. When the generator voltage has been adjusted to give the proper reading on the voltmeter connected to the secondary circuit, the wattmeter is read. These settings are repeated twice. An adjustment is then made for a different flux density and readings taken as before. When the magnetizing current is desired, an ammeter is included in the magnetizing circuit, and its indications noted. Whenever a change is made from a higher to a lower flux, the current is reduced gradually to the lower value in order to demagnetize the material, and this descending set of observations is usually made at a frequency of 30 cycles. Whenever the magnetizing circuit has been broken, it is closed through a considerable resistance, which is continuously

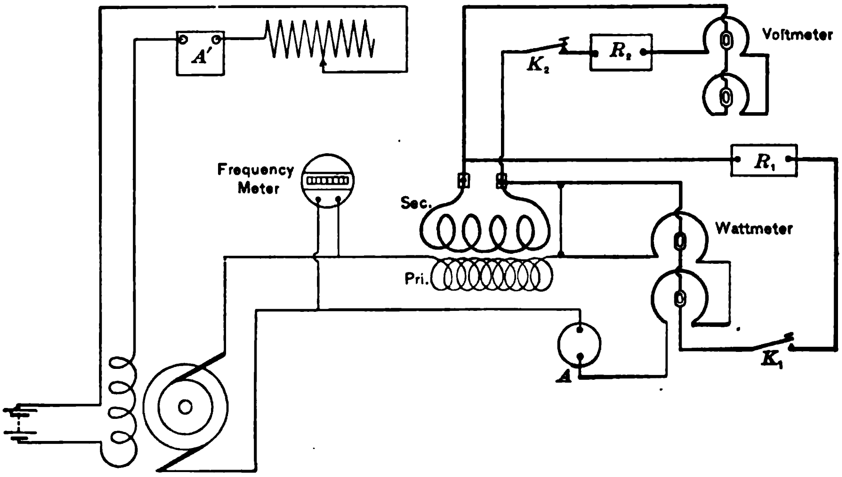

Fig. 3.—Diagram of Connections.

reduced to zero in order to prevent a large first surge and consequent high magnetization, which would require subsequent demagnetization.

Table I gives a specimen set of observations where the corner pieces were of the same material, while in Table II they were of different material.

The total loss is separated into two components, due respectively to hysteresis and eddy currents, as follows, using the Steinmetz equation,

$$W = \eta n B^x + \zeta n^2 B^y$$

where the symbols have the same significance as before, η and ζ being constants of the material. By taking observations at two frequencies, n_1 and n_2, we have

$$\frac{W_1}{n_1} = \eta B^x + \zeta n_1 B^y = a + b n_1$$

$$\frac{W_2}{n_2} = \eta B^x + \zeta n_2 B^y = a + b n_2$$

where a is the hysteresis loss per cycle and bn the eddy current loss per cycle.

$$a = \frac{\frac{W_2}{n_2} n_1 - \frac{W_1}{n_1} n_2}{n_1 - n_2}$$

$$b = \frac{\frac{W_1}{n_1} - \frac{W_2}{n_2}}{n_1 - n_2}$$

If $n_1 = 2\ n_2$ this computation is greatly simplified; for then

$$bn_2 = \frac{W_1}{n_1} - \frac{W_2}{n_2}$$

$$bn_1 = 2bn_2$$

$$a = 2\frac{W_2}{n_2} - \frac{W_1}{n_1} = \frac{W_2}{n_2} - bn_2$$

While the Steinmetz equation, and consequently this separation, is not accurately in accordance with the facts, the errors are very

TABLE III.

Variation of Exponents with Flux Density.

SPECIMEN P_1.

Cycles	Flux Density.	Loss per Cycle	Hysteresis	Eddy Currents	Exponents	
					Hyst.	E. C.
60	12000	842.7	445.7	397		
30	12000	644.2	445.7	198.5	2.12	2.03
60	10000	576.8	302.8	274		
30	10000	439.8	302.8	137	1.75	1.97
60	8000	381.4	204.8	176.6		
30	8000	293.1	204.8	88.3	1.68	1.83
60	6000	230.9	126.5	104.4		
30	6000	178.7	126.5	52.2	1.53	1.97
60	4000	114.9	67.9	47.0		
30	4000	91.4	67.9	23.5	1.51	2.03
60	2000	35.4	23.8	11.6		
30	2000	29.6	23.8	5.8		

TABLE IV.

Variation of Exponents with Flux Density.

SPECIMEN K.

Cycles	Flux Density	Loss per Cycle	Hysteresis	Eddy Currents	Exponents	
					Hyst.	E. C.
60	12500	820	638	182		
30	12500	729	638	91	2.16	1.7
60	10000	518	394	124		
30	10000	456	394	62	1.71	1.9
60	7500	313	241	72		
30	7500	277	241	36	1.53	1.9
60	5000	162.2	129.2	33		
30	5000	145.7	129.2	16.5	1.47	2.0
60	2500	54.6	46.6	8.0		
30	2500	50.6	46.6	4.0		

TABLE V.

Variation of Exponents with Flux Density.

SPECIMEN W_1.

Cycles	Flux Density	Loss per Cycle	Hysteresis	Eddy Currents	Exponents	
					Hyst.	E. C.
60	12500	379	309	70		
30	12500	344	309	35	2.04	2.0
60	10000	241	196	45		
30	10000	218.5	196	22.5	1.68	1.9
60	7500	146.8	120.8	26.0		
30	7500	133.8	120.8	13.0	1.60	2.0
60	5000	74.7	63.1	11.6		
30	5000	68.9	63.1	5.8	1.59	2.2
60	2500	23.6	21.0	2.6		
30	2500	22.3	21.0	1.3		

TABLE VI.

Variation of exponents with Flux Density.

SPECIMEN R.

Cycles	Flux Density	Loss per Cycle	Hysteresis	Eddy Currents	Exponents	
					Hyst.	E. C.
60	14000	606	524	82		
30	14000	565	524	41	1.88	1.6
60	12000	456	392	64		
30	12000	424	392	32	1.71	2.1
60	10000	331	287	44		
30	10000	309	287	22	1.68	1.7
60	8000	227	197	30		
30	8000	212	197	15	1.58	2.2
60	6000	141	125	16		
30	6000	133	125	8	1.65	1.6_5
60	4000	72.2	64	8.2		
30	4000	68.1	64	4.1	1.69	1.8
60	2000	22.1	19.7	2.4		
30	2000	20.9	19.7	1.2		

small in thin sheets. The exponents x and y can be determined by observations at different flux densities, and these have been computed for a number of specimens, as shown in Tables III to VII. Where the eddy current loss is small, as in silicon-steel, the values of y are subject to greater error.

While these exponents do not exhibit any definite and constant value, it will be noticed that the hysteresis exponent does not differ much from 1.6 for flux densities between 5000 and 10000 gausses, while for densities exceeding 10000 it is in the neighborhood of 2, with a definite tendency upward. The exponents for eddy current loss vary rather widely from 2 in some instances, but with specimen C in Table VII, where the eddy current loss is greater and the results consequently more accurate, the values come close to 2.

TABLE VII.

Variation of Exponents with Flux Density.

SPECIMEN C.

Cycles	Flux Density	Joules per Cycle	Hysteresis	Eddy Currents	Exponents	
					Hyst.	E. C.
60	13000	.3554	.2872	.0682		
30	13000	.3213	.2872	.0341	1.82	1.96
60	11000	.2611	.2119	.0492		
30	11000	.2365	.2119	.0246	1.59	2.00
60	9000	.1868	.1540	.0328		
30	9000	.1704	.1540	.0164	1.53	1.98
60	7000	.1247	.1047	.0200		
30	7000	.1147	.1047	.0100	1.48	1.93
60	5000	.0741	.0637	.0104		
30	5000	.0689	.0637	.0052	1.43	1.96
60	3000	.0345	.0307	.0038		
30	3000	.0326	.0307	.0019		

In order to test the proportionality between eddy current loss and frequency, some runs were made with a second generator which gave 180 cycles at normal speed and 90 cycles at half speed. The wave form of this generator is not so pure as that of the one used for the lower frequencies, but the loss could be altered by less than 2 per cent at most from this cause, and probably not over 1 per cent. The error would be less at 180 cycles than at 90 cycles from this cause. The measured loss, however, comes a great deal lower at 180 cycles than is computed from the results at 30 and 60 cycles. This is illustrated by the experiments exhibited in Table VIII, where the hysteresis and eddy current losses are separated by use of the readings at 30 and 60 cycles. The hysteresis is assumed constant, and the resulting eddy current loss does not

TABLE VIII.

Variation of Eddy Current Loss with Frequency.

SPECIMEN C.

B = 5000 *gausses.* *Thickness* = 0.0422 *cm.*

Cycles	Joules per Cycle	Hysteresis	Eddy Currents observed	Eddy Currents computed	Difference per cent
30	.06844	.06221	.00623		
60	.07467	.06221	.01246		
90	.08074	.06221	.01853	.01869	0.9
180	.09626	.06221	.03405	.03738	9.8

SPECIMEN P.

B = 5000 *gausses.* *Thickness* = 0.0437 *cm.*

30	.02385	.01944	.00441		
60	.02826	.01944	.00882		
90	.03249	.01944	.01305	.01323	1.5
180	.04329	.01944	.02385	.02646	11.

SPECIMEN A_1.

B = 3000 *gausses.* *Thickness* = 0.16 *cm.*

30	.0781	.0510	.0271		
60	.1052	.0510	.0542		
90	.1263	.0510	.0753	.0813	7.4
180	.1765	.0510	.1255	.1626	22.8

increase as rapidly as the frequency. The eddy current loss falls off more rapidly, the thicker the specimen. The effect is greatly accentuated in specimen A_1, which consists of sheets 1.6 mm thick, and was tested by using butt joints at the corners of the apparatus. The results for specimen P are plotted in the curve of Fig. 4. The intercept of this curve upon the vertical axis represents the hysteresis loss. A dotted straight line has been drawn through the points for 30 and 60 cycles. We can not be far wrong in assuming that the curve coincides with this at low frequencies, and consequently

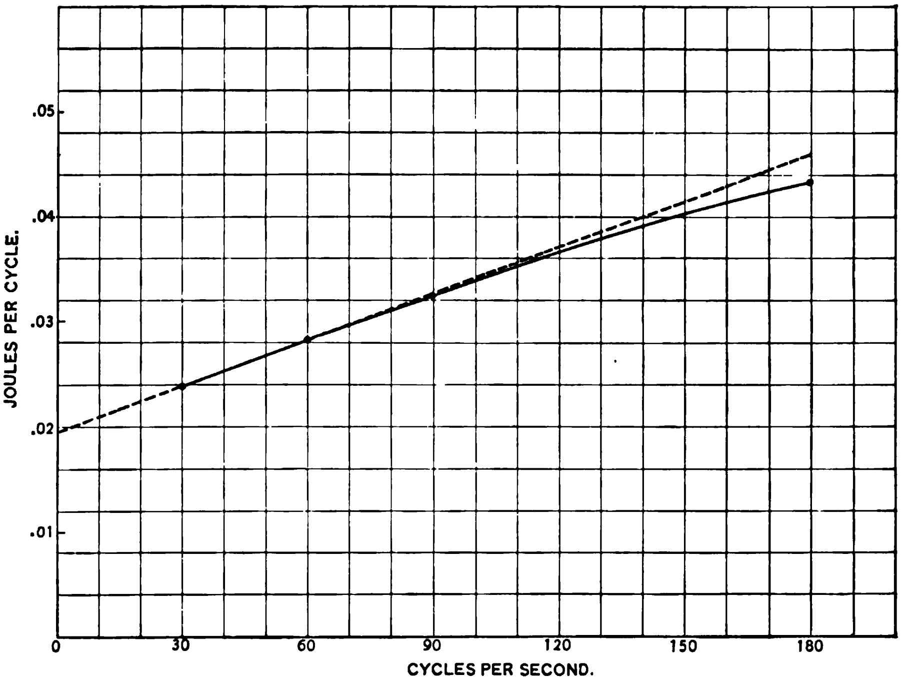

Fig. 4.—Transformer Sheet No. 27 Gage at 5000 gausses.

the method used for the separation of hysteresis and eddy current losses is justified for thin sheets.

The falling off of the eddy current loss at higher frequencies is explained by the consideration that the magnetizing force of the eddy currents reduces the flux in the center and crowds it toward the surfaces of the specimen; the short eddy current paths inclose a smaller flux, while the longest still inclose the same; hence the average emf. of the eddy circuits does not increase as fast as the frequency.

Some experiments were made with thick copper sheets in a solenoid. On account of the low flux densities secured, small quantities of energy had to be measured, and the accuracy was not high, but the results indicated that the eddy current loss did not increase as rapidly as n^2, but somewhat more rapidly than B^2. The latter result is, however, somewhat doubtful.

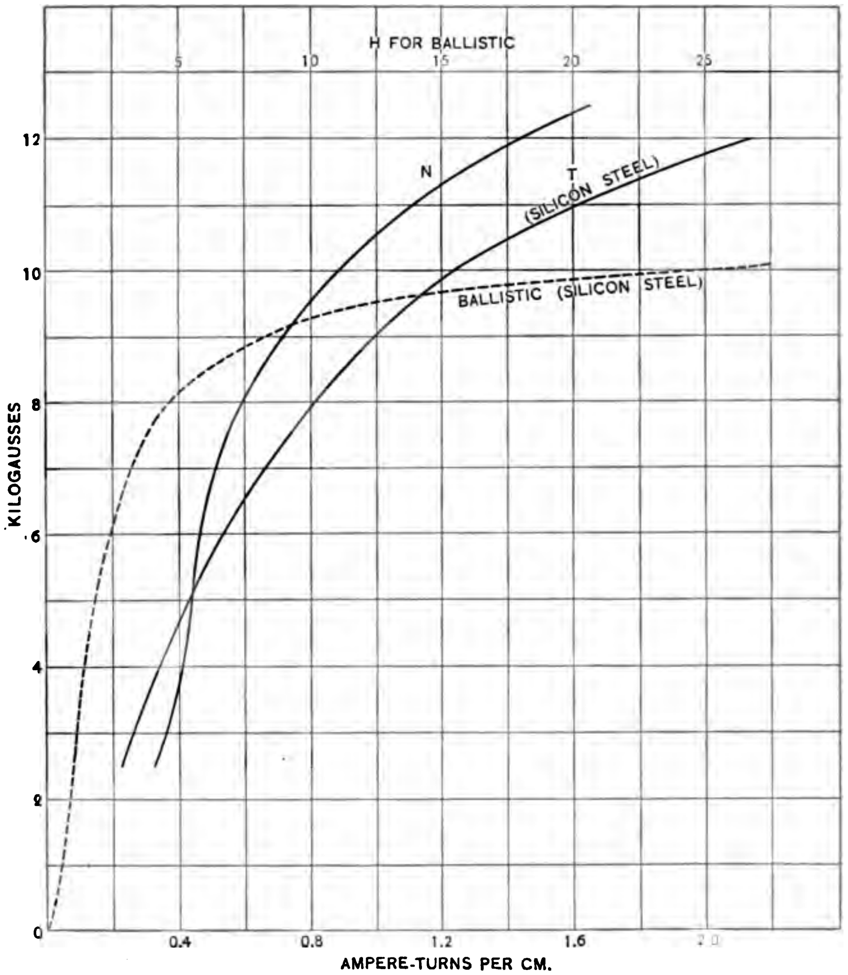

Fig. 5.—Magnetization Curves.

By taking readings upon an ammeter in the magnetizing circuit, it is possible to compute the wattless component of magnetizing current, and a curve of such values is plotted in Fig. 5 in relation to the flux density. Such a curve is just as valuable to the designer, or perhaps more valuable, than a magnetization curve obtained by the ballistic method. A similar curve for silicon-steel is plotted in the same figure. A ballistic curve which has been obtained by Dr. C. W. Burrows for a sample of silicon-steel from the same source, but from a different lot, is also shown, but to a different scale. The low magnetizing current required by silicon-steel at low flux densities makes it particularly suitable for current transformers which must have close regulation, as when used with measuring instruments.

TABLE IX.

SPECIMEN N.

Magnetized Parallel to Direction of Rolling.

Cycles	Flux Density	Ergs per Gram per Cycle	Hysteresis	Eddy Currents
60	10000	531	321	210
30	10000	426	321	105
60	7500	318	196	122
30	7500	257	196	61
60	5000	161	105	56
30	5000	133	105	28

Magnetized Normal to Direction of Rolling.

Cycles	Flux Density	Ergs per Gram per Cycle	Hysteresis	Eddy Currents
60	10000	562	352	210
30	10000	457	352	105
60	7500	332	208	124
30	7500	270	208	62
60	5000	167	110	57
30	5000	138.5	110	28.5

That the hysteresis loss is larger when the steel is magnetized normal to the direction of rolling, than when magnetized parallel to the direction of rolling, is shown by Table IX and the curves of Fig. 6. The eddy current loss in the two cases is practically the same, but the hysteresis is 5 to 10 per cent higher for normal magnetization. This specimen is basic open-hearth steel, but is typical of steel from all sources, though the magnitude of the effect is very variable. Table X shows a similar test upon steel from another source.

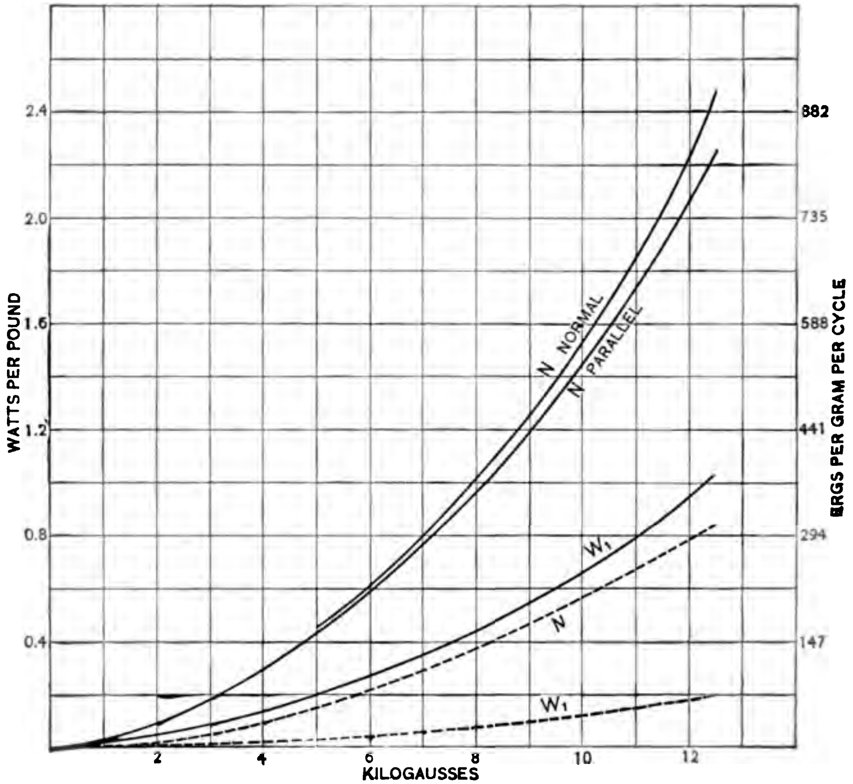

Fig. 6.—Curves showing Relation between Flux Density and Losses when Magnetized parallel and normal to Direction of Rolling. 60 Cycles. Solid Lines represent Total Loss. Dotted Lines show Eddy Current Loss. Difference is due to Hysteresis. N = ordinary steel. W_1 = silicon-steel.

The authors have made tests of sheet iron and steel from a great variety of sources, and have been surprised at the great range in quality of the material in general use by electrical manufacturers, the quality usually having no close relation to the price. Table XI shows some of the results obtained, all the materials having been secured from electrical manufacturers, or from iron mills and dealers supplying the electrical trade. Some foreign samples have been included in the table for comparison. The values of η have been computed from the relation $\frac{W}{n} = \eta B^{1.6}$ where $\frac{W}{n}$ is hysteresis loss in ergs per cc per cycle at 10000 gausses.

TABLE X.

SPECIMEN P_1.

Magnetized Parallel to Direction of Rolling.

Cycles	Flux Density	Ergs per Gram per Cycle	Hysteresis	Eddy Currents
60	10000	576	302	274
30	10000	439	302	137
60	5000	168	96	72
30	5000	132	96	36

Magnetized Normal to Direction of Rolling.

Cycles	Flux Density	Ergs per Gram per Cycle	Hysteresis	Eddy Currents
60	10000	608	336	272
30	10000	472	336	136
60	5000	175	103	72
30	5000	139	103	36

The adjustments can be made and the readings taken so quickly after the circuit is closed, that the specimen becomes heated appreciably only when it is of poor quality or when extremely high frequencies or flux densities are used. The results consequently apply to room temperature. This varies from time to time through several degrees, but it has not been thought necessary to give the temperature in each case. The hysteresis varies only slowly with the temperature, and the eddy current loss, which varies more rapidly, is the smaller part of the total, especially with silicon-steel.

TABLE XI.—Energy Losses due to Alternating

Designation	Thickness	Ergs per Gram per Cycle							
		10000 Gausses				5000 Gausses			
	cm	60 Cycles	30 Cycles	Hysteresis	Eddy Currents at 60~	60 Cycles	30 Cycles	Hysteresis	Eddy Currents at 60~
Unannealed									
A	0.0399	1785	1692	1599	186	608	585	562	46
B	.0326	1290	1223	1156	134	420	402	384	36
C	.0422	1274	1153	1032	242	426	391	356	70
D	.0381	1193	1101	1009	184	401	377	353	48
Annealed									
E	0.0476	971	853	735	236	304	275	246	58
F	.0280	766	716	666	100	247	233.5	220	27
G	.0394	773	668	563	210	247	220	193	54
*H	.0307	558	485	412	146	177.5	158	138.5	39
*H_1	.0277	531	465	399	132	168	150	132	36
J	.0318	543	442	341	202	166.5	139	111.5	55
*K	.0282	518	456	394	124	162	146	130	32
*K_1	.0280	541	479	417	124	170	152	134	36
†L	.0346	565	473	381	184	175	150	125	50
†L_1	.0366	615	516	417	198	192	165	138	54
B_1	.0338	554	454	354	200	173	144.5	116	57
M	.0335	550	461	372	178	173	150	127	46
N	.0340	531	426	321	210	161	133	105	56
N_1	.0312	523	435	347	176	162.5	137.5	112.5	50
P	.0437	518	426	334	184	157	132	107	50
P_1	.0470	576	439	302	274	168	132	96	72
Silicon-Steels									
†Q	0.0361	357	330	303	54	113	105.5	98	15
†Q_1	.0366	390	360	330	60	124	117	110	14
R	.0315	330	309	288	42	104	98.5	93	11
S	.0452	350	314	278	72	108	99	90	18
T	.0338	310	280	250	60	96	87	78	18
U	.0346	312	291	270	42	98	92	86	12
U_1	.0325	322	300	278	44	101	94	87	14
*V	.0310	298.5	275	251.5	47	92	85.5	79	13
*V_1	.0297	303	280	257	46	93	87	81	12
*W	.0305	240	218.5	197	43	74.7	68.5	62.3	12.4
*W_1	.0311	241	218.5	196	45	74.7	68.9	63.1	11.6
X	.0430	265	232.5	200	65	80.8	72.5	64.2	16.6

* German. † English

Magnetization in Various Steels.

x	y	η	Per Cent Silicon	Watts per Pound at 60 Cycles and 10000 Gausses			Designation
				Eddy Current Loss for Gage No. 29 (See note)	Hysteresis	Total	
							Unannealed
1.51	2.02	0.0049		0.41	4.35	4.76	A
1.59	1.89	.00358		0.44	3.14	3.58	B
1.51	1.79	.00319		0.47	2.81	3.28	C
1.52	1.94	.00312		0.44	2.74	3.18	D
							Annealed
1.58	2.02	0.00227		0.36	2.00	2.36	E
1.60	1.88	.00206		0.44	1.81	2.25	F
1.54	1.96	.00174		0.47	1.53	2.00	G
1.58	1.90	.00127		0.54	1.12	1.66	*H
1.60	1.87	.00123		0.60	1.08	1.68	*H_1
1.62	1.88	.00105	0.0	0.70	0.93	1.63	J
1.61	1.90	.00122		0.54	1.07	1.61	*K
1.62	1.82	.00129	0.4	0.55	1.13	1.68	*K_1
1.61	1.88	.00118		0.535	1.035	1.57	†L
1.60	1.87	.00129		0.515	1.135	1.65	†L_1
1.61	1.81	.00110	0.0	0.61	0.96	1.57	B_1
1.55	1.95	.00115		0.55	1.01	1.56	M
1.62	1.90	.00099		0.63	0.87	1.50	N
1.63	1.81	.00107		0.64	0.94	1.58	N_1
1.64	1.88	.00103	1.3	0.34	0.91	1.25	P
1.66	1.92	.00094	0.7	0.43	0.82	1.25	P_1
							Silicon-Steels
1.63		0.00094	3.1	0.14	0.825	0.965	†Q
1.58		.00102		0.16	0.90	1.06	†Q_1
1.64		.00089	3.4	0.15	0.78	0.93	R
1.63		.00086	3.5	0.12	0.755	0.875	S
1.68		.00077	2.8	0.18	0.68	0.86	T
1.66		.00084		0.12	0.735	0.855	U
1.68		.00086	3.9	0.145	0.755	0.90	U_1
1.68		.00078		0.17	0.685	0.855	*V
1.67		.00080	3.8	0.18	0.70	0.88	*V_1
1.67		.00061	3.4	0.16	0.535	0.695	*W
1.64		.00061		0.16	0.535	0.695	*W_1
1.65		.00062	3.2	0.12	0.545	0.665	X

NOTE.—In order to make a fair comparison the eddy current loss has been computed for a thickness of 0.0357 cm (Gage No. 29), assuming the loss proportional to the square of the thickness.

Specimens P and P_1 should perhaps be classed as silicon-steels, although their silicon is not in the proportion which is typical of the alloy. Moreover, they are not put upon the market as an alloy steel, but are sold at about the price of ordinary steel; hence they are classified as such. None of the samples analyzed showed more than the slightest trace of vanadium. Specimen Q contained 0.3 of 1 per cent of aluminum. For the chemical analyses we are indebted to Dr. H. C. P. Weber and Mr. J. R. Cain, of this Bureau.

TABLE XII.

Per Cent Increase in Total Loss at 60 Cycles and 10000 Gausses for Different Periods of Aging.

	TIME IN OVEN.					
	100 hrs.	250 hrs.	500 hrs.	750 hrs.	1000 hrs.	2000 hrs.
G	25	58	67	67	68	67
J	1	5	10	12	15	17
K_1		2	6	9	11	
L_1	−2	−1				
P_1		2	3	5		
Q_1	−2	−1				
R	0	0	0			
T		1	2	4		
U_1		0	0	0		
V_1		−1	0	0	1	
W		−2	−1	0	0	
X		0	0	3		

Artificial aging has been practised upon a number of the specimens, and consists in baking in an oven whose temperature is kept between 90° and 100° C. The baking is only interrupted for the purpose of taking observations, which is done after the specimen has cooled to room temperature. The results are shown in Table XII, where the per cent change in total loss is given after various periods of aging. The time given is only approximate. The silicon steels are almost entirely free from aging, but all the other specimens tested aged considerably, except L_1, and the test upon this was not continued for a very long period.

Caution must be exercised in applying the results of such tests, as one may be easily misled by them. Thus the hysteresis and eddy current losses may be differently affected and, moreover, the hysteresis at different flux densities may be differently altered. In the present cases, measurements were also made at 5000 gausses and at 30 cycles. Separation of the losses after aging indicates which component is active. It is found that the hysteresis is nearly always responsible for the increase, although the eddy current loss may be either increased or diminished.

The decrease in eddy current loss may sometimes mask the increase in hysteresis. Thus specimen Q_1 immediately exhibited a decrease of 14 per cent in eddy current loss, amounting to 2 per cent of the total loss. After 250 hours the hysteresis increased 1 per cent at 10000 gausses, and 2 per cent at 5000 gausses, and yet the total loss still shows a decrease. None of the other silicon-steels showed any change in eddy currents, the slight changes in total loss being entirely due to increased hysteresis.

In the specimens G and J the eddy currents at first increased, but in the second thousand hours showed a marked decrease, which was sufficient to balance the steady increase in hysteresis. The result is a nearly stationary total loss. Specimen L_1 showed a slight initial decrease in hysteresis, and a later increase which was masked by decreasing eddy loss.

Specimen P_1 is a good example of the fact that the loss at different flux densities may be differently affected. The hysteresis throughout rose more rapidly at 5000 than at 10000 gausses, the increases amounting after 750 hours to 22 and 16 per cent respectively. This means that the law of variation of hysteresis with flux density has been changed. The exponent of B to which hysteresis is proportional has been diminished from $x = 1.66$ to $x = 1.60$. In the meantime the eddy currents have decreased, so that the increase in total loss at 10000 gausses appears as only 5 per cent. Meanwhile, the total loss at 30 cycles and 5000 gausses has increased 12.5 per cent. It is thus evident that an aging test should be conducted with measurements at the same flux density as that at which the material is to be used; otherwise the results may not apply to working conditions.

SUMMARY.

The paper contains a discussion of the conditions which should be realized in the measurement of energy losses in sheet iron and steel subjected to alternating magnetization, with a description of a modification of the Epstein method and apparatus, which is believed to better satisfy these conditions and to give an accuracy of 1 per cent with the use of less than 2 kilograms (4.4 pounds) of material.

Results are given showing a wide range in the quality of material in general use, quality having slight connection with price. Several foreign specimens of ordinary steel and silicon-steel are included for comparison with the American product. Silicon-steels contain from 3 to 4 per cent silicon, the quality not depending upon the exact percentage of silicon.

By making measurements at two frequencies, the eddy current and hysteresis losses have been separated and the variation of each with flux density studied. The values of the hysteresis constant and the total loss in watts per pound at 60 cycles and 10000 gausses are tabulated.

The effects of artificial aging are shown to depend upon the flux density selected for test, the hysteresis increasing more for 5000 than for 10000 gausses. Tests should therefore be made at the density to be used. The aging is usually negligible in silicon-steels.

WASHINGTON, January 29, 1909.

A NEW METHOD OF DETERMINING THE FOCAL LENGTH OF A CONVERGING LENS.[1]

By Irwin G. Priest.

INTRODUCTION.

Incident to some work with a modified form of the Fabry-Perot interferometer, there has occurred to me a method for determining the focal length of a converging lens by measuring the linear diameter of a circular fringe in the real image formed by the lens in question. I have formulated the theory of the method and subjected it to experimental test. It is the purpose of this paper to present the method and discuss some of its features.

The method makes use of the system of concentric circular fringes obtained by the interference of the light reflected from a partial silver film with that reflected from a heavy polished silver film parallel to the first and behind it, the incident and reflected beams being perpendicular to these mirrors. The procedure for producing these fringes is precisely similar to that for producing the Fabry-Perot fringes except as to the degree of silvering and the fact that the reflected instead of the transmitted light is examined. Fringes by reflection from parallel silvered mirrors have been used by Hamy; and their theory and method of production have been given by him.[2] But they do not appear to have received as extensive an application in interference measurements as they merit. I have made some study of these fringes with apparatus somewhat different from that of Hamy; the silver films being supported on two plane plates instead of on a plane plate and a lens as was done by Hamy, and the light thrown into the apparatus by a glass plate set at 45° with the normal to the films. The fringes were then observed by looking through the glass plate in a direction normal to the films. Under these conditions the fringes, as seen by the eye, are localized at infinity and a real image of them will be formed in the principal focal plane of a lens placed in front of the apparatus with its axis normal to the films. (See Fig. 1.) The important characteristics of these fringes are: (1) The outer edges of the bright rings are exceedingly sharp and well defined, so that measurements can be made upon them with great precision; (2) the appearance or disappearance of the bright point at the center, due to variation of the distance between the mirrors, is also very sharply defined, so that the difference of path at the center can be adjusted to this point of disappearance with great nicety. These properties of the fringes render them available for the present purpose. These peculiar properties of the fringes vary with the thickness of the partial silver film and with the intensity of the incident light. I have found that films transmitting less than 20 per cent of the incident light give the most satisfactory fringes. As the percentage transmission of the partial film is decreased, the fringes become sharper; but when reduced below 10 per cent the glare of light reflected from the partial film begins to interfere with the observation.

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[1] The substance of this paper, in a slightly modified form, was presented at a meeting of the Philosophical Society of Washington, February 13, 1909.

[2] J. de Physique, 4 série, 5, pp. 789–809; 1906.

2. SUMMARY OF ADVANTAGES.

(*a*) The method gives the focal length properly so defined, i. e., the distance between the second principal point and the principal focus. (*b*) It is of general applicability for lenses of widely varying focal length. (*c*) It gives the focal length for a particular wave-length (spectrum line) very conveniently. Several determinations for selected wave-lengths thus give a definite and precise measure of the chromatism of the lens. The method therefore affords a convenient, definite, and simple test for achromatism. (*d*) Both the observation and calculation involved are exceedingly simple. The constant of the apparatus having been calculated once for all, subsequent determinations require the measurement of only one quantity, a length conveniently measured by a micrometer. The focal length is immediately given by the product of this measured length and the constant before mentioned. It is not necessary to make any measurement from any point on the lens or its mount. (*e*) The method does not require an optical bench, nor a long working distance, nor an "object."

3. THEORY OF THE METHOD.

The following argument assumes a previous knowledge of the theory of the Fabry-Perot fringes as given by Fabry and Perot,[3] and Eversheim.[4]

The method is justified by the following considerations:

(*a*) The real image of the fringe system due to the interference of the light from two accurately parallel mirrors is formed in the principal focal plane of the lens used to produce the image.

(*b*) The focal length, F, of the lens is given by[5]

$$F = R_r \cot \frac{x_r}{2} \tag{1}$$

where $x_r \equiv$ angle subtended at the second principal point by the diameter, in the image, of the r-th ring from the center, counting the central bright point, zero;

$R_r \equiv$ linear radius of the same ring in the image, when the order of interference at the center is given by

$$P = r + N \tag{2}$$

where N is the order of interference of the r-th ring from the center, r being, by definition, an integer.

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[3] Ann. de Chim. et de Phys., 7 série, **16**, p. 115; 1899.

[4] Zs. für wiss. Photog. **5**, 152–180. Trans. in Astrophys. J. **26**, 172; 1907.

[5] The formula for wave lengths itself,

$$\lambda' = \frac{P\lambda}{P'}\left(1 + \frac{D^2 - D'^2}{8F^2}\right)$$

might be transformed to give F, thus:

$$F = \sqrt{\frac{P\lambda\,(D^2 - D'^2)}{8\,(P'\lambda' - P\lambda)}}$$

where P is the order of interference for the ring of diameter, D, due to wave length λ, and P', D' λ' are similarly related; but the advantages of the simpler formula presented in this paper are evident. To apply the above formula to determine F to $\frac{1}{1000}$ would require all the labor and care necessary to determine λ' to $\frac{1}{5000000}$.

(*c*) The factor $\cot \frac{x_r}{2}$ may be calculated from the formula

$$\cot \frac{x_r}{2} = \frac{N}{\sqrt{P^2 - N^2}}, \tag{3}$$

or

$$\cot \frac{x_r}{2} = \frac{P-r}{\sqrt{2Pr - r^2}}. \tag{4}$$

For, assuming[6]

$$P \cos \frac{x_r}{2} = N,$$

we have

$$\cos \frac{x_r}{2} = \frac{N}{P};$$

and, therefore,

$$\cot \frac{x_r}{2} = \frac{\frac{N}{P}}{\sqrt{1 - \frac{N^2}{P^2}}}$$

$$= \frac{N}{\sqrt{P^2 - N^2}}$$

or

$$\cot \frac{x_r}{2} = \frac{P-r}{\sqrt{2Pr - r^2}} \qquad \text{[from (3) and (2)].}$$

(*d*) For large values of P, the value of $\frac{\delta \cot x_r/2}{\delta P}$ (P and N both varying so that r remains constant) is very small. In other words, for different values of P, the corresponding values of the diameter of the r-th ring from the center differ very little. This is true to such a degree that, except for lenses of very short focal length, it will be possible to choose P so that a value of it uncertain by several units will still be sufficiently accurate to serve in determining $\cot x_r/2$ with the desired accuracy. Thus the labor of an exact determination of P is not, in general, involved in the method.

(*e*) The peculiar character of the fringes by reflection affords a means of making the adjustments and measurements with the accuracy requisite to make the method practicable.

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[6] See Fabry and Perot, Ann. de Chim. et de Phys. **16**, p. 121; 1899; and **25**, p. 111; 1902. Make proper translation of symbols.

To summarize: The first two considerations justify the formula for focal length. The third consideration gives the formula for calculating the constant of the apparatus and shows that, for subsequent determinations, only one quantity need be measured. The fourth consideration justifies an approximation which enables the experimental part of the labor to be curtailed. The fifth consideration affords us the experimental means of making the measurements.

4. EXPERIMENTAL PROCEDURE.

The apparatus required is as follows:

(*a*) The interference apparatus, consisting of the two silver mirrors before mentioned, supported by glass plates separated by three invar buttons, and held in place in a frame by three small springs. It is such a contrivance as is used in wave-length measurements.[7] (*b*) A Ramsden ocular, for examining the real image of the fringe system. (*c*) A micrometer screw and carriage to carry the ocular and move it through measured distances. (*d*) A suitable source of monochromatic light and means for illuminating the interference apparatus. (*e*) Suitable mountings for the interference apparatus and the micrometer permitting of their convenient displacement in both a horizontal and vertical direction, perpendicular to the line of sight.

The apparatus is arranged as shown in Fig. 1. The lens under test should be carried on the same support as the micrometer. The mounting should permit of bringing the axes of the ocular and lens into coincidence. There should be no chance for accidental displacement of the lens relative to the micrometer.

The adjustments to be made preliminary to making a measurement are as follows: (*a*) The axes of the lens and ocular are made coincident for the central position of the ocular, by centering the circle of illumination on the cross-hairs with a low power ocular. (*b*) The micrometer ways and the mirrors are made roughly, parallel by sight. (*c*) The source and the illuminating devices are adjusted to throw the light normally on the mirrors and the lens and ocular are brought into position to receive the reflected light. (*d*) The ocular is focused on the cross-hairs and on the image of the fringes, the same colored light being used for both adjustments. (*e*) The common axis of the lens and ocular is tilted until the center of the ring system is in the center of the field (intersection of the cross-hairs). (*f*) The small springs holding the plates are adjusted until the two mirrors are accurately parallel and the bright point at the center has just disappeared. The test for parallelism is made by displacing the interference apparatus both horizontally and vertically in the plane of its mirrors. The rings should neither expand nor contract.

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[7] For description of such an apparatus, see Fabry and Perot, Ann. de Chim. et de Phys. (7) **25**, 107, or Baly, Spectroscopy, p. 302. We have placed the three buttons at the vertices of an isosceles right-triangle instead of the vertices of an equilateral triangle.

The foregoing adjustments having been made, the diameter of a suitable ring is determined from a series of readings of the micrometer screw for settings of the intersection of the cross-hairs on the sharp edge of the ring at opposite ends of a diameter.[8]

We have yet to consider the determination of P. It has been shown that, in general, P need be determined only to within a few units. Therefore, if t be the perpendicular distance between the mirrors and λ be the wave length, P may be determined with sufficient accuracy from

$$\frac{2t}{\lambda}=P$$

if t be known with an error not greater than a few microns. In some cases t could be determined by careful measurements of the invar buttons with a micrometer. I have determined t by means of the fringes of superposition,[9] thus: The difference of path on the sliding interferometer was made equal to $2t$, and then equal to $4t$. The displacement of the carriage (obtained by reading a scale on the carriage by a microscope mounted on the bed) gave t with an error not greater than 2 microns. This is a very satisfactory method.

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[8] The outside edge is used because it is the best defined line to set the cross-hairs upon, and corresponds, in phase, to the disappearance of the bright point at the center, which is likewise the most definite adjustment that can be made of the difference of path at the center. To apply the formula it is not necessary to assume that the difference of path at either of these points is truly integral, but only that they correspond in phase, i. e., that r is accurately integral.

[9] Fabry and Perot, Ann. de Chim. et de Phys., 7, **12**, p. 475; 1897.

5. DISCUSSION OF PRECISION ATTAINABLE.

(*a*) THEORETICAL CONSIDERATIONS.

We desire to determine to what degree the several errors of adjustment and measurement will affect the result; and to decide upon the most favorable circumstances for making the measurement.

The formula, $F = R_r \cot \frac{x_r}{2}$ shows that the percentage accuracy of F, will be fixed by the percentage accuracy of the factors $\cot \frac{x_r}{2}$ and R_r.

The factor $2R_r$ is a length, measured by the micrometer. Experiment shows that the average error of setting at each end is about 3 microns or less. To obtain an accuracy of about one part in a thousand $2R_r$ should be not less than a few millimeters, say 4 or 5.

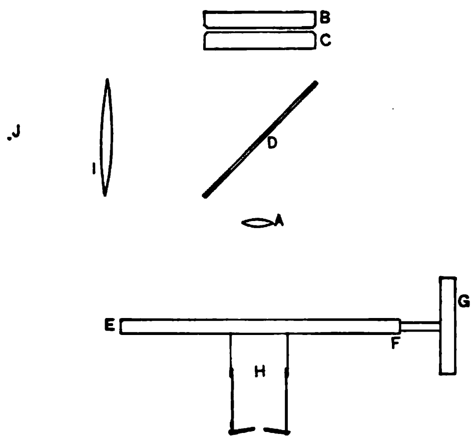

Fig. 1.—Diagrammatic Plan of Arrangement of Apparatus.

A, lens under test; B, C, plates of the interference apparatus; D, plane glass plate; E, F, micrometer ways; G, divided head of micrometer screw; H, ocular; I, condensing lens; J, source, or spectroscope eyepiece-slit, if monochromatic light is obtained by dispersion.

The factor, $\cot \frac{x_r}{2}$ is calculated from the formula,

$$\cot \frac{x_r}{2} = \frac{N}{\sqrt{P^2 - N^2}}$$

and is exact except for the error involved in adjusting P to the value $r + N$. Using partial differentials to express the relation of a small error in P to the resultant error in F, differentiate with respect to P, regarding N as constant. From equations (1) and (3) it follows that

$$\delta F = -F . \frac{P}{P^2 - N^2} \delta P. \qquad (5)$$

Thus the resultant error in F is a fraction of F measured by $\frac{P}{P^2-N^2}\,\delta P$.

When $P-N\equiv r=5$, then $\frac{P}{P^2-N^2}=\frac{P}{10P-25}$.

For $P=100$, we have

$$\frac{P}{10\,P-25}=0.104; \tag{6}$$

and for greater values of P, the approximation to 0.100 becomes closer. Experiment shows that 0.01 is a safe maximum limit to set for δP. We have then,

$$\delta F=-0.001\ F. \tag{7}$$

In words, for $r=5$ and $P>100$, the error in F due to error of adjustment in P is not greater than about $\frac{1}{1000}$. For greater values of r this error will be less.

Having considered the error due to the accidental errors of the observations involved in the form of the formula, we next determine what error may be introduced by the approximation mentioned under 3(*d*). By differentiating with respect to P, regarding r as constant, it follows from equations (1) and (4) that

$$\delta F=F\left(\frac{1}{2P+\frac{r^2}{P}-3r}\right)\delta P \tag{8}$$

which gives the error introduced in F by calculating cot $\frac{x_r}{2}$ from a value of P in error by δP, N being treated as a function of P. If r be taken about 5 or 10, then for all large values of P (say $P>1000$), it is evident that $\frac{1}{2P+\frac{r^2}{P}-3r}=\frac{1}{2P}$ very nearly. We have then,

$$\frac{\delta F}{F}=\frac{\delta P}{2P} \tag{9}$$

from which, selecting the accuracy we desire for F, we can determine the permissible error in P, e. g., if the desired accuracy of F is $\frac{1}{1000}$ and $P = 7000$, the permissible error in P is 14. The permissible error in the distance between the mirrors would then be about 4 microns.

We have now considered all the errors due to the form of the formula and the approximation introduced; and have obtained the expressions which enable us to select favorable conditions for the measurements. If we select $\frac{1}{1000}$ as the desirable accuracy in F, these conditions may be summarized as follows:

R should be greater than 2 mm,
r should be greater than 5;

and if P is determined only with mechanical precision

P should be greater than 5000.

It is to be noted that the conditions for high accuracy require both R_r and r to be large and further, that R_r is made large by making r large, so that these conditions can be met simultaneously. It is also to be noted that the last condition requires P to be large. For a given value or r, this condition would make R_r small; but as r can be chosen greater with advantage, so R can be made greater. As small values of P will need be used for small values of F, it may be necessary to determine P exactly for very short focus lenses.

The method is subject to other possible sources of error as follows:

The error in the adjustment of the parallelism of the mirrors.
The error of focusing the ocular on the image.
The error of adjustment of the axis of lens normal to the plane of the mirrors.
The error in the planeness of the mirrors.
The error due to variation in intensity of illumination.

The effects of these sources of error have been studied experimentally The results so obtained will now be presented.

(b) EXPERIMENTAL RESULTS.

The focal length of a Goerz Dagor lens No. 0000 has been repeatedly determined by this method. Some of these determinations have been made in a way especially suited to obtain information relative to the effects of the sources of error above noted. The results of these determinations and other experiments relative to the same questions will first be given. Then the results not grouped under the above heads will be presented. Finally the results of a series of determinations of the same focal length by one of the old methods will be given as a check.

The following values have been obtained, the parallelism of the mirrors having been readjusted between each two determinations. ($P=3328$. $r=10$. $\lambda=5876$. Sliding interferometer used instead of the invar button standard.)

	42.95 mm
	43.00
	43.00
	43.01
	43.04
Mean	43.00
Average deviation from mean	0.02

The following values have been obtained, the ocular having been refocused on the image between each two determinations. ($P=3327$. $r=10$. $\lambda=5876$. Sliding interferometer.)

	43.14
	43.09
	43.04
	43.09
	43.04
Mean	43.10
Average deviation from mean	0.04

The following values have been obtained, the adjustment of the axis of the lens and ocular normal to the mirrors having been made between each two. ($P = 3328$. $r = 10$. $\lambda = 5876$. Sliding interferometer.)

	43.03
	43.07
	43.08
	43.08
Mean	43.06
Average deviation from mean	0.02

To detect any error due to errors in the plates, which would show in distortion of the rings, mutually perpendicular diameters were measured. A difference of only 0.1 per cent was found. Also, in another case the rear plate was turned about the line of sight through about 90° between successive determinations. The parallelism was readjusted and the ocular refocused. The difference between the two determinations was less than 0.1 per cent.

In order to test the effect of variation in illumination, measurements of the same diameter were made at two widely different intensities (a much greater range than would occur accidentally). A difference of about 0.3 per cent was obtained.

A number of determinations, not grouped under the above heads, have been made. The earliest determinations made without taking due care as to the conditions for accuracy and before practice in the method had been acquired, have been set aside.[10] Results of all determinations made between December 19, 1908, and January 25, 1909, including those grouped above, are presented below. These determinations have been made under varied conditions with different arrangements of apparatus. They are of different degrees of reliability but none have been withheld.

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[10] The mean of a series made about two or three months before the ones here discussed should perhaps be mentioned for the sake of a complete record, and as indicating that determinations made at different times may differ by about 0.5 per cent; but it should not be given great weight. The value was 42.81 mm.

F. in millimeters.

42.49[11]	43.14	43.07	42.96	43.20	
42.95	43.09	43.08	43.05	43.24	
43.00	43.04	43.08	43.00	42.82	[12]
43.00	43.09	43.09	43.09	42.86	
43.01	43.04	42.92	43.00	42.75	
43.04	43.03	42.98	43.00	42.86	
				42.71	

Mean	43.01
Average deviation from mean	**0.09**
P. E. of mean	**±0.013**

As a check on the determinations by the new method, the same focal length has been determined by method (6) p. 180, Kohlrausch, Physical Measurements. As object length, L, the distance between the edges of two white paper riders on a steel meter bar was taken. The image length, l, was measured by the same micrometer used in the new method. The object distance, A, was determined by measuring, with a steel tape, the distance between two points on the table top, the two points having been fixed by dropping a plumb-line from the scale and from a point thought to be about the first principal point of the lens. All of these lengths were of such magnitude that the accidental errors would be less than 0.1 per cent. The focal length is given by

$$F = A\frac{l}{L+l}$$

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[11] This value was obtained without due care as to parallelism of mirrors, and is subject to further error because of uncertainty in P. It is not included in the mean given below.

[12] These values were obtained from measurements made when the apparatus was illuminated by the diffused light from a ground glass, instead of bright light from a condensing lens.

The following values were obtained:

42.76. From mean of two measurements of l for $L=800$ and $A=2913$. Average deviation from mean $=0.04$.
42.94. From mean of three measurements of l for $L=600$ and $A=2913$. Average deviation from mean $=0.03$.
43.02. From mean of five measurements of l for $L=400$ and $A=2913$. Average deviation from mean $=0.11$.
43.06. From mean of five measurements of l for $L=400$ and $A=2857$. Average deviation from mean $=0.01$.

The different measurements of l were each made after refocusing the ocular on the image. Attention is called to the fact that the large values of L gave small values of F. In these cases the ends of the image were near the edge of the circle of illumination and settings were not made under the best conditions. For the last two values, conditions were quite satisfactory. If we assign a weight 1 to each of the first two values and 2 to each of the last two, the most probable value from this data is 42.98 mm.

(*c*) CONCLUSIONS CONCERNING PRECISION OF THE METHOD.

From the discussion just given, the following conclusions are evident:

1. The error due to departure of P from the exact value, $r+N$; and the error due to error in measurement of R can each be made less than 0.1 per cent.
2. The error due to the approximation, *3* (*d*), can generally be made less than 0.1 per cent. The importance of this error can always be estimated, and, if necessary, it may be eliminated.
3. The average error due to readjustment of focus, is less than 0.1 per cent.
4. The average error due to readjustment of the parallelism of mirrors, and the average error due to readjustment of axis of lens normal to plane of mirrors, are each less than 0.05 per cent.
5. With the plates used, the error due to error in planeness of the mirrors, was within the other errors above noted.
6. The difference in results due to a great change in illumination may be a few tenths of 1 per cent. Consequently the differ-

ences due to small departures from a normal working intensity will be very small and within the other errors.

7. The average error in a long series of determinations involving all the sources of error is less than 0.25 per cent.

8. The most probable value by the new method differs from the most probable value by the old method by less than 0.1 per cent.

9. *The results show that a single determination, carefully made, would have given the true value, certainly to within 0.5 per cent; and that a comparatively short series is sufficient to establish the value with a probable error of about ±0.02 per cent.*

6. SUGGESTION AS TO A PROPOSED ROUTINE METHOD.

If it should be desirable to have a rapid, convenient method for determination of focal lengths in a routine way, the manipulation of the method might be still further simplified. Instead of the standard with an air space, a single plane parallel plate of glass or fused quartz heavily silvered on one side and partially silvered on the other might be used. It would not be necessary to know P, nor would P need be integral; for the diameter of any ring in the field might be measured with a lens of *known* focal length and the factor of the apparatus (cot $x/2$) determined from this measurement. Other focal lengths could then be determined by a measurement of the same diameter. Standards of different thicknesses would be prepared to suit different focal lengths. A disadvantage of this modification is that the factor of the apparatus is a function of temperature. However, this factor could be determined at several points over the small range of room temperatures; and within certain limits the correction would be negligible. The modification obviates the readjustments for parallelism. Also, if F of the lens used to determine the factor of the apparatus has been previously determined as the mean of a large number of independent determinations, the error due to readjustment of the difference of path will be eliminated in the modified method, each single determination partaking of the accuracy gained by the multiplication of determinations. Further the accurate parallelism of the surfaces of the plate will be of little consequence, if diameters are always measured in the same direction across the plate.

The method is recommended as a simple, convenient, rapid and precise method for the determination of focal lengths in monochromatic light.

WASHINGTON, March 18, 1909.

A NEW METHOD FOR THE ABSOLUTE MEASUREMENT OF RESISTANCE.

By Edward B. Rosa.

The method of Lorenz is generally considered the best among the various methods that have been employed for the absolute measurement of resistance. It is ideal in its simplicity, and in its method of directly balancing a constant induced electromotive force against the fall of potential in the resistance to be determined. It has, however, a very serious limitation in the very small electromotive force generated, and in the appreciable thermoelectric forces produced at the sliding contacts. And if one were to attempt to get a tenfold greater precision than has hitherto been obtained in absolute resistance measurements by this method, it would probably be found that these sliding contact troubles would be very serious.

In studying the problem of how to secure an accuracy at least ten times as great as has yet been done (for that is what is now demanded in order to keep pace with the possibilities in the absolute measurement of current) it occurred to me that a revolving coil, or two such coils, could be so disposed in the magnetic field of a pair of fixed coils as to yield an electromotive force which could be compared with the fall of potential through a fixed resistance, by means of a differential galvanometer, and so give the absolute value of the resistance. The advantage of this method would be that the electromotive force generated could be a thousand times greater than in the Lorenz apparatus, while the thermoelectric forces at the sliding contacts would be considerably less; for a revolving coil cuts the lines of force four times in each revolution, and two coils of only 125 turns each would therefore generate a thousand times the electromotive force produced by a disk, supposing the field and speed the same, and the area of the coils equal to that of the disk. The thermoelectric forces due to the brushes would be less because the commutator could be much smaller than the disk. The method of securing this result is not quite so simple as in the Lorenz experiment, but it seems to be free from any serious difficulty, and because of the enormous advantage of having an electromotive force of several volts to work with, instead of several thousandths of one volt, the new method seems to merit a careful trial. We are constructing an apparatus of this kind for use at the Bureau of Standards, and I have thought it worth while to give a brief description of the method in order that it might be considered by others interested in the absolute measurement of resistance.

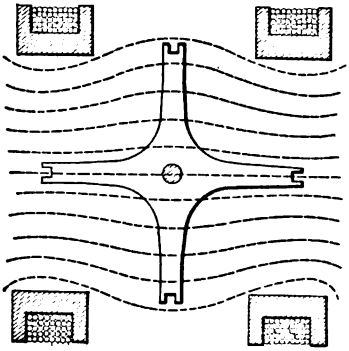

Fig. 1.—Revolving coils and magnetic field of fixed coils.

The two armature coils, at right angles to one another (Fig. 1), rotate in a strong magnetic field produced by two stationary coils, set somewhat farther apart, relatively, than the coils of a Helmholtz galvanometer; the whole constituting a kind of two-phase alternator without iron.

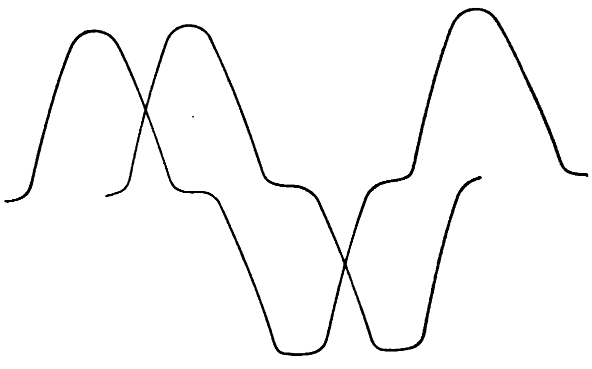

Fig. 2.—Waves produced by the revolving coils of model apparatus.

The wave form of the electromotive force generated is not a sine, as it would be nearly if the armature coils were smaller and the field coils were somewhat nearer together, but has the form shown in Fig. 2. That is, the emf. instead of varying at a maximum rate as it passes through zero, becomes tangential to the axis, permitting the electromotive force to be commutated by means of a two-part commutator, without sensible loss. The lines of force between the field coils swell out as shown in the figure, and the revolving coils slide along the lines very nearly for an appreciable distance at the region of minimum electromotive force, thus giving a very small electromotive force for a considerable angle. It is not practicable to put this commutated electromotive force in series with the constant difference of potential at the terminals of the resistance to be measured, for the commutator would then cut out an uncertain part of the integral emf. of the latter. But they may be compared very accurately by means of a three-circuit differential galvanometer, preferably of the Broca type, as I shall now show.

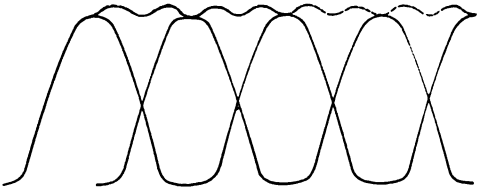

Fig. 3.—**The Two Commutated Emf. Waves and their Resultant.**

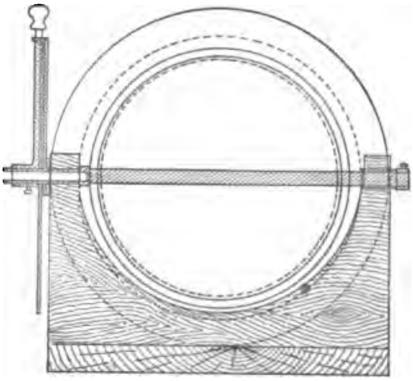

Fig. 4.—**Model Apparatus for Studying Wave Form.**

If we were to use an ordinary two-circuit differential galvanometer, one circuit carrying a constant current from the terminals A B of the resistance *R*, through which the field current *I* passes, and the other a pulsating current (the alternating current generated by one revolving coil rectified by a two-part commutator) the impulses in the needle would be so great that it would be necessary to use a very high speed in the rotating coil, or a very heavy galvanometer needle to prevent vibration of the needle and an indistinct image on the scale. Moreover, such strong impulses might alter the magnetization of the needle, or perhaps produce a deflection even though the integral current in each circuit (going, of course, in opposite directions) were the same. But by using two rotating coils, at right angles to each other, and so disposing the field coils as to make the electromotive force at 45° either side of the maximum equal to one-half the maximum (as can readily be done), the sum of the two currents is constant within about 3 per cent, and this small fluctuation has a frequency four times that of either component (Fig. 3). The effect on the needle therefore, due to the two pulsating currents i_1 and i_2, flowing in two of the three-strand windings of the galvanometer, is the same as though these two currents were combined in a single winding, and is equivalent to a direct current of the same average value. Each circuit has its two-part commutator (set at right angles to each other), and each circuit has the same resistance (perhaps several hundred or a thousand ohms) as the third circuit which carries a constant current.

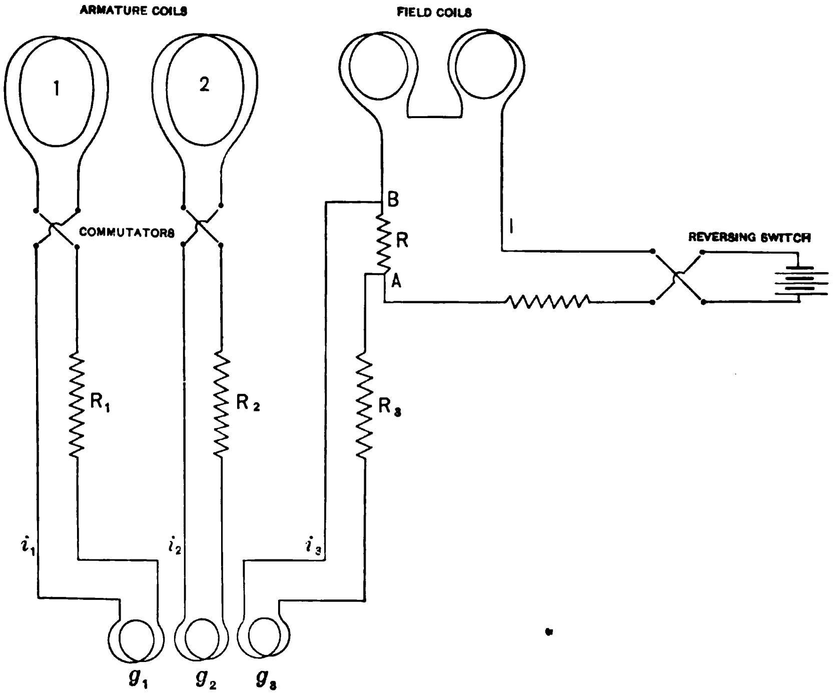

Fig. 5.—The Three Circuits of the Galvanometer with the Connections for measuring the Resistance R.

These resistances do not have to be known, but when the galvanometer is balanced, the sum of the average electromotive forces generated by the two coils is equal to the difference of potential at the terminals of the resistance R, which is the resistance to be determined absolutely.

Let M_1 and M_2 be the mutual inductances of the two revolving coils with respect to the field coils, when each is in the position of maximum inductance, R_1 and R_2 the total resistances of the two circuits of which these coils form a part, including the resistances of the respective galvanometer windings, n the number of revolutions per second of the armature coils, and I the field current. Then the average value of the currents in the two circuits will be

$$i_1 = \frac{4n\, M_1 I}{R_1}$$

$$i_2 = \frac{4n\, M_2 I}{R_2}$$

if the effect of self-inductance is negligible.

The third circuit carries a steady current i_3, due to the difference of potential at the terminals AB of the resistance R through which flows the main current I, which passes through the field coils. Hence

$$i_3 = \frac{RI}{R_3}$$

R_3 is the total resistance of the circuit, including R. We may suppose the windings of the galvanometer not perfectly balanced, so that the currents in any two windings required to give zero deflection are not quite equal.

Then let

$$i'_3 = f_1 i'_1 = f_2 i'_2$$

f_1 and f_2 are factors very nearly unity.

If two circuits are balanced on the same emf E, for example that on the points A B, we should have

$$i'_1 = \frac{E}{R_1} \quad i'_3 = \frac{E}{R_3}$$

and
$$f_1 i'_1 = i'_3 = \frac{E}{R_3}$$

$$\therefore\ i'_1 = \frac{E}{R_1} = \frac{E}{f_1 R_3}$$

Therefore when the galvanometer is balanced, $R_1 = f_1 R_3$ and $R_2 = f_2 R_3$. That is, leaving R_3 constant, the resistance R_1 is altered slightly, if necessary, until there is no deflection, and then the second circuit R_2 is put in place of R_1 and balanced in the same way. It is never necessary either to measure R_1, R_2, R_3 or even to compare them with one another except as is done in occasionally balancing the galvanometer in the manner described above. Any variations due to temperature changes or other causes will thus be corrected.

From what precedes

$$i_1 = \frac{4nM_1 I}{f_1 R_3}$$

$$i_2 = \frac{4nM_2 I}{f_2 R_3}$$

And since $i_3 = f_1 i_1 + f_2 i_2$ when running regularly, and the constant current i_3 is balancing the pulsating currents in the other two circuits

$$\frac{RI}{R_3} = (4nM_1 + 4nM_2)\frac{I}{R_3}$$

or

$$R = 4n(M_1 + M_2) = 4nM$$

where R is the resistance whose absolute value is to be determined, M is the sum of the maximum values of the mutual inductances of the two revolving coils, with respect to the field coils, and n is the speed, or number of revolutions per second. (In the Lorenz apparatus the formula is $R = nM$.)

If $M_1 = M_2 = 25$ millihenrys, and therefore $M = .050$ henrys, and $n = 25$ per second

$$R = 4 \times 25 \times .050 = 5 \text{ ohms},$$

and if the current I is one ampere, the electromotive force E at the terminals of the resistance A B is 5 volts. It would not be difficult to make M much larger than 50 millihenrys, but in a precision apparatus it is desirable to keep the resistances of the field coils as low as practicable to reduce the heating, and the armature coils should be made of small resistance (relative to R_3) and of relatively few turns to reduce the mean temperature coefficient and the self-inductance of the circuit. If the commutators and brushes are properly made, the thermoelectric forces at the sliding contacts will be wholly negligible in comparison with the electromotive forces generated in each coil, and the variation of resistance at the brushes will be negligible in comparison with 500 or 1000 ohms, the total resistance of each circuit. This is partly due to the fact that the commutators will be of much smaller diameter than the large disc of a Lorenz apparatus, and therefore the surface speed will be much less. The temperature changes of resistance will be very slight, for probably 99 per cent of the resistance can be manganin. The three galvanometer windings will always change together, if the temperature changes, and the two windings on the armature will likewise change together. A third winding of copper, having a resistance equal to that of one of the armature coils, in the i_3 circuit could be employed to balance any slight effect of varying room temperature. But that would probably be a needless refinement, if the room were kept at nearly constant temperature, as it should be for other reasons.

These details are mentioned to show that the use of a differential galvanometer carrying an appreciable current does not introduce sources of error that would be troublesome. It seems as though the manipulation of the apparatus would be simple and straightforward, and that a very high accuracy in the results would depend chiefly on measuring n and M with sufficient precision.

From our experience in previous work, I am confident that the uncertainty in the speed need not exceed one part in 100,000. The use of the direct reading chronograph, which was developed especially for this kind of service, and the same method of maintaining constant speeds that we have employed for several years leaves little to be desired in this respect. The burden of the problem therefore is to determine M, the mutual inductance, with sufficient precision.

It has been the practice in all determinations by the Lorenz method to obtain the mutual inductance by calculation from the dimensions of the apparatus. There are some decided advantages in obtaining it, not directly in this way, but by comparison with a standard of mutual inductance, the latter having its value computed from its dimensions. These advantages are as follows:

1. The resistance machine may thus be more compact, and its field coils may be wound with many layers of wire, thus giving a stronger field, and therefore a higher electromotive force, and the machine will be lighter and less expensive to construct.

2. Being more compact, the stray field of the machine is relatively less, and having a strong field the disturbing effects of the earth's field are reduced.

3. The mutual inductances M_1 and M_2 can be measured under working conditions. If there are any magnetic impurities in the shaft or bearings or any other part of the machine, or in any part of the room, hidden or exposed, which render the permeability of the circuit a little greater than unity, they will be taken into account in measuring M_1 and M_2 by comparison with a standard. That is, no assumption is made as to absence of magnetic impurities from the machine, or that the effect of the iron in the driving motor, or other neighboring apparatus, is zero.

4. The mutual inductance can be redetermined by comparison with the standard every time a run is made, if desired, as it will be but a few moments task. Any change due to changes in the windings or temperature effects will therefore be detected. In other words, the machine, which cannot be supposed to remain as constant as a standard of mutual inductance, will not be assumed to have its windings remain indefinitely of constant dimensions. The standard of mutual inductance may be constructed of pure marble and copper wire, and designed to permit the maximum accuracy in the determination of its inductance

from its dimensions. There are several forms which could be used advantageously. The coaxial solenoids of Fig. 6 constitute one of the best known forms. The lengths of the coils must be accurately known, as well as the diameters, and the winding must be very uniform.

The Campbell form of mutual inductance (Fig. 7) is an improvement over this in one important respect. The primary is a single layer coil in two parts, formed by omitting a section in the center, say one-third or one-half the windings. The secondary is outside the primary and of such diameter that it occupies the neutral region, where the magnetic field is nearly zero and the mutual inductance is a maximum. It results from this that the dimensions of the secondary need not be accurately known, and it may consist of a coil of many layers, and may be subdivided to give several values of the inductance.

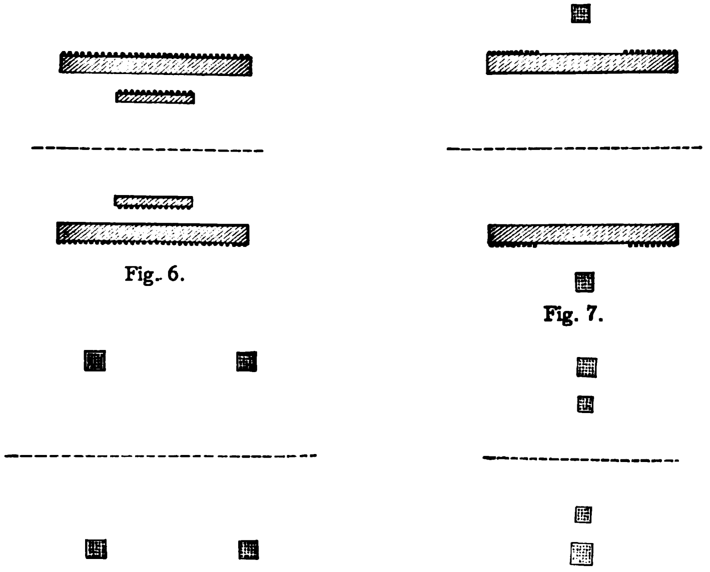

Fig. 6.

Fig. 7.

Fig. 8.

Fig. 9.

Different Forms of Mutual Inductance Standards.

Fig. 6, two coaxial solenoids; Fig. 7, Campbell form; Fig. 8, two equal coaxial coils; Fig. 9, two coaxial coils in the same plane.

A pair of parallel coils of equal radii and rectangular section can be used (Fig. 8). The distance can be very accurately determined, if the precaution be taken to interchange them in use. If they be carefully wound with enameled wire, they can be measured very accurately as wound. But a better method is to compare their radii electrically with a single layer coil, wound on an accurate cylindrical surface. If the number of turns and dimensions be such that the same current can be passed in series through the two coils to neutralize each other's field at the center, the comparison is easily and accurately made.

Another form which I am now having constructed is a pair of coaxial coils in the same plane. (Fig. 9.) The mean radii will be determined electrically by comparison with a standard single layer coil. Being in the same plane, there are no measurements of distance to make. The secondary can be most accurately centered electrically, being in a minimum position with respect to radial displacements, and in a maximum position with respect to axial displacements. The accuracy of the electrical comparisons is probably at least 1 in 100,000, so that the precision of the determinations of radii depend chiefly on the measurements of the standard coil. This can be measured at least as accurately as any single layer winding.

The advantage of this form is that it avoids the measurement of lengths of coils or pitch of windings, and gives a very compact standard which may be of quite large value if desired. The electrical comparisons of radii can easily be repeated, and by winding each coil with a pair of wires which can be joined in series or parallel, one can have three values, as for example 50, 25 and 12.5 millihenrys, merely by changing the connections.

The best way is to build at least two different forms of mutual inductance of the same values, and having the same value as the resistance apparatus, so that they can be compared with one another and with the resistance apparatus by sending the same current in series through their primaries, and connecting their secondaries in opposition through a sensitive galvanometer. One primary may be shunted by a high resistance to secure an exact balance. This comparison can be made very quickly, and with standards of the above values there need not be an error exceeding one part in a million in the comparisons.

By varying the speed from 10 to 25 revolutions per second, and joining the armature windings in series or parallel, various values of *R* from 1 to 10 ohms can be measured with the values of *M* mentioned above. With the recent improvements in the construction and comparison of resistance standards, one is now justified in taking the trouble to get absolute measurements of the highest precision.

At the recent International Electrical Conference in London, Lord Rayleigh expressed the hope that the time was not far distant when a resistance could be measured absolutely so conveniently and so accurately that wire standards could be directly standardized against an absolute resistance machine, and the use of mercury ohms as primary standards of resistance eliminated. If that can be done, it seems probable that an agreement could be reached to check the Weston Normal cell from time to time against an absolute current balance, a procedure which would probably be more accurate, as well as more convenient, than undertaking to check them against the silver coulometer. However that may be, the more accurate determination of the absolute value of our legal ohm is a problem of prime interest and importance, and this new revolving coil method is here proposed with the hope that it may be found useful in this connection.

WASHINGTON, February 27, 1909.

THE THEORY OF COUPLED CIRCUITS.

By Louis Cohen.

INTRODUCTION.

Professor Braun,[1] in seeking for some method whereby he could increase the amount of energy available for radiation in wireless telegraphy, without employing excessively high voltages, conceived the idea of using coupled circuits. The advantages of this method over the old method—direct excited antennæ—are quite obvious, and are familiar to all students of this subject. By the use of an auxiliary circuit of low resistance and large capacity, we can store up a large amount of energy, and at the same time produce a more prolonged oscillation, which is very important for syntonization.

The coupling may be accomplished in two different ways, electromagnetically or direct, as indicated in Figs. 1 and 2. In either case the advantages to be gained are the same.

Owing to its great practical importance, several able investigators have worked on this problem, and we find in the literature of the subject during the past few years a number of contributions dealing with this problem from its practical as well as theoretical aspect.

This problem finds its analogy in accoustics. If we mount several tuning forks on a resonating box and excite one, the other tuning forks will also vibrate, but each one will vibrate with several distinct frequencies which are different from the natural period of the tuning forks. The general outline of the problem and the method of its solution were given by Lord Rayleigh.[2]

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[1] F. Braun, Physikalische Zs. **8**, p. 148; 1901.

[2] Lord Rayleigh: Theory of Sound, Vol. **1**, p. 84.

He points out that the whole motion may be resolved into m normal harmonic vibrations of different periods, each of which is entirely independent of the others.

If we denote the displacement of the respective tuning forks by

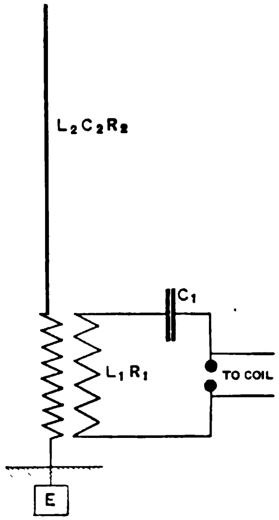

Fig. 1.

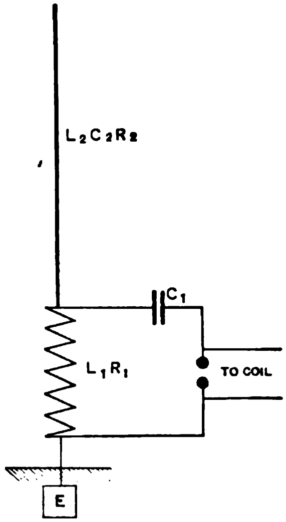

Fig. 2.

$\psi_1, \psi_2, \ldots\ldots \psi_m$, and neglect the frictional resistance, then we may write:

$$\left.\begin{aligned}
\psi_1 &= A_1 \cos(n_1 t - \alpha) + A_2 \cos(n_2 t - \beta) + A_3 \cos(n_3 t - \gamma) + \ldots\ldots \\
\psi_2 &= B_1 \cos(n_1 t - \alpha) + B_2 \cos(n_2 t - \beta) + B_3 \cos(n_3 t - \gamma) + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\psi_m &= C_1 \cos(n_1 t - \alpha) + C_2 \cos(n_2 t - \beta) + C_3 \cos(n_3 t - \gamma) + \ldots
\end{aligned}\right\}$$

where n_1^2, n_2^2, etc., are the m roots of an equation of m^{th} degree in n^2.

If we neglect the resistance, the electrical problem under consideration becomes a very simple matter, for being a system of only two degrees of freedom we shall have only two frequencies, and this will require obtaining the roots of a second-degree equation. If, however, we take the resistance into consideration, which is a very important factor in this case, then the problem becomes much more difficult, for under these conditions we shall have to obtain the roots of a biquadratic instead of a quadratic, as can be seen in the following way. Writing down the differential equations for the potentials in the two circuits, we have:

$$L_1C_1\frac{d^2V_1}{dt^2}+R_1C_1\frac{dV_1}{dt}+V_1+MC_2\frac{d^2V_2}{dt^2}=0$$

$$L_2C_2\frac{d^2V_2}{dt^2}+R_2C_2\frac{dV_2}{dt}+V_2+MC_1\frac{d^2V_1}{dt^2}=0$$

The derivations of these two equations will be given later.

Now assume:

$$V_1=Ae^{\lambda t}$$

$$V_2=Be^{\lambda t}$$

Introducing the values of V_1 and V_2 in the above equations and eliminating $\frac{A}{B}$, we obtain the following equation:

$$C_1C_2(L_1L_2-M^2)\lambda^4+C_1C_2(R_1L_2+R_2L_1)\lambda^3+(L_1C_1+L_2C_2+R_1R_2C_1C_2)\lambda^2+(R_1C_1+R_2C_2)\lambda+1=0$$

and the solution of our problem reduces itself to the determination of the roots of the biquadratic.

Theoretically, of course, it is possible to obtain the roots of a fourth-degree equation; but practically the task is not an easy one, and this is more so in this problem when the coefficients happen to be somewhat complicated expressions.

The problem of electromagnetically coupled circuits is from the mathematical standpoint precisely the same as that of the Tesla transformer, and a solution of the problem was attempted some thirteen years ago by Oberbeck,[3] Galitzin,[4] and Domalys and Kolacek.[5] Later on, very able contributions to the subject were made by Wien[6] and Drude.[7] All these investigations have in general followed the method indicated by Lord Rayleigh, and to avoid the difficulty of obtaining the roots of a biquadratic, they were obliged to limit themselves to consider some special cases only, or at the best get some approximate results. It may also be noted that the method that has been adopted by these investigators offer considerable difficulties in the evaluation of the constants of integration. Furthermore, all discussions of the problem were limited to the case of electromagnetic coupling; and to my knowledge the problem of direct coupled circuits has not been considered at all, except the very special case of no resistance which has been developed by Seibt,[8] and of course if we neglect the resistance, we eliminate from our discussion the most important part, namely, the determination of the damping factors.

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[3] A. Oberbeck, Wied. Ann. **55**, p. 623; 1895.

[4] Fürst B. Galitzin, Petersb. Ber., May, June; 1895.

[5] R. Domalys and F. Kolacek, Wied. Ann. **57**, p. 731; 1896.

[6] M. Wien, Wied. Ann. **61**, p. 151; 1897.

[7] P. Drude, Ann. der Physik. **13**, p. 512; 1904.

In this paper I have developed a method which enables me to obtain the complete solution to the problem, and yet avoid the difficulty of getting the roots of a biquadratic equation. My method is applicable to the problem of direct coupled circuits as well as the electromagnetic coupled circuits, and the evaluation of the constants can be obtained with considerable ease. I have worked out both cases quite fully, obtaining the frequency constants as well as the damping factors. A discussion of the results and the comparative merits of the two systems will be given later, and I trust that the results obtained will be of interest to those engaged in the work of wireless telegraphy and telephony.

ELECTROMAGNETICALLY COUPLED CIRCUITS.

Let us consider two circuits, each having resistance, R, inductance, L, and capacity, C, and specify these quantities for the two circuits by suffixes 1 and 2. Let these two circuits have a mutual inductance, M, then denote by I_1 and I_2 the respective currents at any instant, v_1 and v_2 the respective potentials of the condensers of the two circuits, and we have the following fundamental equations:

$$\left.\begin{aligned} L_1\frac{dI_1}{dt}+R_1I_1+v_1+M\frac{dI_2}{dt}=0 \\ L_2\frac{dI_2}{dt}+R_2I_2+v_2+M\frac{dI_1}{dt}=0 \end{aligned}\right\} \qquad (1)$$

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[8] G. Seibt, Physik. Zs., Aug. 1, 1904.

We have also the relations:

$$I_1 = C_1 \frac{dv_1}{dt}, \qquad I_2 = C_2 \frac{dv_2}{dt}.$$

Introducing these values in (1) we obtain:

$$L_1C_1 \frac{d^2v_1}{dt^2} + R_1C_1 \frac{dv_1}{dt} + v_1 + MC_2 \frac{d^2v_2}{dt^2} = 0$$

$$L_2C_2 \frac{d^2v_2}{dt^2} + R_2C_2 \frac{dv_2}{dt} + v_2 + MC_1 \frac{d^2v_1}{dt^2} = 0$$

or using the following abbreviations:

$$\left.\begin{aligned} a_1 &= L_1C_1, \qquad & b_1 &= R_1C_1, \qquad & d_1 &= MC_2 \\ a_2 &= L_2C_2, \qquad & b_2 &= R_2C_2, \qquad & d_2 &= MC_1 \end{aligned}\right\} \tag{2}$$

We may write the above equations in the following form:

$$\left.\begin{aligned} a_1 \frac{d^2v_1}{dt^2} + b_1 \frac{dv_1}{dt} + v_1 + d_1 \frac{d^2v_2}{dt^2} &= 0 \\ a_2 \frac{d^2v_2}{dt^2} + b_2 \frac{dv_2}{dt} + v_2 + d_2 \frac{d^2v_1}{dt^2} &= 0 \end{aligned}\right\} \tag{3}$$

Let us assume:

$$\left.\begin{aligned} v_1 &= m_1w_1 + m_2w_2 \\ v_2 &= m_3w_1 + m_4w_2 \end{aligned}\right\} \tag{4}$$

where m_1, m_2, m_3, m_4 are constants, and w_1, w_2 are different functions of t. Introducing the values of v_1 and v_2 as given by (4) into (3) we obtain:

$$\left.\begin{aligned} (a_1m_1 + d_1m_3) \frac{d^2w_1}{dt^2} + (a_1m_2 + d_1m_4) \frac{d^2w_2}{dt^2} + b_1m_1 \frac{dw_1}{dt} + b_1m_2 \frac{dw_2}{dt} + {} \\ m_1w_1 + m_2w_2 = 0 \\ (a_2m_3 + d_2m_1) \frac{d^2w_1}{dt^2} + (a_2m_4 + d_2m_2) \frac{d^2w_2}{dt^2} + b_2m_3 \frac{dw_1}{dt} + b_2m_4 \frac{dw_2}{dt} + {} \\ m_3w_1 + m_4w_2 = 0 \end{aligned}\right\} \tag{5}$$

Eliminating from these two equations, first the term containing

$\frac{d^2w_1}{dt^2}$ and then the term containing $\frac{d^2w_2}{dt^2}$,

we shall obtain the following two equations:

$$\left.\begin{aligned}
&\Big\{(a_1m_2+d_1m_4)(a_2m_3+d_2m_1)-(a_2m_4+d_2m_2)(a_1m_1+d_1m_3)\Big\}\frac{d^2w_2}{dt^2}+\\
&\qquad\Big\{b_1m_1(a_2m_3+d_2m_1)-b_2m_3(a_1m_1+d_1m_3)\Big\}\frac{dw_1}{dt}+\\
&\qquad\Big\{b_1m_2(a_2m_3+d_2m_1)-b_2m_4(a_1m_1+d_1m_3)\Big\}\frac{dw_2}{dt}+\\
&\qquad\{m_1(a_2m_3+d_2m_1)-m_3(a_1m_1+d_1m_3)\}w_1+\\
&\qquad\{m_2(a_2m_3+d_2m_1)-m_4(a_1m_1+d_1m_3)\}w_2=0
\end{aligned}\right\}\quad(6)$$

$$\left.\begin{aligned}
&\Big\{(a_2m_4+d_2m_2)(a_1m_1+d_1m_3)-(a_1m_2+d_1m_4)(a_2m_3+d_2m_1)\Big\}\frac{d^2w_1}{dt^2}+\\
&\qquad\Big\{b_1m_1(a_2m_4+d_2m_2)-b_2m_3(a_1m_2+d_1m_4)\Big\}\frac{dw_1}{dt}+\\
&\qquad\Big\{b_1m_2(a_2m_4+d_2m_2)-b_2m_4(a_1m_2+d_1m_4)\Big\}\frac{dw_2}{dt}+\\
&\qquad\{m_1(a_2m_4+d_2m_2)-m_3(a_1m_2+d_1m_4)\}w_1+\\
&\qquad\{m_2(a_2m_4+d_2m_2)-m_4(a_1m_2+d_1m_4)\}w_2=0
\end{aligned}\right\}\quad(7)$$

Let us now give to the constants m_1, m_2, m_3, m_4 such values as to make the coefficients of w_1 in equation (6) and the coefficient of w_2 in equation (7) separately equal to zero; that is, put:

$$a_2m_1m_3+d_2m_1^2-a_1m_1m_3-d_1m_3^2=0$$

$$a_2m_2m_4+d_2m_2^2-a_1m_2m_4-d_1m_4^2=0$$

From which we get:

$$\frac{m_1}{m_3}=\frac{m_2}{m_4}=\frac{a_1-a_2\pm\sqrt{(a_1-a_2)^2+4d_1d_2}}{2d_2}.$$

Using different signs before the radical for the different ratios, we get:

$$\left.\begin{aligned}\frac{m_1}{m_3} &= k_1 = \frac{a_1 - a_2 + \sqrt{(a_1 - a_2)^2 + 4d_1d_2}}{2d_2}\\[2ex] \frac{m_2}{m_4} &= k_2 = \frac{a_1 - a_2 - \sqrt{(a_1 - a_2)^2 + 4d_1d_2}}{2d_2}\end{aligned}\right\} \quad (8)$$

Expanding now the various coefficients in equations (6) and (7) and dividing through by m_3m_4, we get:

$$\left.\begin{aligned}&(a_1a_2 - d_1d_2)(k_2 - k_1)\frac{d^2w_2}{dt^2} + (a_2b_1k_2 + b_1d_2k_1k_2 - a_1b_2k_1 - b_2d_1)\frac{dw_2}{dt} +\\ &\qquad (a_2k_2 + d_2k_1k_2 - a_1k_1 - d_1)\,w_2 +\\ &\qquad \frac{m_1}{m_4}\left(a_2b_1 + d_2b_1k_1 - a_1b_2 - \frac{b_2d_1}{k_1}\right)\frac{dw_1}{dt} = 0\\ &(a_1a_2 - d_1d_2)(k_1 - k_2)\frac{d^2w_1}{dt^2} + (a_2b_1k_1 + b_1d_2k_1k_2 - a_1b_2k_2 - b_2d_1)\frac{dw_1}{dt} +\\ &\qquad (a_2k_1 + d_2k_1k_2 - a_1k_2 - d_1)w_1 +\\ &\qquad \frac{m_2}{m_3}\left(a_2b_1 + b_1d_2k_2 - a_1b_2 - \frac{d_1b_2}{k_2}\right)\frac{dw_2}{dt} = 0.\end{aligned}\right\} \quad (9)$$

Suppose we put now for brevity $\frac{m_4}{m_1} = k_3$ and

$$\frac{m_2}{m_3} = \frac{m_4}{m_1}\,\frac{m_1}{m_4}\,\frac{m_2}{m_3} = k_3k_1k_2.$$

Then we may write equations (9) in the following form:

$$\left.\begin{aligned}P_1\frac{d^2w_2}{dt^2} + P_2\frac{dw_2}{dt} + P_3w_2 + \frac{1}{k_3}P_4\frac{dw_1}{dt} &= 0\\[2ex] T_1\frac{d^2w_1}{dt^2} + T_2\frac{dw_1}{dt} + T_3w_1 + k_3T_4\frac{dw_2}{dt} &= 0\end{aligned}\right\} \quad (10)$$

where:

$$\left.\begin{array}{l|l} P_1 = (a_1a_2 - d_1d_2)\ (k_2 - k_1) & T_1 = (a_1a_2 - d_1d_2)(k_1 - k_2) = -P_1\\ P_2 = a_2b_1k_2 + b_1d_2k_1k_2 - a_1b_2k_1 & T_2 = a_2b_1k_1 + b_1d_2k_1k_2 - a_1b_2k_2\\ \qquad - b_2d_1 & \qquad - b_2d_1\\ P_3 = a_2k_2 + d_2k_1k_2 - a_1k_1 - d_1 & T_3 = a_2k_1 + d_2k_1k_2 - a_1k_2 - d_1\\ P_4 = a_2b_1 + d_2b_1k_1 - a_1b_2 - \dfrac{b_2d_1}{k_1} & T_4 = k_1k_2\left(a_2b_1 + b_1d_2k_2 - a_1b_2 - \dfrac{b_2d_1}{k_2}\right)\end{array}\right\} \quad (11)$$

Let us assume now that,

$$\left.\begin{aligned} w_1 &= Ae^{-at}w_1' \\ w_2 &= Be^{-at}w_2' \end{aligned}\right\} \tag{12}$$

where w_1' and w_2' are functions of t, and a is an arbitrary constant, we have,

$$\frac{dw_1}{dt} = Ae^{-at}\frac{dw_1'}{dt} - aAe^{-at}w_1'$$

$$\frac{d^2w_1}{dt^2} = Ae^{-at}\frac{d^2w_1'}{dt^2} - 2Aae^{-at}\frac{dw_1'}{dt} + Aa^2e^{-at}w_1'$$

and similar expressions for $\frac{dw_2}{dt}$ and $\frac{d^2w_2}{dt^2}$.

Introducing these values in equations (10) we get:

$$\left.\begin{aligned} &\left\{P_1\frac{d^2w_2'}{dt^2} - 2P_1a\frac{dw_2'}{dt} + P_1a^2w_2' + P_2\frac{dw_2'}{dt} - P_2aw_2' + \right. \\ &\qquad \left. P_3w_2'\right\}Be^{-at} + \frac{1}{k_3}\left(P_4\frac{dw_1'}{dt} - P_4aw_1'\right)Ae^{-at} = 0 \\ &\left\{T_1\frac{d^2w_1'}{dt^2} - 2T_1a\frac{dw_1'}{dt} + T_1a^2w_1' + T_2\frac{dw_1'}{dt} - T_2aw_1' + \right. \\ &\qquad \left. T_3w_1'\right\}Ae^{-at} + k_3\left(T_4\frac{dw_2'}{dt} - T_4aw_2'\right)Be^{-at} = 0 \end{aligned}\right\} \tag{13}$$

Referring back to equations (10) we find that each one of the equations contains w_1 and w_2, and, in order that these equations may hold true for all values of time, it is evident that w_1 and w_2 must be similar functions of time, and therefore the ratio of $\frac{w_1}{w_2}$ or $\frac{dw_1}{dw_2}$ will be a constant. We can therefore choose such a value for a which will make the following relations obtain:

$$\left.\begin{aligned} B\left\{P_2 - 2P_1a\right\}\frac{dw_2'}{dt} + \frac{AP_4}{k_3}\frac{dw_1'}{dt} &= 0 \\ A\left\{T_2 - 2T_1a\right\}\frac{dw_1'}{dt} + BT_4k_3\frac{dw_2'}{dt} &= 0 \end{aligned}\right\} \tag{14}$$

Eliminating $\dfrac{A\dfrac{dw_1'}{dt}}{B\dfrac{dw_2'}{dt}}$, from the above equations we get:

$$\frac{(P_2-2P_1a)k_3}{P_4}=\frac{T_4k_3}{T_2-2T_1a}$$

or

$$T_2P_2-2(P_1T_2+T_1P_2)a+4T_1P_1a^2=T_4P_4$$

and

$$a=\frac{2(P_1T_2+T_1P_2)\pm\sqrt{4(P_1T_2+T_1P_2)^2-16T_2P_2T_1P_1+16T_1P_1T_4P_4}}{8P_1T_1} \tag{15}$$

Since, however, $T_1=-P_1$, equation (15) will reduce to the following:

$$a=\frac{(T_2-P_2)\pm\sqrt{(T_2+P_2)^2-4P_4T_4}}{4T_1} \tag{16}$$

Since a has two distinct values, equations (12) will be written in the following form:

$$\left.\begin{aligned} w_1&=\{A_1e^{-a_1t}+A_2e^{-a_2t}\}w_1' \\ w_2&=\{B_1e^{-a_1t}+B_2e^{-a_2t}\}w_2' \end{aligned}\right\} \tag{17}$$

From equations (13) and (14) we obtain the following two equations:

$$\left.\begin{aligned} &\left\{P_1\frac{d^2w_2'}{dt^2}+P_1a^2w_2'-P_2aw_2'+P_3w_2'\right\}B-\frac{aP_4}{k_3}Aw_1'=0 \\ &\left\{T_1\frac{d^2w_1'}{dt^2}+T_1a^2w_1'-T_2aw_1'+T_3w_1'\right\}A-k_3T_4Baw_2'=0 \end{aligned}\right\} \tag{18}$$

Put for brevity

$$\left.\begin{aligned} P_1a^2-P_2a+P_3&=N_1 \\ T_1a^2-T_2a+T_3&=N_2 \end{aligned}\right\} \tag{19}$$

Then equations (18) will become:

$$\left.\begin{aligned} &\left\{P_1\frac{d^2w_2'}{dt^2}+N_1w_2'\right\}B-A\frac{aP_4}{k_3}w_1'=0 \\ &\left\{T_1\frac{d^2w_1'}{dt^2}+N_2w_1'\right\}A-BaT_4k_3w_2'=0 \end{aligned}\right\} \tag{20}$$

Let us assume the following solutions,

$$\left.\begin{aligned} w_1' &= De^{i\lambda t} \\ w_2' &= Fe^{i\lambda t} \end{aligned}\right\} \tag{21}$$

Introducing these values in (20) we obtain:

$$\left.\begin{aligned} \left\{-P_1\lambda^2 + N_1\right\} BF - \frac{aP_4}{k_3} AD &= 0 \\ \left\{-T_1\lambda^2 + N_2\right\} AD - aT_4k_3BF &= 0 \end{aligned}\right\} \tag{22}$$

Eliminating from these two equations $\frac{AD}{BF}$ we get

$$\frac{(-P_1\lambda^2 + N_1)k_3}{aP_4} = \frac{aT_4k_3}{-T_1\lambda^2 + N_2}$$

or

$$T_1P_1\lambda^4 - (N_1T_1 + N_2P_1)\lambda^2 + N_1N_2 - a^2P_4T_4 = 0$$

and

$$\lambda^2 = \frac{N_1T_1 + N_2P_1 \pm \sqrt{(N_1T_1 + N_2P_1)^2 - 4N_1N_2P_1T_1 + 4T_1P_1P_4T_4a^2}}{2T_1P_1}$$

Remembering that $T_1 = -P_1$, the above equation will reduce to,

$$\lambda^2 = \frac{N_2 - N_1 \pm \sqrt{(N_2 + N_1)^2 - 4P_4T_4a^2}}{2T_1} \tag{23}$$

Since λ has two distinct values, we shall have

$$w_1' = D_1e^{i\lambda_1 t} + D_2e^{i\lambda_2 t}$$

$$w_2 = F_1e^{i\lambda_1 t} + F_2e^{i\lambda_2 t}$$

and the complete solutions of w_1 and w_2 will therefore be

$$\left.\begin{aligned} w_1 &= \left\{A_1e^{-a_1 t} + A_2e^{-a_2 t}\right\}\left\{D_1e^{i\lambda_1 t} + D_2e^{i\lambda_2 t}\right\} \\ w_2 &= \left\{B_1e^{-a_1 t} + B_2e^{-a_2 t}\right\}\left\{F_1e^{i\lambda_1 t} + F_2e^{i\lambda_2 t}\right\} \end{aligned}\right\} \tag{24}$$

Put now

$$\begin{array}{llll} A_1D_1=K_1, & A_1D_2=K_2, & A_2D_1=K_3, & A_2D_2=K_4 \\ B_1F_1=K_5, & B_1F_2=K_6, & B_2F_1=K_7, & B_2F_2=K_8 \end{array}$$

Equations (24) will reduce to

$$\left.\begin{array}{l} w_1=K_1e^{-a_1t}e^{i\lambda_1t}+K_2e^{-a_1t}e^{i\lambda_2t}+K_3e^{-a_2t}e^{i\lambda_1t}+K_4e^{-a_2t}e^{i\lambda_2t} \\ w_2=K_5e^{-a_1t}e^{i\lambda_1t}+K_6e^{-a_1t}e^{i\lambda_2t}+K_7e^{-a_2t}e^{i\lambda_1t}+K_8e^{-a_2t}e^{i\lambda_2t} \end{array}\right\} \quad (25)$$

and by equation (4) we have,

$$\left.\begin{array}{l} v_1=m_1w_1+m_2w_2 \\ \quad=(m_1K_1+m_2K_5)\ e^{-a_1t}e^{i\lambda_1t}+(m_1K_2+m_2K_6)e^{-a_1t}e^{i\lambda_2t}+ \\ \qquad(m_1K_3+m_2K_7)\ e^{-a_2t}e^{i\lambda_1t}+(m_1K_4+m_2K_8)\ e^{-a_2t}e^{i\lambda_2t} \\ \\ V_2=m_3w_1+m_4w_2 \\ \quad=(m_3K_1+m_4K_5)\ e^{-a_1t}e^{i\lambda_1t}+(m_3K_2+m_4K_6)\ e^{-a_1t}e^{i\lambda_2t}+ \\ \qquad(m_2K_3+m_4K_7)e^{-a_2t}e^{i\lambda_1t}+(m_3K_4+m_4K_8)e^{-a_2t}e^{i\lambda_2t} \end{array}\right\} \quad (26)$$

Equations (26) give the complete expressions for the potentials in the primary and secondary, and the expressions for the currents can be obtained, of course, from equations (26) by the aid of equations (2).

In the evaluations of the constants a_1, a_2, λ_1, and λ_2 it will be shown that they are all real quantities, and hence it is evident that a_1 and a_2 are the damping factors, and λ_1 and λ_2 are the frequency constants.

The values of a and λ are given by equations (16) and (23), respectively, their evaluation in terms of the constants of the circuits for the most general case is somewhat laborious, and I shall therefore consider only the most important case which is usually adopted in practice, that of resonance. In that case we have $a_2=a_1$, and therefore by equation (8), we get

$$k_1=-k_2=\sqrt{\frac{d_1}{d_2}} \quad (27)$$

Using these values in equations (16), we have

$$T_2 - P_2 = 2(a_2b_1 + a_1b_2)\sqrt{\frac{d_1}{d_2}}$$

$$T_2 + P_2 = -2d_1(b_1 + b_2)$$

$$P_4T_4 = -\frac{d_1}{d_2}\left\{(a_2b_1 - a_1b_2)^2 - d_1d_2(b_1 - b_2)^2\right\}$$

and therefore

$$a = \frac{2(a_2b_1+a_1b_2)\sqrt{\frac{d_1}{d_2}} \pm \sqrt{4d_1^2(b_1+b_2)^2 + 4\frac{d_1}{d_2}\left\{(a_2b_1-a_1b_2)^2 - d_1d_2(b_1-b_2)^2\right\}}}{8(a_1a_2 - d_1d_2)\sqrt{\frac{d_1}{d_2}}}$$

$$= \frac{a_2b_1 + a_1b_2 \pm \sqrt{(a_2b_1 - a_1b_2)^2 + 4d_1d_2b_1b_2}}{4(a_1a_2 - d_1d_2)} \qquad (28)$$

Substituting the values of the various constants and simplifying we get:

$$a = \frac{L_2R_1 + L_1R_2 \pm \sqrt{(L_2R_1 - L_1R_2)^2 + 4M^2R_1R_2}}{4(L_1L_2 - M^2)} \qquad (29)$$

If we designate the natural damping factors of the primary and secondary by γ_1 and γ_2, respectively; that is, put $\frac{R_1}{2L_1} = \gamma_1$ and $\frac{R_2}{2L_2} = \gamma_2$, and also denote the coefficient of coupling $\frac{M^2}{L_1L_2}$ by μ^2, then equations (29) may be written in the following form:

$$\left.\begin{aligned} a_1 &= \frac{\gamma_1 + \gamma_2 + \sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\mu^2}}{2(1 - \mu^2)} \\ a_2 &= \frac{\gamma_1 + \gamma_2 - \sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\mu^2}}{2(1 - \mu^2)} \end{aligned}\right\} \qquad (30)$$

When the coefficient of coupling is zero, then the above two equations will reduce to $a_1 = \gamma_1$ and $a_2 = \gamma_2$
That is, the damping factors will be the natural damping factors of the primary and secondary.

To obtain the expressions for λ_1 and λ_2 in terms of the constants of the circuits, we note that

$$N_2-N_1=-2P_1a^2+(P_2-T_2)a+T_3-P_3$$
$$=4(a_1a_2-d_1d_2)\,k_1a^2+2(a_2b_1+a_1b_2)\,k_1a+2(a_1+a_2)\,k_1$$
$$N_2+N_1=-(P_2+T_2)a+P_3+T_3=2d_1(b_1+b_2)a+4d_1$$
$$P_4T_4=-\frac{d_1}{d_2}\Big\{(a_2b_1-a_1b_2)^2+d_1d_2(b_1-b_2)^2\Big\}$$

Introducing these values in equation (23) we obtain:

$$\begin{aligned}\lambda^2&=\frac{4(a_1a_2-d_1d_2)k_1a^2+2(a_2b_1+a_1b_2)k_1a+2(a_1+a_2)k_1}{4\,(a_1a_2-d_1d_2)k_1}\\&\pm\frac{\sqrt{\{2d_1(b_1+b_2)a-4d_1\}^2+\frac{4d_1}{d_2}\{(a_2b_1-a_1b_2)^2+d_1d_2(b_1-b_2)^2\}a^2}}{4(a_1a_2-d_1d_2)k_1}\\&=\frac{2(a_1a_2-d_1d_2)a^2+(a_2b_1+a_1b_2)a+(a_1+a_2)}{2\,(a_1a_2-d_1d_2)}\\&\pm\frac{\sqrt{d_1d_2\{(b_1+b_2)a-2\}^2+\{(a_2b_1-a_1b_2)^2+d_1d_2(b_1-b_2)^2\}a^2}}{2(a_1a_2-d_1d_2)}\end{aligned}\qquad(31)$$

It is seen from equation (31) that all the terms in the expression of λ^2 are positive, and therefore when we use the positive sign before the radical; that is, the value of λ_1 will certainly be positive, but even when we use the negative sign before the radical—that is, the value of λ_2 will also be positive, because the principal term in the part of the expression outside the radical is a_1+a_2, the magnitude of which is of the order LC, while the principal term under the radical is $4d_1d_2$ and its magnitude is of the order MC, hence the expression of λ^2 as given by (31) will be always positive, and therefore λ is a real quantity. It is also seen from equation (28) that a is a real quantity, hence the a's as given by equation (28) are the damping factors, and the λ's as given by equation (31) are the frequency constants.

It may also be noted that in all cases which may arise in practice, the terms in equation (31) which contain a as a factor are usually very small as compared with the other terms. If we neglect the resistance entirely, then equation (31) will reduce to the following:

$$\lambda^2 = \frac{a_1 + a_2 \pm \sqrt{4d_1d_2}}{2(a_1a_2 - d_1d_2)} \tag{32}$$

which agrees, of course, with the results of Oberbeck[9] who developed the theory on the assumption of no resistance.

Having determined the damping factors, and the frequency constants, it remains yet to evaluate the constants of integration to obtain the complete solution of the problem.

The initial conditions are:

When $t=0, \qquad v_1=E, \qquad v_2=0, \qquad I_1=0, \qquad I_2=0$

The expressions for v_1 and v_2 are given by equation (26).

$$\left.\begin{aligned} v_1 = {} & (m_1K_1+m_2K_5)e^{-a_1t}e^{i\lambda_1t} + (m_1K_2+m_2K_6)e^{-a_1t}e^{i\lambda_2t} + \\ & (m_1K_3+m_2K_7)e^{-a_2t}e^{i\lambda_1t} + (m_1K_4+m_2K_8)e^{-a_2t}e^{i\lambda_2t} \\ v_2 = {} & (m_3K_1+m_4K_5)e^{-a_1t}e^{i\lambda_1t} + (m_3K_2+m_4K_6)e^{-a_1t}e^{i\lambda_2t} + \\ & (m_3K_3+m_4K_7)e^{-a_2t}e^{i\lambda_1t} + (m_3K_4+m_4K_8)e^{-a_2t}e^{i\lambda_2t} \end{aligned}\right\} \quad (26 \text{ bis})$$

In order that the values of w_1 and w_2 as given by equations (25) should satisfy equations (22) there must exist a certain relation between the various K's which enter in equations (25). If we introduce the values of w_1 and w_2 as given by equations (25) into (22) we find that the relation between the constants are as follows:

$$\left.\begin{aligned} \frac{K_5}{K_1} &= \frac{-T_1\lambda_1^2+N_2}{a_1T_4k_3} \text{ or } K_5 = \frac{-T_1\lambda_1^2+N_2}{a_1T_4k_3}K_1 = \frac{Q_1}{k_3}K_1 \\ \frac{K_6}{K_2} &= \frac{-T_1\lambda_2^2+N_2}{a_1T_4k_3} \text{ or } K_6 = \frac{-T_1\lambda_2^2+N_2}{a_1T_4k_3}K_2 = \frac{Q_2}{k_3}K_2 \\ \frac{K_7}{K_3} &= \frac{-T_1\lambda_1^2+N_2}{a_2T_4k_3} \text{ or } K_7 = \frac{-T_1\lambda_1^2+N_2}{a_2T_4k_3}K_3 = \frac{Q_3}{k_3}K_3 \\ \frac{K_8}{K_4} &= \frac{-T_1\lambda_2^2+N_2}{a_2T_4k_3} \text{ or } K_8 = \frac{-T_1\lambda_2^2+N_2}{a_2T_4k_3}K_4 = \frac{Q_4}{k_3}K_4 \end{aligned}\right\} \quad (33)$$

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[9] A. Oberbeck, Wied. Ann. 55, p. 623; 1895.

Introducing these values in equation (26) and remembering that $k_3 = \frac{m_4}{m_1}$, we get:

$$\left.\begin{aligned} v_1 = m_1K_1(1+k_2Q_1)e^{-a_1t}e^{i\lambda_1t} + m_1K_2(1+k_2Q_2)e^{-a_1t}e^{i\lambda_2t} + \\ m_1K_3(1+k_2Q_3)e^{-a_2t}e^{i\lambda_1t} + m_1K_4(1+k_2Q_4)e^{-a_2t}e^{i\lambda_2t} \\ v_2 = m_3K_1(1+k_1Q_1)e^{-a_1t}e^{i\lambda_1t} + m_3K_2(1+k_1Q_2)e^{-a_1t}e^{i\lambda_2t} + \\ m_3K_3(1+k_1Q_3)e^{-a_2t}e^{i\lambda_1t} + m_3K_4(1+k_1Q_4)e^{-a_2t}e^{i\lambda_2t} \end{aligned}\right\} \quad (34)$$

The expressions for the currents in the primary and secondary are as follows:

$$\left.\begin{aligned} I_1 = c_1m_1\{K_1(1+k_2Q_1)(i\lambda_1 - a_1)e^{-a_1t}e^{i\lambda_1t} + \\ K_2(1+k_2Q_2)(i\lambda_2 - a_1)e_1{}^{-a_1t}e^{i\lambda_2t} + \\ K_3(1+k_2Q_3)(i\lambda_1 - a_2)e^{-a_2t}e^{i\lambda_1t} + \\ K_4(1+k_2Q_4)(i\lambda_2 - a_2)e^{-a_2t}e^{i\lambda_2t}\} \\ I_2 = c_2m_3\{K_1(1+k_1Q_1)(i\lambda_1 - a_1)e^{-a_1t}e^{i\lambda_1t} + \\ K_2(1+k_1Q_2)(i\lambda_2 - a_1)e^{-a_1t}e^{i\lambda_1t} + \\ K_3(1+k_1Q_3)(i\lambda_1 - a_2)e^{-a_2t}e^{i\lambda_1t} + \\ K_4(1+k_1Q_4)(i\lambda_2 - a_2)e^{-a_2t}e^{i\lambda_2t}\} \end{aligned}\right\} \quad (35)$$

When $t = 0$, we have:

$$\left.\begin{aligned} m_1\{K_1(1+k_2Q_1) + K_2(1+k_2Q_2) + K_3(1+k_2Q_3) + \\ K_4(1+k_2Q_4)\} = E \\ m_3\{K_1(1+k_1Q_1) + K_2(1+k_1Q_2) + K_3(1+k_1Q_3) + \\ K_4(1+k_1Q_4\} = 0 \\ c_1m_1\{K_1(1+k_2Q_1)(i\lambda_1 - a_1) + K_2(1+k_2Q_2)(i\lambda_2 - a_1) + \\ K_3(1+k_2Q_3)(i\lambda_1 - a_2) + K_4(1+k_2Q_4)(i\lambda_2 - a_2)\} = 0 \\ c_2m_3\{K_1(1+k_1Q_1)(i\lambda_1 - a_1) + K_2(1+k_1Q_2)(i\lambda_2 - a_1) + \\ K_3(1+k_1Q_3)(i\lambda_1 - a_2) + K_4(1+k_1Q_4)(i\lambda_2 - a_2)\} = 0 \end{aligned}\right\} \quad (36)$$

We have here four equations from which we can by the use of determinates obtain the values of the four constants K_1, K_2, K_3, and K_4. The results, however, as thus obtained will be extremely complicated, and it will therefore be advisable to get approximate values of the constants if we will hereby simplify matters materially. As a matter of fact, the approximations that we will adopt here will be very close to the absolute values, and, furthermore, in our final results it is the ratio of the secondary potential to the primary potential that we seek to determine, and since these constants enter in the expression of v_1 and v_2, hence the approximation will affect the numerator and denominator alike, and the ratio will be a still closer approximation to the absolute value.

In examining equations (35) we find that each term of the right-hand sides contain a factor $(i\lambda - a)$. Now, in practice, a is generally very small compared with λ, but it must be further noted that if we take the real part (35), the term $i\lambda - a$ will become $\sqrt{\lambda^2 + a^2}$, and a^2 is generally negligible compared with λ^2. Hence we may neglect a compared with $i\lambda$, and under these conditions equations (36) will reduce to the following:

$$\left.\begin{aligned}
&\{K_1(1+k_2Q_1)+K_3(1+k_2Q_3)\}+\\
&\qquad\{K_2(1+k_2Q_2)+K_4(1+k_2Q_4)\}=\frac{E}{m_1}\\
&i\lambda_1\{K_1(1+k_2Q_1)+K_3(1+k_2Q_3)\}+\\
&\qquad i\lambda_2\{K_2(1+k_2Q_2)+K_4(1+k_2Q_4)\}=0\\
&\{K_1(1+k_1Q_1)+K_3(1+k_1Q_3)\}+\\
&\qquad\{K_2(1+k_1Q_2)+K_4(1+k_1Q_4)\}=0\\
&i\lambda_1\{K_1(1+k_1Q_1)+K_3(1+k_1Q_3)\}+\\
&\qquad i\lambda_2\{K_2(1+k_1Q_2)+K_4(1+k_1Q_4)\}=0
\end{aligned}\right\}\qquad(37)$$

In order that the last two equations of (37) be both satisfied, we must have each one of the bracket expressions separately equal to zero; that is, we have the following relations:

$$K_1(1+k_1Q_1)+K_3(1+k_1Q_3)=0$$
$$K_2(1+{}_1kQ_2)+K_4(1+k_1Q_4)=0$$

and therefore,

$$\frac{K_1}{K_3}=-\frac{1+k_1Q_3}{1+k_1Q_1}\quad \frac{K_2}{K_4}=-\frac{1+k_1Q_4}{1+k_1Q_2} \tag{38}$$

From the first two equations of (37) we get:

$$\left.\begin{aligned} K_2(1+k_2Q_2)+K_4(1+k_2Q_4)&=\frac{\frac{E}{m_1}\lambda_1}{\lambda_1-\lambda_2}\\ K_1(1+k_2Q_1)+K_3(1+k_2Q_3)&=\frac{\frac{E}{m_1}\lambda_2}{\lambda_2-\lambda_1}\end{aligned}\right\} \tag{39}$$

From (38) and (39) we can determine the values of the constants, which are as follows:

$$\left.\begin{aligned} K_1&=\frac{\frac{E}{m_1}\lambda_2(1+k_1Q_3)}{(\lambda_2-\lambda_1)\{(1+k_2Q_1)(1+k_1Q_3)-(1+k_2Q_3)(1+k_1Q_1)\}}\\ &=\frac{\frac{E}{m_1}\lambda_2(1+k_1Q_3)}{S_1}\\ K_2&=\frac{\frac{E}{m_1}\lambda_1(1+k_1Q_4)}{(\lambda_1-\lambda_2)\{(1+k_2Q_2)(1+k_1Q_4)-(1+k_2Q_4)(1+k_1Q_2)\}}\\ &=\frac{\frac{E}{m_1}\lambda_1(1+k_1Q_4)}{S_2}\\ K_3&=-\frac{\frac{E}{m_1}\lambda_2(1+k_1Q_1)}{S_1}\qquad K_4=-\frac{\frac{E}{m_1}\lambda_1(1+k_1Q_2)}{S_2}\end{aligned}\right\} \tag{41}$$

Introducing the values of the constants as given by equation (41) into equations (34) and (35), we shall have the complete expressions for the potentials and currents in the primary and secondary circuits, which are as follows:

$$\left.\begin{aligned} v_1 = {} & \frac{E\lambda_2}{S_1}(1+k_2Q_1)(1+k_1Q_3)e^{-a_1t}e^{i\lambda_1t}+ \\ & \frac{E\lambda_1}{S_2}(1+k_1Q_4)(1+k_1Q_2)e^{-a_1t}e^{i\lambda_2t}- \\ & \frac{E\lambda_2}{S_1}(1+k_1Q_1)(1+k_2Q_3)e^{-a_2t}e^{i\lambda_1t}- \\ & \frac{E\lambda_1}{S_2}(1+k_1Q_2)(1+k_2Q_4)e^{-a_2t}e^{i\lambda_2t} \end{aligned}\right\} \qquad (42)$$

$$\left.\begin{aligned} v_2 = {} & \frac{E\lambda_2}{k_1S_1}(1+k_1Q_3)(1+k_1Q_1)e^{-a_1t}e^{i\lambda_1t}+ \\ & \frac{E\lambda_1}{k_1S_2}(1+k_1Q_4)(1+k_1Q_2)e^{-a_1t}e^{i\lambda_2t}- \\ & \frac{E\lambda_2}{k_1S_1}(1+k_1Q_1)(1+k_1Q_3)e^{-a_2t}e^{i\lambda_1t}- \\ & \frac{E\lambda_1}{k_1S_2}(1+k_1Q_2)(1+k_1Q_4)e^{-a_2t}e^{i\lambda_2t} \end{aligned}\right\} \qquad (43)$$

$$\left.\begin{aligned} I_1 = {} & \frac{C_1E\lambda_2}{S_1}(1+k_2Q_1)(1+k_1Q_3)i\lambda_1e^{-a_1t}e^{i\lambda_1t}+ \\ & \frac{C_1E\lambda_1}{S_2}(1+k_2Q_2)(1+k_1Q_4)i\lambda_2e^{-a_1t}e^{i\lambda_2t}- \\ & \frac{C_1E\lambda_2}{S_1}(1+k_2Q_3)(1+k_1Q_1)i\lambda_1e^{-a_2t}e^{i\lambda_2t}- \\ & \frac{C_1E\lambda_1}{S_2}(1+k_2Q_4)(1+k_1Q_2)i\lambda_2e^{-a_2t}e^{i\lambda_2t} \end{aligned}\right\} \qquad (44)$$

$$\left.\begin{aligned} I_2 = {} & \frac{C_2E\lambda_2}{k_1S_1}(1+k_1Q_1)(1+k_1Q_3)i\lambda_1e^{-a_1t}e^{i\lambda_1t}+ \\ & \frac{C_2E\lambda_1}{k_1S_2}(1+k_1Q_2)(1+k_1Q_4)i\lambda_2e^{-a_1t}e^{i\lambda_2t}- \\ & \frac{C_2E\lambda_2}{k_1S_1}(1+k_1Q_3)(1+k_1Q_1)i\lambda_1e^{-a_2t}e^{i\lambda_1t}- \\ & \frac{C_2E\lambda_1}{k_1S_2}(1+k_1Q_4)(1+k_1Q_2)i\lambda_2e^{-a_2t}e^{i\lambda_2t} \end{aligned}\right\} \qquad (45)$$

Let us now use the following abbreviations:

$$\left.\begin{aligned}
\frac{E\lambda_2(1+k_2Q_1)(1+k_1Q_3)}{S_1}&=H_1, & \frac{E\lambda_1(1+k_1Q_2)(1+k_2Q_4)}{S_2}&=H_4\\
\frac{E\lambda_1(1+k_1Q_4)(1+k_2Q_2)}{S_2}&=H_2, & \frac{E\lambda_2(1+k_1Q_3)(1+k_1Q_1)}{k_1S_1}&=H_5\\
\frac{E\lambda_2(1+k_1Q_1)(1+k_1Q_3)}{S_1}&=H_3, & \frac{E\lambda_1(1+k_1Q_4)(1+k_1Q_2)}{k_1S_2}&=H_6
\end{aligned}\right\}\quad(46)$$

Then the real parts of equations (42)–(45) may be written in the following form:

$$v_1=H_1e^{-a_1t}\cos\lambda_1t+H_2e^{-a_1t}\cos\lambda_2t-H_3e^{-a_2t}\cos\lambda_1t-H_4e^{-a_2t}\cos\lambda_2t\quad(47)$$

$$v_2=H_5e^{-a_1t}\cos\lambda_1t+H_6e^{-a_1t}\cos\lambda_2t-H_5e^{-a_2t}\cos\lambda_1t-H_6e^{-a_2t}\cos\lambda_2t\quad(48)$$

$$\left.\begin{aligned}I_1=C_1H_1\lambda_1e^{-a_1t}\sin\lambda_1t+C_1H_2\lambda_2e^{-a_1t}\sin\lambda_2t-C_1H_3\lambda_1e^{-a_2t}\sin\lambda_1t-\\ C_1H_4\lambda_2e^{-a_2t}\sin\lambda_2t\end{aligned}\right\}\quad(49)$$

$$\left.\begin{aligned}I_2=C_2H_5\lambda_1e^{-a_1t}\sin\lambda_1t+C_2H_6\lambda_2e^{-a_1t}\sin\lambda_2t-C_2H_5\lambda_1e^{-a_2t}\sin\lambda_1t-\\ C_2H_6\lambda_2e^{-a_2t}\sin\lambda_2t\end{aligned}\right\}\quad(50)$$

or we may write the above equations in the following form:

$$\left.\begin{aligned}
v_1&=\left\{H_1e^{-a_1t}-H_3e^{-a_2t}\right\}\cos\lambda_1t+\left\{H_2e^{-a_1t}-H_4e^{-a_2t}\right\}\cos\lambda_2t\\
v_2&=H_5\left\{e^{-a_1t}-e^{-a_2t}\right\}\cos\lambda_1t+H_6\left\{e^{-a_1t}-e^{-a_2t}\right\}\cos\lambda_2t\\
I_1&=C_1\lambda_1\left\{H_1e^{-a_1t}-H_3e^{-a_2t}\right\}\sin\lambda_1t+C_1\lambda_2\left\{H_2e^{-a_1t}-H_4e^{-a_2t}\right\}\sin\lambda_2t\\
I_2&=C_2\lambda_1H_5\left\{e^{-a_1t}-e^{-a_2t}\right\}\sin\lambda_1t+C_2\lambda_2H_6\left\{e^{-a_1t}-e^{-a_2t}\right\}\sin\lambda_2t
\end{aligned}\right\}\quad(51)$$

Equations (52) give the complete solution to the problem of electromagnetic coupled systems. It is seen that the potential and current in the primary and secondary circuits consist of two distinct waves, each one of which has a different periodicity, and each wave has two different damping factors. It is also evident that each wave in the primary circuit will produce a corresponding wave in the secondary circuit.

We may thus write:

$$\left.\begin{aligned}v_1&=v_1'+v_1''\\ v_2&=v_2'+v_2''\end{aligned}\right\}\quad(52)$$

Where v_1' and v_2' correspond to the terms of frequency λ_1, and v_1'', v_2'' correspond to the terms of frequency λ_2.

If we should introduce the values of the electrical constants that are generally used in practice in equations (33), it will be found that the constants Q_1, Q_2, Q_3, and Q_4 are large quantities, their magnitude being of the order of one hundred or more. If we neglect unity compared with Q, we shall find by referring to equations (46) that $H_1 = H_3$ and $H_2 = H_4$.

The first and third equations of (51) will therefore reduce to,

$$\left.\begin{aligned} v_1 &= H_1(e^{-a_1t} - e^{-a_2t}) \cos \lambda_1 t + H_2(e^{-a_1t} - e^{-a_2t}) \cos \lambda_2 t \\ I_1 &= C_1\lambda_1 H_1(e^{-a_1t} - e^{-a_2t}) \cos \lambda_1 t + C_1\lambda_2 H_2(e^{-a_1t} - e^{-a_2t}) \cos \lambda_2 t \end{aligned}\right\} \quad (53)$$

If we assume that there is no resistance in the circuits; that is, $a_1 = a_2 = 0$, then the values of the various Q's as given by equations (33) will be infinite, and we may therefore neglect unity as compared with kQ in equations (46). Under these conditions we find that,

$$\frac{v_2'}{v_1'} = \frac{v_2''}{v_1''} = \frac{1}{k_1} = \sqrt{\frac{C_1}{C_2}} \quad (54)$$

which agrees with the result obtained in the development of the simple theory on the assumption of no resistance in the circuits.

It will perhaps be of some interest to work out one or two examples so as to get an idea of the relative magnitude of the various constants which enter in the final equations.

Let us assume the following values for the resistances, inductance, and capacity of the primary and secondary circuits:

$R_1 = 1$ ohm $= 10^9$ cm, $L_1 = 6200$ cm, $C_1 = 5300 \times 10^{-21}$

$R_2 = 50$ ohm $= 50 \times 10^9$ cm, $L_2 = 73000$ cm, $C_2 = 450 \times 10^{-21}$

and let us also consider two different cases of different degrees of coupling $\mu = 0.1$ and $\mu = 0.4$. Then we shall have the following values for our various constants (see Table I).

From the values given in the table, it is seen that the differences between H_1 and H_3, H_2 and H_4, are small, so that our assumption in equation (53) that $H_1 = H_3$ is justifiable. We also

TABLE I.

Calculated Values of Constants for Two Degrees of Coupling.

μ	$L_1C_1=L_2C_2$	λ_1	λ_2	a_1	a_2	Q_1	Q_2	Q_3	Q_4	S_1	S_2
0.1	33×10^{-15}	5.5×10^6	5.1×10^6	38×10^4	9×10^4	−91	161	− 375	662	-68×10^6	120×10^6
0.4	33×10^{-15}	7.1×10^6	4.65×10^6	20×10^4	5.4×10^4	−3461	1440	−11815	5369	-12280×10^6	5775×10^6

μ	H_1	H_2	H_3	H_4	H_5	H_6	$\frac{H_5}{H_1}$	$\frac{H_6}{H_2}$	$\sqrt{\frac{C_1}{C_2}}$
0.1	192	−433	218	−444	− 733	1503	3.8	3.5	3.42
0.4	1530	−662	1527	−664	−5090	2215	3.33	3.34	

see that, in the case of loose coupling ($\mu = 0.1$), the frequencies do not differ much from each other, and the effect will be approximately the same as if we had an oscillation of one frequency. In the case, however, of strong coupling ($\mu = 0.4$) the frequency constants differ materially from each other, so in this case we must consider the effect as that due to two distinct oscillations of different frequencies and different amplitudes.

DIRECT COUPLED CIRCUITS.

The method that was used in the discussion of the problem of electromagnetically coupled circuits is in general applicable to the discussion of the problem of direct coupled circuits. The treatment of this problem is so closely analogous to that of the preceding case that it may appear almost like a repetition, yet the results obtained, we trust, justify the additional labor involved in the development of this problem. As was said in the introduction, no adequate treatment of this problem has, so far as we know, ever been given; in fact, not even an approximate determination of the damping factors has ever been attempted, and it is hoped therefore that the results obtained will be of considerable interest.

In the discussion of this problem we shall use similar symbols, as in the previous case, but they may have the same or different meanings attached to them, and therefore in reading this part of the paper it must be remembered that it is a distinct and separate part, and all the symbols used will be defined in their proper place.

Let us denote by I_1 the current in the primary coil, I_2 the current in the antennæ, and I_3 the current in the condenser of the primary circuit; let V_1 denote the potential across the condensers, and V_2 the potential of the antenna. If L_1, R_1, C_1 and L_2, R_2, C_2 are the inductance, resistance, and capacity of the primary circuit and antenna, respectively, we shall have the following relations:

$$\left.\begin{aligned}
&L_1\frac{dI_1}{dt}+R_1I_1=V_1\\
&L_2\frac{dI_2}{dt}+R_2I_2+V_2=V_1\\
&I_3=C_1\frac{dV_1}{dt},\quad I_2=C_2\frac{dV_2}{dt},\quad I_1+I_2+I_3=0
\end{aligned}\right\} \tag{55}$$

From the last relation given in equation (55) we have

$$I_1=-(I_2+I_3)=-C_2\frac{dV_2}{dt}-C_3\frac{dV_1}{dt}$$

and hence we can easily obtain the following two equations:

$$\left.\begin{aligned}
&L_1C_2\frac{d^2V_2}{dt^2}+L_1C_1\frac{d^2V_1}{dt^2}+R_1C_2\frac{dV_2}{dt}+R_1C_1\frac{dV_1}{dt}+V_1=0\\
&L_2C_2\frac{d^2V_2}{dt^2}+R_2C_2\frac{dV_2}{dt}+V_2-V_1=0
\end{aligned}\right\} \tag{56}$$

or we may write the above two equations in this form:

$$\left.\begin{aligned}
&a_1\frac{d^2V_1}{dt^2}+b_1\frac{dV_1}{dt}+d_1\frac{d^2V_2}{dt^2}+h_1\frac{dV_2}{dt}+V_1=0\\
&a_2\frac{d^2V_2}{dt^2}+b_2\frac{dV_2}{dt}+V_2-V_1=0
\end{aligned}\right\} \tag{57}$$

where

$$\left.\begin{aligned}
&a_1=L_1C_1,\quad b_1=R_1C_1,\quad d_1=L_1C_2,\quad h_1=R_1C_2\\
&a_2=L_2C_2,\quad b_2=R_2C_2
\end{aligned}\right\} \tag{58}$$

As in the previous case we will assume,

$$\left.\begin{aligned}
V_1&=m_1w_1+m_2w_2\\
V_2&=m_3w_1+m_4w_2
\end{aligned}\right\} \tag{59}$$

where m_1, m_2, m_3, and m_4 are arbitrary constants, and w_1 and w_2 are functions of t.

Introducing these values in (57) we obtain

$$\left.\begin{aligned}
&(a_1m_1+d_1m_3)\frac{d^2w_1}{dt^2}+(a_1m_2+d_1m_4)\frac{d^2w_2}{dt^2}+(b_1m_1+h_1m_3)\frac{dw_1}{dt}+\\
&\qquad(b_1m_2+h_1m_4)\frac{dw_2}{dt}+m_1w_1+m_2w_2=0\\
&a_2m_3\frac{d^2w_1}{dt^2}+a_2m_4\frac{d^2w_2}{dt^2}+b_2m_3\frac{dw_1}{dt}+b_2m_4\frac{dw_2}{dt}+(m_3-m_1)\,w_1+\\
&\qquad(m_4-m_2)\,w_2=0
\end{aligned}\right\}\quad(60)$$

Eliminating first w_1 and then w_2 we get the following two equations:

$$\left.\begin{aligned}
&\Big\{(m_3-m_1)(a_1m_1+d_1m_3)-a_2m_1m_3\Big\}\frac{d^2w_1}{dt^2}+\\
&\Big\{(m_3-m_1)(a_1m_2+d_1m_4)-a_2m_1m_4\Big\}\frac{d^2w_2}{dt^2}\\
&+\Big\{(m_3-m_1)(b_1m_1+h_1m_3)-b_2m_1m_3\Big\}\frac{dw_1}{dt}+\\
&\Big\{(m_3-m_1)(b_1m_2+h_1m_4)-b_2m_1m_4\Big\}\frac{dw_2}{dt}+\\
&\Big\{m_2(m_3-m_1)-m_1(m_4-m_2)\Big\}w_2=0\\
&\Big\{(m_4-m_2)(a_1m_1+d_1m_3)-a_2m_2m_3\Big\}\frac{d^2w_1}{dt^2}+\\
&\Big\{(m_4-m_2)(a_1m_2+d_1m_4)-a_2m_2m_4\Big\}\frac{d^2w_2}{dt^2}\\
&+\Big\{(m_4-m_2)(b_1m_1+h_1m_3)-b_2m_2m_3\Big\}\frac{dw_1}{dt}+\\
&\Big\{(m_4-m_2)(b_1m_2+h_1m_4)-b_2m_2m_4\Big\}\frac{dw_2}{dt}+\\
&\Big\{m_1(m_4-m_2)-m_2(m_3-m_1)\Big\}w_1=0
\end{aligned}\right\}\quad(61)$$

Let us now choose the constants m_1, m_2, m_3, and m_4 so as to make in the first equation of (61) the coefficient of $\frac{d^2w_1}{dt^2}$ equal to zero, and in the second equation the coefficient of $\frac{d^2w_2}{dt^2}$ equal to zero.

That is, we shall put

$$a_1m_1m_3 + d_1m_3^2 - a_1m_1^2 - d_1m_1m_3 - a_2m_1m_3 = 0$$

$$a_1m_2m_4 + d_1m_4^2 - a_1m_2^2 - d_1m_2m_4 - a_2m_2m_4 = 0$$

From which we obtain

$$\frac{m_1}{m_3} = \frac{m_2}{m_4} = \frac{a_1 - a_2 - d_1 \pm \sqrt{(a_1 - a_2 - d_1)^2 + 4a_1d_1}}{2a_1}$$

Using different signs before the radical for the two different ratios we get,

$$\left.\begin{aligned} \frac{m_1}{m_3} = k_1 &= \frac{a_1 - a_2 - d_1 + \sqrt{(a_1 - a_2 - d_1)^2 + 4a_1d_1}}{2a_1} \\ \frac{m_2}{m_4} = k_2 &= \frac{a_1 - a_2 - d_1 - \sqrt{(a_1 - a_2 - d_1)^2 + 4a_1d_1}}{2a_1} \end{aligned}\right\} \tag{62}$$

To simplify matters we shall consider in this problem only the case when the two circuits are in resonance. This is in fact the most important case, which usually occurs in practice. Under these conditions we have

$$L_1C_1 = (L_1 + L_2)C_2 \tag{63}$$

or in our notation $a_1 = a_2 + d_1$

and therefore equations (62) will reduce to

$$k_1 = \sqrt{\frac{d_1}{a_1}}, \qquad k_2 = -\sqrt{\frac{d_1}{a_1}} \tag{64}$$

Expanding now the various coefficients in equations (61) and dividing through by m_3m_4 we get

$$\left.\begin{aligned} &\left\{a_1k_2 + d_1 - a_1k_1k_2 - d_1k_1 - a_2k_1\right\}\frac{d^2w_2}{dt^2} + \\ &\left\{b_1k_2 - b_1k_1k_2 + h_1 - h_1k_1 - b_2k_1\right\}\frac{dw_2}{dt} \\ + &\left\{k_2 - k_1\right\}w_2 + \frac{m_1}{m_4}\left\{b_1 - b_1k_1 + \frac{h_1}{k_1} - h_1 - b_2\right\}\frac{dw_1}{dt} = 0 \end{aligned}\right\} \tag{65}$$

$$\left.\begin{array}{l}\left\{a_1k_1-a_1k_1k_2+d_1-d_1k_2-a_2k_2\right\}\dfrac{d^2w_1}{dt^2}+\\ \left\{b_1k_1-b_1k_1k_2+h_1-h_1k_2-b_2k_2\right\}\dfrac{dw_1}{dt}\\ +\left\{k_1-k_2\right\}w_1+\dfrac{m_2}{m_3}\left\{b_1-b_1k_2+\dfrac{h_1}{k_2}-h_1-b_2\right\}\dfrac{dw_2}{dt}=0\end{array}\right\} \quad (65)$$

Put for brevity $\frac{m_1}{m_4}=k_3$ and $\frac{m_2}{m_3}=\frac{m_4}{m_1}\frac{m_1}{m_4}\frac{m_2}{m_3}=\frac{k_1k_2}{k_3}$ and then make the following abbreviations:

$$\left.\begin{array}{ll}P_1=(a_1+a_2)k_2+2d_1-d_1k_1 & T_1=(a_1+a_2)k_1+2d_1-d_1k_2\\ P_2=(b_1+b_2+h_1)k_2-b_1k_1k_2+h_1 & T_2=(b_1+b_2+h_1)k_1-b_1k_1k_2+h_1\\ P_3=2k_2 & T_3=2k_1\\ P_4=b_1-h_1-b_2-b_1k_1+\dfrac{h_1}{k_1} & T_4=k_1k_2\left\{b_1-h_1-b_2-b_1k_2+\dfrac{h_1}{k_2}\right\}\end{array}\right\}(66)$$

We may then write equations (65) in the following form:

$$\left.\begin{array}{l}P_1\dfrac{d^2w_2}{dt^2}+P_2\dfrac{dw_2}{dt}+P_3w_2+k_3P_4\dfrac{dw_1}{dt}=0\\ T_1\dfrac{d^2w_1}{dt^2}+T_2\dfrac{dw_1}{dt}+T_3w_1+\dfrac{T_4}{k_3}\dfrac{dw_2}{dt}\quad=0\end{array}\right\} \quad (67)$$

Comparing equations (67) with equations (10) we see that they are of identical forms except that the constants in the two cases have different values. It is evident therefore that the solutions must also be of the same form. It is not necessary to repeat the work. We can write down at once the values of w_1 and w_2, which are given by equations (25),

$$\left.\begin{array}{l}w_1=K_1e^{-a_1t}e^{i\lambda_1t}+K_2e^{-a_1t}e^{i\lambda_2t}+K_3e^{-a_2t}e^{i\lambda_1t}+K_4e^{-a_2t}e^{i\lambda_2t}\\ w_2=K_5e^{-a_1t}e^{i\lambda_1t}+K_6e^{-a_1t}e^{i\lambda_2t}+K_7e^{-a_2t}e^{i\lambda_1t}+K_8e^{-a_2t}e^{i\lambda_2t}\end{array}\right\} \quad (68)$$

Of course, the various constants which enter in equations (68) have entirely different values from the corresponding constants in equations (25), and the most important point is the evaluation of these constants in terms of the electrical constants of the circuits.

The expressions for a and λ are, of course, of the same form as given by equations (15) and (23); that is,

$$a=\frac{P_1T_2+T_1P_2\pm\sqrt{(P_1T_2-P_2T_1)^2+4P_1T_1P_4T_4}}{4P_1T_1} \tag{69}$$

$$\lambda^2=\frac{N_1T_1+N_2P_1\pm\sqrt{(N_1T_1-N_2P_1)^2+4P_1T_1P_4Ta^2}}{2T_1P_1} \tag{70}$$

$$N_1=P_1a^2-P_2a+P_3 \qquad N_2=T_1a^2-T_2a+T_3$$

The values of the various constants are given by equations (66), and introducing their values we shall have

$$P_1T_2+T_1P_2=-\frac{2d_1}{a_1}\left\{(a_1+a_2)(b_1+b_2+h_1)+d_1(b_2-b_1+h_1)-2h_1a_1\right\}$$

$$P_1T_2-T_1P_2=4d_1(b_1+b_2+h_1)k_1+2b_1(a_1+a_2)\frac{d_1}{a_1}k_2-2d_1b_1\frac{d_1}{a_1}k_1+$$
$$2h_1(a_1+a_2)-2h_1d_1k_1$$

$$P_4T_4=-\frac{d_1}{a_1}\left\{(b_1-b_2)^2-b_1^2k_1^2\right\}$$

$$P_1T_1=4d_1(d_1-a_1)$$

now $\quad h_1=R_1C_2, \qquad b_1^2k_1^2=R_1^2C_1C_2, \qquad \frac{b_1}{a_1}d_1=R_1C_2=h_1$

and generally in practice R_1 and C_2 are each comparatively very small quantities, so that we may neglect all the terms containing h_1 as a factor, and also the term $b_1^2\,k_1^2$ and the term containing $\frac{b_1}{a_1}d_1$. Under these conditions we get

$$\left.\begin{aligned}
&P_1T_2+P_2T_1=-\frac{2d_1}{a_1}\left\{(a_1+a_2)(b_2+b_1)-d_1(b_1-b_2)\right\}\\
&P_1T_2-T_1P_2=2d_1\left\{2b_2+b_1-b_1\frac{a_2}{a_1}\right\}k_1\\
&P_4T_4=-\frac{d_1}{a_1}(b_1-b_2)^2, \qquad P_1T_1=4d_1(d_1-a_1)
\end{aligned}\right\} \tag{71}$$

Introducing these values in (69) we obtain

$$a=\frac{-\frac{2d_1}{a_1}\left\{(a_1+a_2)(b_2+b_1)-d_1(b_1-b_2)\right\}\pm\sqrt{\frac{4d_1^3}{a_1}\left(2b_2+b_1-b_1\frac{a_2}{a_1}\right)^2-\frac{16d_1^2}{a_1}(d_1-a_1)(b_1-b_2)^2}}{16d_1(d_1-a_1)}\qquad(72)$$

If we expand the terms under the radical, and neglect terms containing b_1^2 as being very small compared with the terms containing b_2^2, we get,

$$a=\frac{(a_1+a_2)(b_1+b_2)-d_1(b_1-b_2)\pm\sqrt{4a_1^2b_2^2-4b_1b_2(a_2d_1+a_1d_1-2a_1^2)}}{8(a_1^2-a_1d_1)}\qquad(73)$$

Substituting the values of *a*, *b*, and *d* as given by equations (58) and simplifying, we get

$$a=\frac{2L_1R_2+2L_2R_1\pm\sqrt{4L_1^2R_2^2-4R_1R_2\left(\frac{L_1^2L_2}{L_1+L_2}-L_1^2-2L_1L_2\right)}}{8L_1L_2}\qquad(74)$$

If we should neglect R_1 entirely, which, as a matter of fact, will introduce only a small error, since in practice R_1 is usually only about 1 per cent of R_2, then equation (74) will reduce to

$$a_2=\frac{R_2}{2L_2}\text{ and }a_1=0\qquad(75)$$

It is seen therefore that in the direct coupled circuit there is practically only one damping factor which is approximately the natural damping of the antenna. It remains yet to evaluate the frequency constants in terms of the electrical constants of the circuits. Referring back to equation (70) we have

$$N_1T_1+N_2P_1=2P_1T_1a^2-(P_1T_2+P_2T_1)a+(T_1P_3+P_1T_3)$$

$$N_1T_1-N_2P_1=(P_1T_2-T_1P_2)a+(T_1P_3-P_1T_3)$$

Introducing the values of P and T from equations (66) and neglecting the terms containing R_1 as a factor as being very small compared with the other terms, we shall have

$$N_1T_1+N_2P_1=4d_1(d_1-a_1)\,a^2+4d_1b_2a-8d_1$$

$$N_1T_1-N_2P_1=4d_1b_2k_1a+8d_1k_2$$

$$P_1T_1P_4T_4=-\frac{4d_1^2}{a_1}b_2^2\,(d_1-a_1)$$

Introducing the above values in equation (70) we get

$$\left.\lambda^2=\frac{4d_1(d_1-a_1)\,a^2+4d_1b_2a-8d_1\pm\sqrt{(4d_1b_2k_1a+8d_1k_2)^2-\frac{16d_1^2}{a_1}(d_1-a_1)\,b_2^2a^2}}{8d_1(d_1-a_1)}\right\} \qquad (76)$$

Introducing the values of *a*, *b*, and *d* as given by equations (58) and simplifying we get

$$\left.\lambda^2=\frac{L_1C_1\frac{L_2}{L_1+L_2}a^2-R_2C_2a+2\pm\sqrt{\frac{L_1}{L_1+L_2}(R_2C_2a-2)^2+\frac{L_2}{L_1+L_2}R_2^2C_2^2a^2}}{2L_1C_1\frac{L_2}{L_1+L_2}}\right\} \qquad (77)$$

If we neglect the resistance of the antenna, the above equation will reduce to

$$\lambda^2=\frac{2\pm2\sqrt{\frac{L_1}{L_1+L_2}}}{2L_1C_1\frac{L_2}{L_1+L_2}}=\frac{1}{L_1C_1}\left\{\left(1+\frac{L_1}{L_2}\right)\left(1\pm\sqrt{\frac{L_2}{L_1+L_2}}\right)\right\} \qquad (78)$$

The value of λ^2 as given by equation (78) agrees with the results obtained by Seibt[1] who developed the same problem, but on the assumption that the resistance is negligible, which, of course, is a comparatively simple matter.

We have seen by equation (75) that in a direct coupled circuit practically only one damping factor exists, hence in this case the equations corresponding to (17) for the electromagnetically coupled circuits will be

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[1] Loc. cit.

$$\left.\begin{aligned} w_1 &= Ae^{-at}w_1' \\ w_2 &= Be^{-at}w_2' \end{aligned}\right\} \qquad (79)$$

and therefore equation (68) will reduce to

$$\left.\begin{aligned} w_1 &= e^{-at}\{K_1 e^{i\lambda_1 t} + K_2 e^{i\lambda_2 t}\} \\ w_2 &= e^{-at}\{K_5 e^{i\lambda_1 t} + K_6 e^{i\lambda_2}\} \end{aligned}\right\} \qquad (80)$$

The values of the potentials in the primary and antenna will be

$$\left.\begin{aligned} V_1 &= e^{-at}\{(m_1K_1 + m_2K_5)e^{i\lambda_1 t} + (m_1K_2 + m_2K_6)e^{i\lambda_2 t}\} \\ V_2 &= e^{-at}\{(m_3K_1 + m_4K_5)e^{i\lambda_1 t} + (m_2K_2 + m_4K_6)e^{i\lambda_2 t}\} \end{aligned}\right\} \qquad (81)$$

The expressions for the currents can be easily obtained from (81) by the aid of equation (55). The constants of integration can be evaluated by following the same method that was adopted for the case of electromagnetically coupled circuits.

CONCLUSION.

To summarize briefly the results obtained, we may say that for the case of electromagnetically coupled circuits, the equations (47)–(50) are the most complete expressions for the potential and current in the primary and secondary circuits. It will be seen that the potential and current in the primary and secondary circuits consist of two distinct oscillations, and each oscillation has two different damping factors. The expressions for the damping factors and frequency constants are given by equations (30), (31), and (32). If we shall put for brevity

$$\frac{1}{\sqrt{L_1C_1}} = p_1, \frac{1}{\sqrt{L_2C_2}} = p_2$$

then we find by referring to equation (32),

$$\lambda_1^2 - \lambda_2^2 = \frac{\mu p_1 p_2}{1 - \mu^2} \qquad (82)$$

and from equation (30) we get

$$a_1 - a_2 = \frac{\sqrt{(\gamma_2 - \gamma_1 + 4\gamma_1\gamma_2\mu^2}}{2(1 - \mu^2)} \tag{83}$$

If μ is of such a magnitude that we may neglect μ^2 compared with unity, then we have

$$\left.\begin{aligned} \lambda_1^2 - \lambda_2^2 &= \mu p_1 p_2 \\ a_1 - a_2 &= \frac{1}{2}\sqrt{(\gamma_1 - \gamma_2)^2 + 4\gamma_1\gamma_2\mu^2} \end{aligned}\right\} \tag{84}$$

which shows that the difference between the two frequency constants varies more rapidly with the degree of coupling than that of the damping factors.

The constants of integration have been completely determined, and though the expressions for the constants are somewhat complicated, and it is a little difficult to see in which way they depend on the frequency constants and damping factors, yet their numerical evaluation in practical problems is comparatively simple.

In the case of direct coupled circuits, we have shown that the results obtained are in a general way analogous to those for the electromagnetically coupled circuits, except that one of the damping factors is very small and may be neglected. So in this case we have in each circuit two oscillations of different frequencies, and both have the same damping factor which is approximately the natural damping factor of the antenna itself.

The writer wishes to extend his thanks to Prof. J. S. Ames for his kindness in reading the manuscript and making some valuable suggestions.

WASHINGTON, February 1, 1909.

A VOLT SCALE FOR A WATTS-PER-CANDLE METER.

By Herbert E. Ives.

Volt scales, indicating the voltage to give a desired candlepower, have been used on photometers in many incandescent-lamp factories. They are better adapted to the requirements of the lamp trade than are candlepower reading scales, because the demand is not for a large range of candlepowers at one voltage, but for few candlepowers with considerable latitude in voltages. Recently, the introduction of the graphitized and new metallic filament lamps has been marked by a new and more rational classification, namely, by watts per candle and total watts. To the writer's knowledge, however, there has not been developed a method of reading directly from the photometer the voltage to give the desired watts per candle and the watts corresponding to that voltage. Candlepower, current, and voltage readings are made, none of which is necessarily the final value adopted, and sorting is done by rough calculation, aided by limit tables and by experience. It would obviously be advantageous to obtain the voltage giving the correct watts per candle by direct reading, and, if possible, the corresponding watts or candlepower at the same time. The sorting of lamps would then be more quickly and probably more accurately done than by an indirect method. The efficiency-meter volt scale here described accomplishes this object.

A watts-per-candle or efficiency meter, designed by Hyde and Brooks, is part of the photometric equipment of the Bureau of Standards.[1] It consists essentially of a wattmeter with a variable pressure-circuit resistance, so arranged that as the photometer screen moves the wattmeter reads watts per candle. This instrument lends itself to the operation of finding the volts for a given efficiency by the method of trial and error. Where the number of lamps is small and time not too short, it affords a convenient and exact means of finding the voltage in question. In the extensive testing done by the Bureau for various Government departments, under the Bureau's specifications, carbon lamps are run on life test at a uniform efficiency of 3.1 watts per mean horizontal candle. This necessitates determining for each lamp the voltage which gives this efficiency. Finding the voltage by trial, with the efficiency meter, proved too long an operation to be practicable with a large number of lamps, and a shorter procedure was sought.

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[1] This Bulletin, 2, p. 145.

Experiment showed it possible to calculate with considerable accuracy, from the watts per candle at rated voltage (an initial run at rated voltage is part of the regular test), the voltage to give 3.1 watts per candle, by use of the relation $\left(\frac{V_2}{V_1}\right)^k = \frac{wpc_1}{wpc_2}$ where for the range from 3.6 to 3.1 wpc., for treated carbon filaments, k has the value 3.5. The results of such a calculation may be used in the form of percentage tables, or the actual volt values may be calculated for each rated voltage met with, and then tabulated. From such a table, for instance, it can be read that a 110-volt lamp operating at 3.5 watts per candle must be raised to 113.7 volts to operate at 3.1 wpc. Tables so calculated greatly reduce the work of finding the 3.1 voltage. It seemed desirable, however, to further reduce the number of operations, and this was accomplished by placing a scale in the efficiency meter which indicates at once the data of the tables.

This scale is a volt scale, similar in idea to the volt scales used in assigning lamps to the 16 or 32 candlepower voltage, but differing in that it indicates the voltage that will give a certain watts per candle. With a photometer operated at one voltage—say 110—it is only necessary to place in the efficiency meter a new scale which shall read 110 at the 3.1 point (assuming 3.1 to be the watts per candle desired), 111 at 3.21, 112 at 3.315, etc., in accordance with the voltage-watts-per-candle relation. The position of the photometer screen for intensity match then gives on the new scale the 3.1 voltage.

In the Bureau of Standards work a complication arose from the fact that lamps must be measured at rated voltage, and the number of rated voltages is so large that separate scales for each can not be conveniently placed within the instrument. Resort was had to an optical device to accomplish the same purpose. A series of volt scales for each voltage from 80 to 130 were drawn, one above another, on a narrow strip of paper. Their 3.1 points being in a straight line, the volt marks when joined formed smooth curves, and the resultant complete scale consisted of a series of converging volt lines crossed at right angles by curved lines numbered to indicate the corresponding rated voltages. This scale was fastened on a small cylinder mounted to permit of rotation about a horizontal axis above the efficiency meter scale. On top of the latter instrument was fastened a silvered glass mirror from whose lower surface the silver had been removed up to a curved line, whose curvature was approximately that of the instrument scale. This mirror, so placed that no parallax exists between the needle and the image of the outside scale, permits of reading the volts from any volt line which has been turned to a fixed pointer, by merely placing the eye in position to see the volt line at the edge of the silver mirror, and observing the needle past the edge. This operation is no more difficult than the usual sighting of a needle over its image, and as quick. Since there is no parallax, special pains need not be taken to have the eye exactly centered laterally; in fact, both eyes may be used. The instrument in this form has been used with complete success in the Bureau's testing of carbon lamps.

A possible application of the volt scale efficiency meter is in the rating and sorting of lamps according to watts per candle, the quantity now recognized as of most importance. By having factory photometers operated at definite voltages, as is done with present volt-scale photometers, it is at once evident that volt-scale efficiency meters would indicate at one reading the voltage at which lamps should be rated. A further requirement must be met before the lamp is completely rated, namely, the candlepower or watts at the voltage indicated should be given. This can readily be done by an extension of the volt-scale idea. It is only necessary, if we desire candlepower indication, to have scales on the photometer bench giving the candlepowers at various voltages corresponding to candlepowers at the voltage used. For instance, the 16-candlepower point for the 110-volt scale would be the 19.5-candlepower point on the 114-volt scale. These scales, whose similarly numbered points may be joined by continuous lines, may be conveniently mounted on a cylinder, as is done at present with different candlepower scales where sector disks are used. When a voltage is indicated by the efficiency meter scale, the candlepower scale should be turned to the point corresponding to that voltage—a line on the cylinder, or a division on the turning mechanism. The candlepower at rated voltage is then shown by the pointer attached to the photometer screen.

If watts are desired, a similar watt scale may be made whose readings are the candlepowers in the above scale multiplied by the common watts per candle. In this case the watt reading would of course be subject to the errors of photometric setting. An alternative form of watt scale could be made by multiplying the actual efficiency reading by the candlepower reading, by the use of scales, each corresponding to an efficiency, mounted upon a cylinder as described above. In this way the candlepower reading is eliminated, and watts at the photometer voltage may be read independent of errors of candlepower estimation. Instead of actual watts, watts at the voltage corresponding to the efficiency meter indication may be plotted in a manner similar to that described for candlepowers.

It is therefore possible by combining a volt scale on a watts-per-candle meter with proper scales on the photometer, to read from one photometer setting—

1. Voltage to give a certain efficiency.
2. Candlepower or watts at that voltage.

In short, the quantities now desired in rating incandescent lamps are obtained at one operation.

It is of practical interest to know how accurately lamps may be placed at a desired watts per candle by the use of a volt scale efficiency meter. It is obvious that this depends upon the uniformity with which individual lamps conform to the voltage-watts-per-candle relation used in plotting the scale. Experiment with about 2000 lamps of several different makes has shown that those of any one make can be placed regularly within a range from 3.05 to 3.15 wpc. from readings made in the neighborhood of 3.50 wpc. With different makes the amount of "treatment" of the filament varies, calling for a different voltage exponent. This change of exponent has been found to be accounted for by the changed voltage-candlepower exponent; that is, if the exponent in the equation $\left(\frac{V_2}{V_1}\right)^k = \frac{cp_2}{cp_1}$ is 5.7 instead of 5.5, the exponent in the equation $\left(\frac{V_2}{V_1}\right)^k = \frac{wpc_1}{wpc_2}$ is 3.7 instead of 3.5.[2] In other words, any shortcoming in the efficiency meter volt scale plan is only such as would be met with in using a volt scale for candlepower work with the same lamps. In any one factory where the product is reasonably uniform, a watts-per-candle volt scale should be quite as practical as the present volt-scale photometer.

WASHINGTON, February 27, 1909.

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[2] See article by F. E. Cady, Illuminating Engineering Society Convention, October, 1908.

THE COEFFICIENT OF REFLECTION OF ELECTRICAL WAVES AT A TRANSITION POINT.

By Louis Cohen.

The subject of the propagation of electrical waves along conductors is becoming of great practical importance, problems of this nature presenting themselves in nearly all branches of electrical engineering. A discussion, therefore, of any phase of the many problems in electrical wave propagation which may arise in practice will, I trust, be of some interest.

The discussion of the problems as they are usually treated in text-books assumes a uniform conductor; this, however, is rarely the case. In practice we always have to deal with a complex network of conductors, and an interesting question arises as to what happens at a transition point when an electric wave passes over from one conductor to another of different electrical constants. We know, of course, that the velocity of propagation depends on the constants of the electrical circuits, and so for different circuits the velocity will be different; hence we may expect that, at a transition point, the wave will be partly reflected and partly transmitted. It is not always clear, however, as to what are likely to be the relative magnitudes of the reflected and transmitted waves as compared with the incoming wave.

In this brief communication I wish to develop an expression for the ratios of the reflected and transmitted waves to the incoming wave at a transition point.

Dr. O. Heaviside, who has contributed so much to our knowledge of electricity, has also indicated briefly a method for the solution of the problems under discussion. Heaviside makes use of the differential operator, the theory of which he has extensively developed and which has proven to be a powerful tool for the solution of this class of problems. In this particular problem, however, we can obtain our result in a very simple way without the aid of the differential operator.

Let us denote by V_1 the potential on a line of the incoming wave, by V_2 and V_3 the reflected and transmitted waves respectively. We assume that the potential on the line consists of a single wave, which is not usually the case, as the potential will generally be the resultant of the superposition of several waves, but as far as the transition point is concerned what is true of one wave will be true of any number of waves; hence we may limit our discussion to the case of a single wave.

The general expression for a single potential wave developed on a line of uniform inductance and capacity is as follows:

$$V = Ae^{-\alpha s} \cos (pt - \beta s)$$

where α is the damping factor, β the velocity constant, and A is the amplitude. In this particular problem, however, we are mainly concerned with the potentials and currents at points close to the transition point, the amount of damping therefore in passing from a point on one side of the transition point to a point on the other side will be very small; hence we may neglect the damping factor. If we denote by β_1 and β_2 the velocity constants of the two sections of the line on the different sides of the transition point we have the following expressions for the potentials:

$$\left.\begin{aligned} V_1 &= A_1 \cos (pt - \beta_1 s) \\ V_2 &= A_2 \cos (pt - \beta_1 s) \\ V_3 &= A_3 \cos (pt - \beta_2 s) \end{aligned}\right\} \qquad (1)$$

The currents corresponding to these potentials can be derived of course by the aid of the well-known relation:

$$C\frac{dV}{dt} = -\frac{dI}{ds}$$

hence we get,

$$\left.\begin{aligned} I_1 &= \frac{C_1 p}{\beta_1} A_1 \cos (pt - \beta_1 s) \\ I_2 &= -\frac{C_1 p}{\beta_1} A_2 \cos (pt - \beta_1 s) \\ I_3 &= \frac{C_2 p}{\beta_2} A_3 \cos (pt - \beta_2 s) \end{aligned}\right\} \qquad (2)$$

At the transition point, the potential will be the sum of the incoming and reflected waves, while the current will be represented by the difference of two waves, since they travel in opposite directions. Since there is no discontinuity of any sort at the transition point we must have the following relations satisfied:

$$\left.\begin{aligned} (A_1+A_2)\cos(pt-\beta_1 s) &= A_3\cos(pt-\beta_2 s) \\ \frac{C_1 p}{\beta_1}(A_1-A_2)\cos(pt-\beta_1 s) &= \frac{C_2 p}{\beta_2}A_3\cos(pt-\beta_2 s) \end{aligned}\right\} \tag{3}$$

If we take the origin at the transition point, s will be zero in the above equations, and we have by dividing the first equation by the second,

$$\frac{A_1+A_2}{A_1-A_2}=\frac{C_1\beta_2}{C_2\beta_1} \tag{4}$$

and

$$\frac{A_1+A_2}{A_1-A_2}+1=\frac{2A_1}{A_1-A_2}=\frac{C_1\beta_2+C_2\beta_1}{C_2\beta_1}$$

$$\frac{A_1-A_2}{2A_1}=\frac{1}{2}-\frac{A_2}{2A_1}=\frac{C_2\beta_1}{C_1\beta_2+C_2\beta_1}$$

$$\frac{A_2}{A_1}=\frac{C_1\beta_2-C_2\beta_1}{C_1\beta_2+C_2\beta_1} \tag{5}$$

$\frac{A_2}{A_1}$ is the ratio of the amplitudes of the reflected to the incoming wave. In case of very high frequencies we may neglect the resistance as compared with the reactance and the expressions for β_1 and β_2 will be $p\sqrt{L_1C_1}$ and $p\sqrt{L_2C_2}$; replacing β_1 and β_2 in equation (5) by their corresponding values we get:

$$\frac{A_2}{A_1}=\frac{\sqrt{\frac{L_2}{C_2}}-\sqrt{\frac{L_1}{C_1}}}{\sqrt{\frac{L_2}{C_2}}+\sqrt{\frac{L_1}{C_1}}} \tag{6}$$

The ratio of the transmitted wave to the incoming wave can be obtained in the following way:

We have by equation (3),

$$A_1 + A_2 = A_3$$

Replacing A_2 by its value in terms of A_1 from equation (5) we get,

$$A_1\left\{1 + \frac{C_1\beta_2 - C_2\beta_1}{C_1\beta_2 + C_2\beta_1}\right\} = A_3$$

Therefore,

$$\frac{A_3}{A_1} = \frac{2C_1\beta_2}{C_1\beta_2 + C_2\beta_1} = \frac{2\sqrt{\frac{L_2}{C_2}}}{\sqrt{\frac{L_2}{C_2}} + \sqrt{\frac{L_1}{C_1}}} \qquad (7)$$

We see by equation (7) that if we pass from a line of low inductance and high capacity into a line of high inductance and low capacity the voltage will be increased. As an example, we may consider the following case: Suppose we connect an underground cable, whose constants are $L = 0.4 \times 10^{-3}$ henry and $C = 0.6 \times 10^{-6}$ farads, with that of an overhead line whose constants are $L = 1.95 \times 10^{-3}$ henry and $C = 0.0162 \times 10^{-6}$ farads, in passing from the underground cable to the overhead line we shall have,

$$\frac{A_3}{A_1} = \frac{2\sqrt{\frac{1.95}{0.0162} \times 10^3}}{\sqrt{\frac{1.95}{0.0162} \times 10^3} + \sqrt{\frac{0.4}{0.6} \times 10^3}} = 1.9$$

The potential will rise to nearly double its value.

The results as given by equations (6) and (7) are in agreement with some of the results obtained by Professor Steinmetz[1] in his discussion of the "General Equation of the Electric Circuit."

The relations between the reflected and transmitted waves to the incoming wave as given by equations (6) and (7) are only applicable to the case where the wave in passing the transition point continues its travel in the form of a wave; that is, we have distributed inductance and capacity on both sides of the transition point. In the case, however, where at the transition point we have a lumped inductance and capacity the conditions are different.

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[1] C. P. Steinmetz: Proc. Am. Inst. of Elect. Eng., **27**, p. 1190.

Let us denote the impedance of the apparatus at the transition point by Z, and as before we will denote the potential of the incoming wave by $V_1 = A_1 \cos (pt - \beta s)$; that of the reflected wave by $V_2 = A_2 \cos (pt - \beta s)$. The currents corresponding to these two waves will be

$$\frac{Cp}{\beta} A_1 \cos (pt - \beta s) \text{ and } \frac{Cp}{\beta} A_2 \cos (pt - \beta s).$$

Now, at the transition point, the potential is the sum of the incoming and reflected waves, and this must be equal to the drop of potential at the terminal apparatus.

We have, therefore,

$$V_1 + V_2 = ZI$$

But the current at the transition point is the difference of the currents of the incoming and reflected waves, since they travel in opposite directions; hence we have,

$$V_1 + V_2 = Z(I_1 - I_2)$$

Introducing the values of V_1, V_2, I_1 and I_2, we get,

$$(A_1 + A_2) \cos (pt - \beta s) = \frac{pC}{\beta} Z (A_1 - A_2) \cos (pt - \beta s)$$

From which we obtain,

$$\frac{A_1 + A_2}{A_1 - A_2} = \frac{pCZ}{\beta}$$

Proceeding in the same way as in deriving equation (5), we obtain,

$$\frac{A_2}{A_1} = \frac{pCZ - \beta}{pCZ + \beta} \tag{8}$$

If we denote the inductance and capacity of the terminal apparatus by L_1, C_1 and neglect the resistance, we have $Z = pL_1 - \frac{1}{pC_1}$.

Replacing also β by its corresponding value $p\sqrt{LC}$, equation (8) will become

$$\frac{A_2}{A_1}=\frac{C\left(pL_1-\frac{1}{p_1C}\right)-\sqrt{LC}}{C\left(pL_1-\frac{1}{p_1C}\right)+\sqrt{LC}} \tag{9}$$

In the case of a free oscillation, we have $p=\frac{1}{\sqrt{LC}}$ and therefore,

$$\frac{A_2}{A_1}=\frac{\left(\frac{L_1}{\sqrt{LC}}-\frac{\sqrt{LC}}{C_1}\right)-\sqrt{\frac{L}{C}}}{\left(\frac{L_1}{\sqrt{LC}}-\frac{\sqrt{LC}}{C_1}\right)+\sqrt{\frac{L}{C}}} \tag{10}$$

The rise of potential at the terminal, or the ratio of the potential at the transition point to that of line will be,

$$\frac{A_2+A_1}{A_1}=\frac{2\left(\frac{L_1}{\sqrt{LC}}-\frac{\sqrt{LC}}{C_1}\right)}{\left(\frac{L_1}{\sqrt{LC}}-\frac{\sqrt{LC}}{C_1}\right)+\sqrt{\frac{L}{C}}} \tag{11}$$

If the inductance of the terminal apparatus is very high, we may write approximately equation (11) in the following form:

$$\frac{A_2+A_1}{A_1}=\frac{2\,L_1}{L_1+L}$$

The voltage may therefore rise to nearly twice the voltage of the line

WASHINGTON, February 27, 1909.

A TUNGSTEN COMPARISON LAMP IN THE PHOTOMETRY OF CARBON LAMPS.

By Herbert E. Ives and L. R. Woodhull.

In the photometry of incandescent lamps the substitution method is commonly used. That is, the lamp measured is not compared directly with a standard, but with a "comparison lamp" which is carefully calibrated against a candlepower standard or standards at the beginning of a run, and at frequent intervals. This method is adopted not only to obtain the greater accuracy of a substitution method, but to avoid operating a valuable standard continuously, with consequent comparatively rapid change in its value. After the standard check, the results of the measurement depend entirely upon the correctness and constancy of the comparison lamp. Since, however, the comparison lamp is subject to the changes inherent in the type of filament, it is necessary with carbon lamps to use only seasoned ones, and to keep close watch on their candlepower by checking with standards.

Some time ago, Professor Rosa, who is in charge of the photometric work of the Bureau, requested us to try a Tungsten lamp, operated at voltages to give carbon lamp colors, as a comparison lamp in the testing of carbon filament lamps. The object in view was to obtain a long-lived and constant comparison light which at the same time would give perfect color match with all types of carbon lamps. It was expected that this would be found the case with the Tungsten lamp, because, first, the candlepower changes with this filament are small during life, even at rated voltage; secondly, the life of all glow lamps is greatly increased at high watts per candle, and the Tungsten lamp matches the carbon lamps in color at efficiencies near 3 watts per mean horizontal candle, or two and one-half times the normal rating of 1.25 watts per candle. If the relationship holding between watts per candle and life for carbon lamps is approximately true for the Tungsten as well, this means an increase in life of at least twenty times. Consequently the qualities of longevity and constancy could be predicted. The further advantage of color match is of great importance where accurate photometric work is required. Concordant readings by different observers can not be expected where a color difference exists in the lights measured. In the Bureau's commercial work, separate standards for 3.5 watts per mean horizontal candlepower, and 3.1 watts per candle are used. The values of these are the means of readings made against the primary 4-watt standards by all the members of the photometric section. With lamps of each efficiency, the appropriate standards and appropriate comparison lamp color are used. In this way the color difference difficulty is eliminated, except in the preparation of the standards, where it is met by securing once for all the mean and most probable value obtainable. Since the Tungsten lamp at low voltages will match either 2.5, 3.1, 3.5, or 4 watts-per-candle color it lends itself well to this plan.

Two months' experience with the Tungsten comparison lamp has shown it to fully equal expectations, and to be much superior for the purpose to carbon lamps.

The details of installation are here given as a guide to any who may desire to duplicate the arrangement. An 80-watt, 120-volt lamp was taken, and the voltages determined experimentally, at which it gave the same color as 4, 3.5, 3.1, and 2.5 (Gem) watts-per-candle lamps. These voltages were 79, 84, 87, and 97, respectively, corresponding approximately to 4.5, 3.7, 3.4 and 2.6 watts per mean spherical candle. Obviously the candlepower varied greatly for these different voltages. With a candlepower scale, as used on the Bureau of Standards commercial photometer, it is necessary to have the comparison lamp accurately 16 candlepower. A means to accomplish this with the Tungsten lamp was imperative, and was found in the use of diaphragms, since the candlepower of the 80-watt lamp at all colors was greater than 16. It was impossible to place the diaphragms before the lamp itself since the distance of the diaphragm from the center of radiation, due to the size of the bulb, would permit an error of 3 per cent in the candlepower readings at the ends of the scale. Two means of overcoming this difficulty presented themselves. First, an image of the filament could be formed by a lens or concave mirror, and the image diaphragmed. Secondly, a ground-glass screen could be placed in front of the lamp, and the diaphragms placed over this. The latter plan was adopted as most convenient, although it was then necessary to back the lamp with a mirror to secure sufficient light. With a larger lamp the mirror would be unnecessary. Lamp, mirror, and ground-glass screen were mounted together on a stand, the ground glass being held in a groove wide enough to receive the several diaphragms which cut down the light to very closely 16 candlepower for each color. These diaphragms vary from 2 to 3.5 inches square, and are all small enough so that the error due to considering the glass surface as a point source is negligible.

In order to set the lamp to give exactly 16 candlepower, the same procedure was adopted as in the Bureau's work with carbon lamps. The voltage is adjusted by trial (or by knowledge from previous work) to give very nearly the correct candlepower reading. Then a number of standards are read, and from the mean deviation of the readings from the standards' true values the change in voltage which will give the correct reading is calculated. For carbon lamps (3.5 watts per candle) the percentage change is given by the relation $\left(\frac{V_2}{V_1}\right)^{5.6} = \frac{cp_2}{cp_1}$, or in the differential form, $5.6\, dV = d(cp)$. In order to follow the same plan with the Tungsten lamp, it was necessary to know the voltage-candlepower relation at the efficiencies used. This was obtained from data given by Mr. F. E. Cady,[1] from which it follows that from 2.6 to 4.5 watts per mean spherical candle the voltage exponent ranges from 3.8 to 4; the relationship to use is therefore $3.9\, dV = d(cp)$. By the use of this relation (actual numerical values are kept tabulated on the photometer table) the comparison voltage is accurately fixed.

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[1] Transactions Illuminating Engineering Society, October, 1908.

Because of the comparatively low voltage of the Tungsten lamp it has been found convenient to place in series with it an adjustable rheostat, and measure voltage across the two. In this way, by placing stops on the rheostat corresponding to 3.1, 3.5, etc., the comparison voltage is made uniformly 100 volts, or some other convenient voltage if desired. For instance, with a run of 110-volt lamps, the voltage of each side of the photometer can be 110, and may be checked without moving from the same potentiometer post.

The Tungsten comparison lamp has proved eminently convenient and practical, and after two months' daily running of from four to seven hours, chiefly at 3.5 and 3.1 colors, the voltage to give 16 candlepower has changed less than two-tenths of 1 per cent.

WASHINGTON, February 27, 1909.

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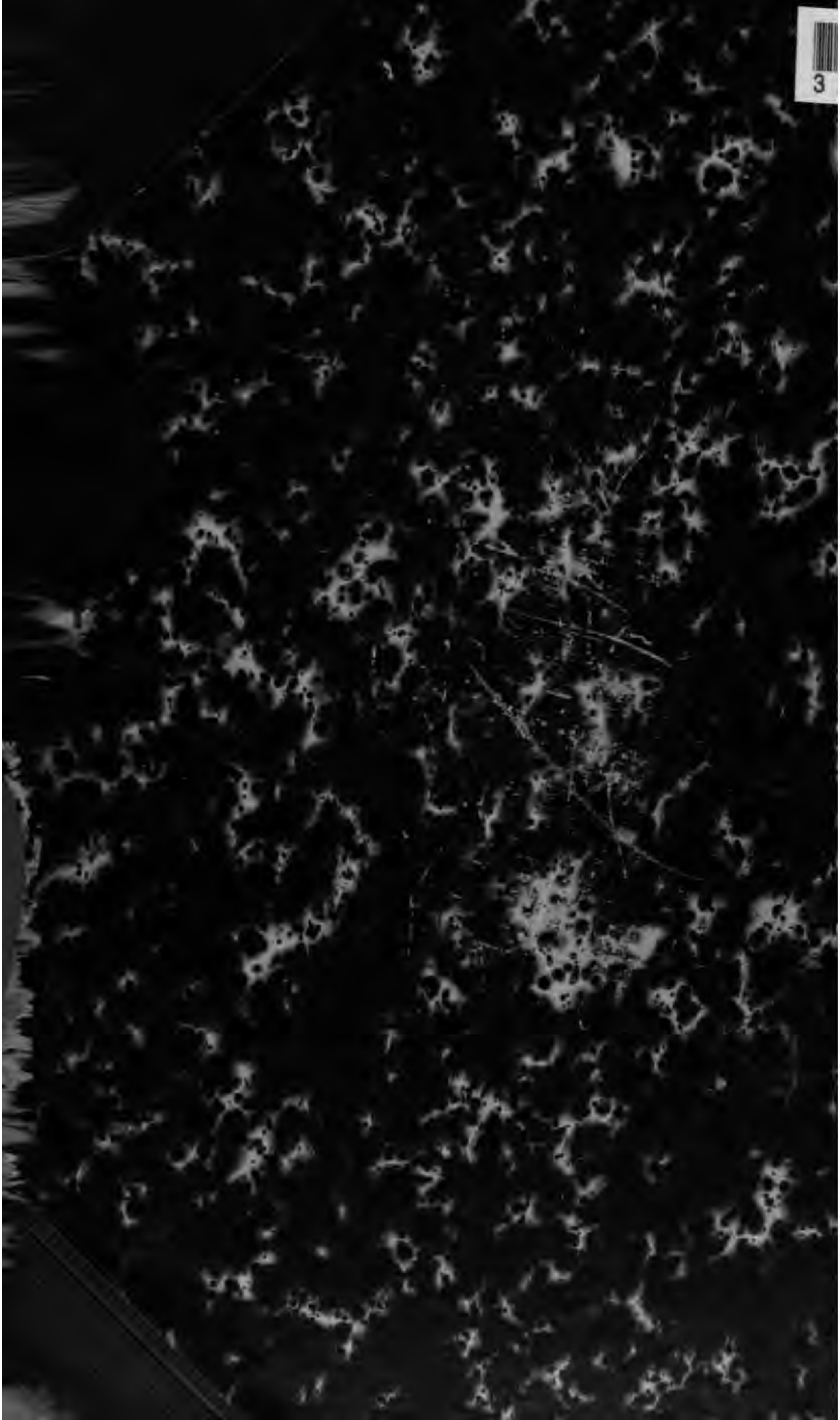

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